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SOME PROPERTIES OF THE DENSITY TOPOLOGY WITH RESPECT TO AN EXTENSION OF THE LEBESGUE MEASURE

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Abstract: We investigate some properties of the μ-density topology where μ is an extension of the Lebesgue measure.

Let μ denote an extension of the Lebesgue measure l over the real line $\mathbb{R}$. Let S_μ denote the domain of μ and let $\mathcal{L}$ be the σ-field of all Lebesgue measurable sets. Let $\mathcal{I}_\mu$ be the σ-ideal of μ-null sets and L the σ-ideal of Lebesgue null sets. By $\mathcal{T}_d$ we denote the density topology and by $\mathcal{T}_0$ - the natural topology in $\mathbb{R}$. We recall that $x \in \mathbb{R}$ is a density point of a μ-measurable set X if

$$\lim_{h\to 0} \frac{\mu(X \cap [x-h, x+h])}{2h} = 1.$$

Let $\Phi_\mu(X) = \{x \in \mathbb{R};\quad x \text{ is a density point of } X\}$. Let us define a family $\mathcal{T}_\mu$ in the following way:

$$\mathcal{T}_\mu = \{X \in S_\mu, X \subset \Phi_\mu(X)\}.$$

Theorem A (cf. [5], [6]). *$\mathcal{T}_\mu$ is a topology in $\mathbb{R}$.*

If $\mu = l$,then the family $\mathcal{T}_l$ is the topology in $\mathbb{R}$, usually called density topology and labelled by $\mathcal{T}_d$ (cf. [9]). It was observed in [5] that every $\mathcal{T}_\mu$-open set is a member of the σ-field $\mathcal{L}\triangle\mathcal{I}_\mu$ and the topology $\mathcal{T}_\mu$ is generated by the density topology $\mathcal{T}_d$ and the σ-ideal $\mathcal{I}_\mu$ (cf. [4]). Namely, we have

Theorem B (cf. [5]). *Every $\mathcal{T}_\mu$-open set X has the form $U - Z$ where U is $\mathcal{T}_d$-open and $Z \in \mathcal{I}_\mu$ (abbr. $\mathcal{T}_\mu = \mathcal{T}_d \ominus \mathcal{I}_\mu$). Moreover, the family of all meager sets in the topology $\mathcal{T}_\mu$ is identical with the family $\mathcal{I}_\mu$.*

The important role in our further considerations is played by the consequence of Theorems 4 and 5 in [8] which we can establish in the following form:

Theorem C. *Let $(X, \mathcal{T})$ be an arbitrary topological space, (Y, τ) a regular topological space and $\mathcal{I}$ a σ-ideal of subsets of X free from nonempty $\mathcal{I}$-open sets and such that family of sets*

$$\mathcal{T} \ominus \mathcal{I} = \{Z \subset X : Z = U - P, \quad U \in \mathcal{T}, \quad P \in \mathcal{I}\}$$

forms a topology (called the Hashimoto topology). Then

$$C\big((X,\mathcal{T}),(Y,\tau)\big) = C\big((X,\mathcal{T}\ominus\mathcal{I}),(Y,\tau)\big)$$

where $C\big((X,\mathcal{T}),(Y,\tau)\big)$ is family of all continuous functions acting from space $(X,\mathcal{T})$ to (Y,τ) and $C\big((X,\mathcal{T}\ominus\mathcal{I}),(Y,\tau)\big)$ is the family of all continuous functions acting from the space $(X,\mathcal{T}\ominus\mathcal{I})$ to the space (Y,τ).

