Mathematica Pannonica
9/2 (1998), 153–164

CONVEX DIFFERENTIABLE SET-VALUED FUNCTIONS

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Received: March 1997

MSC 1991: 26E25, 26B25, 54C60, 58C25

Keywords: Set-valued functions, differentiability, convex set-valued functions, cancellation law, Hausdorff metric, embedding theorem.

Abstract: We present necessary and sufficient conditions under which a differentiable set-valued function is convex.

1. Let L be a normed linear space and let C be a convex and open subset of L. We define $f : C \to \mathbb{R}$ to be *convex* on C if

$$f((1-\lambda)x + \lambda y) \le (1-\lambda)f(x) + \lambda f(y)$$

for $\lambda \in (0,1)$ and $x, y \in C$. Our main goal is to give a generalization of the following well-known theorem:

Theorem. (cf., e.g., [5], p. 98) *Suppose that $f : C \to \mathbb{R}$ is convex on C and differentiable at x_0 (i.e. f has a Fréchet derivative at x_0). Then for $x \in C$*

$$f(x) - f(x_0) \ge f'(x_0)(x - x_0). \tag{1}$$

If f is differentiable throughout C, then f is convex if and only if (1) holds for all $x, x_0 \in C$.

First the above theorem will be extended to convex functions $f : C \to M$, where M is an ordered normed space. Further we shall transfer the above theorem to set-valued convex functions with suitable adapted definition of differentiability.

2. Let L and M be normed linear spaces and let C be an open set in L. A transformation $T : L \to M$ is said to be *homogeneous* if $T(th) = tT(h)$ for all $h \in L$ and $t \in \mathbb{R}$. Assume that a function f is defined on C and takes values in M.

Definition 1. A function $f : C \to M$ is said to be *differentiable at* $x_0 \in C$ if there is a homogeneous transformation $T : L \to M$ such that

$$\frac{\varphi(h)}{\|h\|} \to 0 \quad \text{as} \quad h \to 0, \tag{2}$$

where $\varphi(h) := f(x_0 + h) - f(x_0) - T(h)$ for each $h \in L$ such that $x_0 + h \in C$. The homogeneous transformation T is called the *derivative* and it is denoted by $f'(x_0)$.

We do not assume that T is additive or continuous. So our definition is essentially weaker than the Fréchet differentiability. For example, the function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x) = f(x_1, x_2) = \sqrt[3]{x_1^3 + x_2^3}$ is differentiable at (0,0) with respect to Def. 1, whereas it is not Fréchet differentiable in this point.

It is easy to see that above definition of differentiability is correct. Indeed, if T and S are homogeneous transformations from L to M such that (3) holds, then $T = S$.

It is evident that every function $f : (a, b) \to M$, where (a, b) is an interval of $\mathbb{R}$ and differentiable at $x_0 \in (a, b)$ with respect to Def. 1 has to have the ordinary derivative, i.e., there exists

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

and it is equal to $T(1)$. Conversely, if $f : (a, b) \to M$ has the ordinary derivative $f'(x_0)$ at $x_0 \in (a, b)$, then it is differentiable in this point with respect to our definition and $T(h) = hf'(x_0), h \in \mathbb{R}$.

Theorem 1. *Let $f : C \to M$ be differentiable at x_0. Then f is continuous at x_0 if and only if $f'(x_0)$ is continuous at zero.*

Proof. Let $f'(x_0)$ be continuous at zero. The differentiability of f and the inequality

$$\|f(x_0 + h) - f(x_0)\| \leq \|f(x_0 + h) - f(x_0) - f'(x_0)(h)\| + \|f'(x_0)(h)\|$$

yield the continuity of f at x_0. Conversely, if f is continuous at x_0, then the continuity of $f'(x_0)$ at zero follows from the inequality

$$\|f'(x_0)(h)\| \leq \|f'(x_0)(h) + f(x_0) - f(x_0 + h)\| + \|f(x_0 + h) - f(x_0)\|. \quad \diamond$$

Denote by $S(\delta)$ the open ball in L centered in zero and with the radius δ.