Now, we present some properties of $\mathcal{T}_\mu$-topology in the context of the properties of $\mathcal{T}_d$-topology and $\mathcal{I}$-density topology, contained in [1]. As an obvious conclusion of Th. C we have

Property 1. *The family of all real continuous functions with respect to the topology $\mathcal{T}_\mu$ is identical with the family of all approximate continuous functions.*

Since the topology $\mathcal{T}_d$ is connected, we see that:

Property 2. *The space $(\mathbb{R}, \mathcal{T}_\mu)$ is connected.*

We easy conclude that

Property 3. *The space $(\mathbb{R}, \mathcal{T}_\mu)$ is Hausdorff.*

Property 4. *A set X is closed and discrete in $\mathcal{T}_\mu$ if and only if $X \in \mathcal{I}_\mu$.*

Proof. Let $X \in \mathcal{I}_\mu$. Then X is $\mathcal{T}_\mu$-closed and, since the measure μ is complete, any subset of X is a μ-null and $\mathcal{T}_\mu$-closed set. Let us suppose that X is closed and discrete in the topology $\mathcal{T}_\mu$, and $X \notin \mathcal{I}_\mu$. Hence $\operatorname{Int}X = \emptyset$ because, otherwise, the set $\operatorname{Int}X$ being open and closed in $\mathcal{T}_\mu$ which is connected would coincide with $\mathbb{R}$. This would contradict the fact that $\mathcal{T}_\mu$ is connected. In such a way, X is nowhere dense in $\mathcal{T}_\mu$, which implies, by Th. B that X is a μ-null set. $\Diamond$

Property 5. *The space $(\mathbb{R}, \mathcal{T}_\mu)$ is neither separable nor possesses the Lindelöf property.*

Proof. Any countable set is a μ-null set. This implies by Th. B, that it is closed in the topology $\mathcal{T}_\mu$. Hence $\mathbb{R}$ is not separable with respect to $\mathcal{T}_\mu$. Let X be the Cantor set. It is clear that X is a μ-null set and thus $\mathcal{T}_\mu$-closed. Then each set $U_x = (\mathbb{R} - C) \cup \{x\}$ is $\mathcal{T}_\mu$-open and $\bigcup_{x \in C} U_x = \mathbb{R}$, but there does not exist a countable subfamily $\{U_x\}_{x \in C}$ covering $\mathbb{R}$. $\Diamond$

Lemma 1. *The space $(\mathbb{R}, \mathcal{T}_\mu)$ is regular if and only if, for an arbitrary set $X \in \mathcal{I}_\mu$ and any point $x \notin X$, there exist disjoint $\mathcal{T}_d$-open sets V_1, V_2 such that $X \subset V_1$ and $x \in V_2$.*

Proof. Let $X \in \mathcal{I}_\mu$ and $x \notin X$. Since $\mathcal{T}_\mu$ is regular, there exist $\mathcal{T}_\mu$-open sets W_1, W_2 such that $X \subset W_1$, $x \in W_2$ and $W_1 \cap W_2 = \emptyset$. By Th. 2, $W_1 = V_1 - Z_1$, $W_2 = V_2 - Z_2$, where V_1, $V_2 \in \mathcal{T}_d$ and Z_1, $Z_2 \in \mathcal{I}_\mu$. We see that $V_1 \cap V_2 = \emptyset$ if and only if $W_1 \cap W_2 = \emptyset$. In fact, $V_1 \cap \cap V_2 = W_1 \cap W_2 - (Z_1 \cup Z_2) = \emptyset$ implies that $W_1 \cap W_2 \subset Z_1 \cup Z_2$ and $0 = \mu(W_1 \cap W_2) = l(W_1 \cap W_2)$. Hence $W_1 \cap W_2 = \emptyset$ because, otherwise, $l(W_1 \cap W_2) > 0$. If $W_1 \cap W_2 = \emptyset$, then $V_1 \cap V_2 = \emptyset$. Hence the sets W_1 and W_2 separate the sets X and $\{x\}$.

Now, let F be $\mathcal{T}_\mu$-closed and let $x \notin F$. The set F is the union of a $\mathcal{T}_d$-closed set F_1 and a μ-null set X. Since $x \notin F_1$ and the topology $\mathcal{T}_d$ is regular, then there exist $\mathcal{T}_d$-open sets V_1, V_2 such that $F_1 \subset V_1$ and $V_1 \cap V_2 = \emptyset$. By the assumption, there exist $\mathcal{T}_\mu$-open sets V_3, V_4 such that $X \subset V_3$, $x \in V_4$ and $V_3 \cap V_4 = \emptyset$. Putting $V_1 \cup V_3$ and $V_2 \cap V_4$, we have $\mathcal{T}_\mu$-open sets separating the sets F and $\{x\}$. $\Diamond$

Property 6. *The space $(\mathbb{R}, \mathcal{T}_\mu)$ is regular if and only if $\mathcal{T}_\mu = \mathcal{T}_d$.*

Proof. Sufficiency is a consequence of the fact that the $\mathcal{T}_d$-topology is regular (cf. [2]).

Necessity. Let us suppose that $\mathcal{T}_\mu$ is regular and $\mathcal{T}_\mu \neq \mathcal{T}_d$. Hence there exists a set $X \in \mathcal{I}_\mu \backslash L$. It is clear that $X \notin \mathcal{L}$. Let S be a measurable cover of X. We see that $\Phi_l(S) \backslash X \neq \emptyset$ because, otherwise, $\Phi_l(S) \subset X \subset S$, and X would be Lebesgue measurable. Let $x \in \in \Phi_l(S) \backslash X$. Since $\mathcal{T}_\mu$ is regular, by Lemma 1, there exist $\mathcal{T}_d$-open sets V_1 and V_2 such that $V_1 \supset X$, $x \in V_2$ and $V_1 \cap V_2 = \emptyset$. Then

$$V_2 \subset \mathbb{R} - V_1$$

and

$$S \cap V_2 \subset S - V_1 \subset S - X.$$

From the definition of the cover S we conclude that $l(S - V_1) = 0$ and $\Phi_l(S \backslash V_1) = \emptyset$. This implies that $\Phi_l(S \cap V_2) = \Phi_l(S) \cap \Phi_l(V_2) = \emptyset$. At the same time, $x \in \Phi_l(S) \cap V_2 \subset \Phi_l(S) \cap \Phi_l(V_2) = \emptyset$. This contradiction ends the proof. $\Diamond$

Remark. We have proved that there exist a $\mathcal{T}_\mu$-closed set X and a point $x \notin X$, such that the sets X and $\{x\}$ cannot be separated by $\mathcal{T}_d$-open sets.

We are able to demonstrate a much stronger result. Namely, there exists a $\mathcal{T}_\mu$-closed set X^* such that, for each point $x \notin X^*$, the sets X and $\{x\}$ cannot be separated by $\mathcal{T}_d$-open sets. Let $X \in \mathcal{I}_\mu \backslash L$ and let S be a Lebesgue measurable cover of X such that $X \subset \Phi_l(S)$. Putting $X^* = X \cup (\mathbb{R} - \Phi_l(S))$, we see that X^* is $\mathcal{T}_\mu$-closed. Then $x \notin X^*$ if and only if $x \in \Phi_l(S) \backslash X$ and, analogously as in the proof of the above property, we conclude that the set X^* has the desired property.

Property 7. *The space $(\mathbb{R}, \mathcal{T}_\mu)$ is completely regular if and only if $\mathcal{T}_\mu = \mathcal{T}_d$.*

Proof. Sufficiency is a consequence of the fact that the $\mathcal{T}_d$-topology is completely regular (cf. [3]). Necessity is a consequence of Prop. 6. $\Diamond$

Property 8. *The space $(\mathbb{R}, \mathcal{T}_\mu)$ is not normal.*

Proof. The topology $\mathcal{T}_d$ is not normal (cf. [3]). Let $\mathcal{T}_\mu \neq \mathcal{T}_d$. If $\mathcal{T}_\mu$ where normal, then $\mathcal{T}_\mu$ would be completely regular and, by Prop. 7, we have the contradiction with the fact that $\mathcal{T}_\mu = \mathcal{T}_d$. $\Diamond$

Property 9. *A set X is $\mathcal{T}_\mu$-compact if and only if it is finite.*

Proof. If X finite, then it is $\mathcal{T}_\mu$-compact. Let X be $\mathcal{T}_\mu$-compact. Then we claim that X is finite. Let us suppose that X is infinite. Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence of distinct elements of X. Putting $U_n = \mathbb{R} - \{x_k : k \geq n\}$, we have that $\mathbb{R} \supset \bigcup_{n=1}^{\infty} U_n$, but there does not exist a finite subfamily of $\{U_n\}_{n \in \mathbb{N}}$ covering $\mathbb{R}$. $\Diamond$

Lemma 2. *A $\mathcal{T}_\mu$-open set X is $\mathcal{T}_\mu$-regular open if and only if $X = \Phi_\mu(X)$*

Proof. First of all, we prove that, for each $X \in \mathcal{T}_\mu$, the set $\Phi_\mu(X)$ is $\mathcal{T}_\mu$-regular open. We see that $\Phi_\mu(X)$ is $\mathcal{T}_\mu$-open. It is a consequence of the fact that it is sufficient to consider the case where $S_\mu = \mathcal{L} \triangle \mathcal{I}_\mu$ (see [5]) and then $\Phi_\mu(\Phi_\mu(X)) = \Phi_\mu(X)$. Now, we show that $\Phi_\mu(X) = \mathrm{Int}\overline{\Phi_\mu(X)}$ with respect to $\mathcal{T}_\mu$. Since $\Phi_\mu(X)$ is $\mathcal{T}_\mu$-open, then $\overline{\Phi_\mu(X)} = \Phi_\mu(X) \cup Z$ where $Z = Fr(\Phi_\mu(X))$ is nowhere dense and thus $Z \in \mathcal{I}_\mu$. Let U be any open set in $\mathcal{T}_\mu$. Then $U = V - Y$ where $V \in \mathcal{T}_d$, $Y \in \mathcal{I}_\mu$. We can assume that $V = \Phi_l(W)$ where $W \in \mathcal{L}$. We see that if $U \subset \overline{\Phi_\mu(X)}$, then

$$\Phi_l(W) - Y \subset \Phi_l(W) = \Phi_\mu(\Phi_l(W) - Y)$$
$$\in \Phi_\mu(\Phi_\mu(X) \cup Z) = \Phi_\mu(\Phi_\mu(X)) = \Phi_\mu(X).$$

Since the set $\Phi_\mu(X)$ is $\mathcal{T}_\mu$-open, therefore $\Phi_\mu(X) = \mathrm{Int}\overline{\Phi_\mu(X)}$. Hence if $X = \Phi_\mu(X)$, then X is regular open. Let X be regular open. Since $\mu(X \triangle \Phi_\mu(X)) = 0$, the set $X \triangle \Phi_\mu(X)$ is nowhere dense. But the sets X and $\Phi_\mu(X)$ are $\mathcal{T}_\mu$-regular open in the Baire space $(\mathbb{R}, \mathcal{T}_\mu)$, whence $X = \Phi_\mu(X)$. ◊

Property 10. *A set X is $\mathcal{T}_\mu$-regular open if and only if X is $\mathcal{T}_d$-regular open.*

Proof. If X is $\mathcal{T}_d$-regular open, then $X = \Phi_l(X) = \Phi_\mu(X)$ and X is $\mathcal{T}_\mu$-regular open. If X is $\mathcal{T}_\mu$-regular open, then $X = \Phi_\mu(X)$. But we may assume that $X \in \mathcal{L} \triangle \mathcal{I}_\mu$; then there exists a Lebesgue measurable set Y such that $X = \Phi_\mu(X) = \Phi_l(Y)$. This implies that X is $\mathcal{T}_d$-regular open. ◊

Property 11. *By assuming C.H., the topological space $(\mathbb{R}, \mathcal{T}_\mu)$ is not a Blumberg space for any complete extension μ of the Lebesgue measure.*

Proof. We shall explore the fact that under the assumption of C.H. the topological space $(\mathbb{R}, \mathcal{T}_d)$ is not a Blumberg space (cf. [1]). Let us suppose that for some complete extension μ of the Lebesgue measure, such that $\mathcal{T}_\mu \neq \mathcal{T}_d$, $(R, \mathcal{T}_\mu)$ is a Blumberg space. This means that, for any function $f : \mathbb{R} \to \mathbb{R}$, there exists a $\mathcal{T}_\mu$-dense set D such that $f/_D$ is a $\mathcal{T}_\mu$-continuous function. Let us fix a function $f : \mathbb{R} \to \mathbb{R}$ and let D be a $\mathcal{T}_\mu$-dense set such that $f/_D$ is continuous. This is equivalent to the fact that f is continuous with respect to the topology $\mathcal{T}_\mu \cap D = \{X \subset \subset \mathbb{R} : X = V \cap D; \quad V \in \mathcal{T}_\mu\}$. Since $\mathcal{T}_\mu$ is the Hashimoto topology of the form $\mathcal{T}_d \ominus \mathcal{I}_\mu$, we have that $\mathcal{T}_\mu \cap D$ is the Hashimoto topology of the form $(\mathcal{T}_d \cap D) \ominus \mathcal{I}_D$ where $\mathcal{I}_D = \mathcal{I}_\mu \cap D = \{X \subset R : X = = Z \cap D, \quad Z \in \mathcal{I}_\mu\}$. We notice that the σ-ideal $\mathcal{I}_D$ is free from the nonempty $\mathcal{T}_d \cap D$-open sets. Otherwise, there exist sets $V \in \mathcal{T}_d$ and $Z \in \mathcal{I}_\mu$ such that $V \cap D = Z \cap D$. Hence $(V - Z) \cap D = \emptyset$. But the set $V - Z$ is nonempty and $\mathcal{T}_\mu$-open. Then if the set D is $\mathcal{T}_\mu$-dense, $(V - Z) \cap D \neq \emptyset$. We get a contradiction.

Now, applying Th. C, we have that

$$C\big((R, (\mathcal{T}_d \cap D) \ominus \mathcal{I}_D), (R, \mathcal{T}_0)\big) = C\big((R, \mathcal{T}_d \cap D), (R, \mathcal{T}_0)\big).$$

This implies that the function f is continuous with respect to the topology $\mathcal{T}_d \cap D$ and thus $f/_D$ is continuous. At the same time, the set D is $\mathcal{T}_d$-dense, and we got a contradiction with the fact that the space

$(R, \mathcal{T}_d)$ is not a Blumberg space. $\Diamond$

Definition 1. (cf. [2]). Let $(X, \mathcal{T})$ be a topological space. If, for any topology $\mathcal{T}'$ on X with the property that the set of continuous selfmaps $f : (X, \mathcal{T}') \to (X, \mathcal{T}')$ contains the set of continuous selfmaps $f : (X, \mathcal{T}) \to (X, \mathcal{T})$, it is also true that $\mathcal{T}' \supset \mathcal{T}$, then $(X, \mathcal{T})$ is called *generated.*

Property 12. *The topological space* $(\mathbb{R}, \mathcal{T}_\mu)$ *for any complete extension* μ *of the Lebesgue measure is not generated.*

Proof. When $\mathcal{T}_\mu = \mathcal{T}_d$, it was proved in [2] that $(\mathbb{R}, \mathcal{T}_d)$ is not generated. Let $\mathcal{T}_\mu \neq \mathcal{T}_d$. Let $C\big((\mathbb{R}, \mathcal{T}_\mu), (\mathbb{R}, \mathcal{T}_d)\big)$ be the family of all continuous functions $f : (\mathbb{R}, \mathcal{T}_\mu) \to (\mathbb{R}, \mathcal{T}_d)$ and let $C\big((\mathbb{R}, \mathcal{T}_d), (\mathbb{R}, \mathcal{T}_d)\big)$ be the family of all continuous functions $f : (\mathbb{R}, \mathcal{T}_d) \to (\mathbb{R}, \mathcal{T}_d)$. Since the space $(\mathbb{R}, \mathcal{T}_d)$ is regular (see [3]) and $\mathcal{T}_\mu$ is the Hashimoto topology $\mathcal{T}_d \ominus \mathcal{I}_\mu$, we infer that

$$C\big((\mathbb{R}, \mathcal{T}_\mu), (\mathbb{R}, \mathcal{T}_d)\big) = C\big((\mathbb{R}, \mathcal{T}_d), (\mathbb{R}, \mathcal{T}_d)\big).$$

Simultaneously, we see that

$$C\big((\mathbb{R}, \mathcal{T}_\mu), (\mathbb{R}, \mathcal{T}_\mu)\big) \subset C\big((\mathbb{R}, \mathcal{T}_\mu), (\mathbb{R}, \mathcal{T}_d)\big)$$

Hence

$$C\big((\mathbb{R}, \mathcal{T}_\mu), (\mathbb{R}, \mathcal{T}_\mu)\big) \subset C\big((\mathbb{R}, \mathcal{T}_d), (\mathbb{R}, \mathcal{T}_d)\big),$$

but it is not true that $\mathcal{T}_\mu \subset \mathcal{T}_d$ because $\mathcal{T}_\mu \neq \mathcal{T}_d$. Hence we conclude that the topological space $(\mathbb{R}, \mathcal{T}_\mu)$ is not generated. $\Diamond$

It is well known that the density with respect to the Lebesgue measure has the following property called the Lusin-Menchoff property:

Theorem D (cf. [3]). *Let* E *be a measurable Lebesgue set and let* F *be a closed set such that* $F \subset E$ *and every point of* F *is the density point of* E *then there exists a perfect set* P *such that* $F \subset P \subset E$ *and every point of* F *is the density point of* P.

The Lusin-Menchoff property was published first time by Bogomolova in 1924 when the density topology $\mathcal{T}_d$ was not known. Now we can interpretate this property as the some property of the topology $\mathcal{T}_d$ introduced in 1952 and described in detail in 1961 (cf. see [3]). We have the following property.

Proposition. *The Lusin-Menchoff property is satisfied if and only if for every set* $U \in \mathcal{T}_d$ *and for every closed set* $F \subset U$ *there exists a* $\mathcal{T}_d$*-open set* V *such that* $F \subset V \subset \bar{V} \subset U$.

Proof. Let U be a nonempty $\mathcal{T}_d$-open set and let F be nonempty closed set such that $F \subset U$. Since U is $\mathcal{T}_d$-open then every point of U is the

Lebesgue density point of U, thus by the Lusin-Menchoff property there exists a perfect set P such that $F \subset P \subset U$ and $F \subset \Phi_l(P)$. Putting $V = P \cap \Phi_l(P)$ we see that V is $\mathcal{T}_d$-open and $F \subset V \subset \bar{V} \subset U$.

Sufficiency. Let E be a Lebesgue measurable set and let F be a closed set such that $F \subset E$ and every point of F is the Lebesgue density point of E. Putting $U = E \cap \Phi_l(E)$ we have that U is $\mathcal{T}_d$-open set. Thus there exists a $\mathcal{T}_d$-open set V such that $F \subset V \subset \bar{V} \subset U$. Let $\bar{V} = P \cup Z$ where P is perfect and Z is the set of all isolated points of $\bar{V}$. We see that P is countable. Moreover $V \cap Z = \emptyset$. Otherwise there exists a member $x \in V \cap Z$. It implies that x is a density point of V and x is an isolated point of V. This contradiction proves that $V \cap Z = \emptyset$. Thus we have that $F \subset P \subset E$ and every point of F is the density point of P. $\Diamond$

According to this theorem we see that we can consider the Lusin-Menchoff property of the density as the property of $\mathcal{T}_d$-topology with respect to the natural topology. This is a good starting point to formulate the Lusin-Menchoff property in more generale situation.

Let τ_1 and τ_2 be the topologies on the space X such that $\tau_2 \supset \tau_1$

Definition 2 (cf. [7]). We shall say that the topology τ_2 has the *Lusin-Menchoff property* with respect to the topology τ_1 if for every pair of disjoint sets $F_{\tau_1}, F_{\tau_2} \subset X$ such that F_{τ_1} is τ_1-closed and F_{τ_2} is τ_2-closed there exist disjoint sets $G_{\tau_1}, G_{\tau_2} \subset X$ such that G_{τ_1} is τ_1-open and G_{τ_2} is τ_2-open and $F_{\tau_1} \subset G_{\tau_2}$, $F_{\tau_2} \subset G_{\tau_1}$.

This definition is equivalent to the following one (see [7]):

Definition 3. We shall say that the topology τ_2 has the *Lusin-Menchoff property* with respect to the topology τ_1 if for every $U_{\tau_2} \in \tau_2$ and every τ_1-closed set F_{τ_1} such that $F_{\tau_1} \subset U_{\tau_2}$ there exists an τ_2-open set V_{τ_2} such that $F_{\tau_1} \subset V_{\tau_2} \subset \bar{V}_{\tau_2}^{(\tau_1)} \subset U_{\tau_2}$.

We investigate the Lusin-Menchoff property of the topology $\mathcal{T}_\mu$ with respect to the topology $\mathcal{T}_d$ and natural topology. Firstly we have the following

Lemma 3. *If τ_1, τ_2 are topologies on X such that $\tau_2 \supset \tau_1$, (X, τ_2) is not regular and (X, τ_1) is T_1-space then the Lusin-Menchoff property of the topology τ_2 with respect to the topology τ_1 does not hold.*

Proof. If τ_2 is not regular then there exist a τ_2-closed set F_{τ_2} and a point $x \notin F_{\tau_2}$ such that the sets $\{x\}$, F_{τ_2} cannot be separated by τ_2-open sets. Since the space (X, τ_1) is T_1 then the set $\{x\} = F_{\tau_2}$ is τ_1-closed. We conclude that the disjoint sets F_{τ_1} and F_{τ_2} cannot be separated by τ_2-open set and τ_1-open set. $\Diamond$

Property 13. *For any complete extension μ of the Lebesgue measure the topology $\mathcal{T}_\mu$ has not the Lusin-Menchoff property with respect to the topology $\mathcal{T}_d$.*

Proof. If $\mathcal{T}_\mu = \mathcal{T}_d$ then $\mathcal{T}_\mu$ has not the Lusin-Menchoff property with respect to the Topology $\mathcal{T}_d$ because otherwise $\mathcal{T}_d$ would be normal. This fact is not true for the topology $\mathcal{T}_d$ (cf. [3]). Let $\mathcal{T}_\mu \neq \mathcal{T}_d$. We have proved in Prop. 6 that $\mathcal{T}_\mu$ is not regular and it is obvious that the topology $\mathcal{T}_d$ is T_1. Thus by Lemma 3 we conclude that the Lusin-Menchoff property of the topology $\mathcal{T}_\mu$ with respect to the topology $\mathcal{T}_d$ does not hold. $\Diamond$

Property 14. *For any complete extension μ of the Lebesgue measure the topology $\mathcal{T}_\mu$ has the Lusin-Menchoff property with respect to the natural topology on the real line if and only if $\mathcal{T}_\mu = \mathcal{T}_d$.*

Proof. Sufficiency is the consequence of the Proposition.

Necessity. Let us suppose that $\mathcal{T}_\mu \neq \mathcal{T}_d$. Then by Prop. 6 we have that the topology $\mathcal{T}_\mu$ is not regular. By Lemma 3 we conclude that the Lusin-Menchoff property of the $\mathcal{T}_\mu$ with respect to the natural topology does not hold. $\Diamond$

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