Theorem 2. *Let C be an open and connected subset of L. Then $f : C \to M$ is constant if and only if it is differentiable on C and $f'(x) = 0$ for $x \in C$.*

Proof. The necessity is obvious. To prove sufficiency we take an $\epsilon > 0$. For each $x \in C$ there is a $\delta > 0$ such that

$$(3) \qquad \|f(x+h) - f(x)\| = \|f(x+h) - f(x) - f'(x)(h)\| < \epsilon\|h\|$$

for $h \in S(\delta)$. Fix an $x_1 \in C$ and consider the function ψ, $\psi(x) := \|f(x) - f(x_1)\|$ for $x \in C$. Of course

$$|\psi(x+h) - \psi(x)| = | \, \|f(x+h) - f(x_1)\| - \|f(x) - f(x_1)\| \, | \leq \leq \|f(x+h) - f(x)\|$$

for all $x \in C$ and all $h \in L$ such that $x+h \in C$. Hence and in virtue (3) the Fréchet derivative $\psi'(x)$ of ψ is equal to zero for all $x \in C$. Therefore ψ is constant since C is connected. It follows from the definition of ψ that $\psi(x) = 0$ on C and clearly $f(x) = f(x_1)$ for all $x \in C$. $\Diamond$

Simple proof of the following theorem is omitted.

Theorem 3. *If $f, g : C \to M$ are differentiable at $x_0 \in C$ and $\lambda \in \mathbb{R}$, then $f+g$ and λf are differentiable at x_0 and*

$$(f+g)'(x_0) = f'(x_0) + g'(x_0), \quad (\lambda f)'(x_0) = \lambda f'(x_0).$$

3. Let $(M, \leq)$ be a normed linear space partially ordered by $\leq$. It means that the binary relation $\leq$ is reflexive, i.e., $x \leq x$ for all x, antisymmetric, i.e., if $x \leq y$ and $y \leq x$, then $x = y$ and transitive, i.e., if $x \leq y$ and $y \leq z$, then $x \leq z$ for all $x, y, z \in M$.

Definition 2. Let $(M, \leq)$ be normed linear space partially ordered by $\leq$. If

- $u \leq v$ implies $u + w \leq v + w$ for all $u, v, w \in M$,
- $u \leq v$ implies $\lambda u \leq \lambda v$ for all $u, v \in M$ and $\lambda > 0$,
- the positive cone $K := \{u \in M : u \geq 0\}$ is a closed subset of M,

then M is said to be an *ordered normed space* (cf. [6]).

In the sequel of this part we shall assume that M is an ordered normed space and L is a normed space whereas C is an open convex subset of L.

Definition 3. A function $f : C \to M$ is said to be *convex (concave)* if

$$f((1-\lambda)x+\lambda y)\leq(1-\lambda)f(x)+\lambda f(y)$$
$$\Big((1-\lambda)f(x)+\lambda f(y)\leq f((1-\lambda)x+\lambda y)\Big)$$
for all $x,y\in C$ and $\lambda\in(0,1)$.

Theorem 4. *If $f:C\to M$ is convex and differentiable at $x_0\in C$, then there exists a $\delta>0$ such that $\varphi(h)\geq 0$ for $h\in S(\delta)$, where φ is given by* Def. 1.

Proof. There exists a $\delta>0$ such that $x_0+S(\delta)\subset C$. Take $h\in$ $\in S(\delta), h\neq 0$ and $\lambda\in(0,1)$. By the convexity of f we have
$$f(x_0+\lambda h)=f((1-\lambda)x_0+\lambda(x_0+h))\leq(1-\lambda)f(x_0)+\lambda f(x_0+h).$$
The differentiability of f yields
$$f(x_0+\lambda h)=f(x_0)+\lambda T(h)+\varphi(\lambda h).$$
Consequently
$$f(x_0)+\lambda T(h)+\varphi(\lambda h)\leq(1-\lambda)f(x_0)+\lambda f(x_0)+\lambda T(h)+\lambda\varphi(h),$$
whence
$$\varphi(\lambda h)\leq\lambda\varphi(h).$$
Thus
$$\varphi(h)-\frac{\varphi(\lambda h)}{\lambda}\geq 0.$$
On the other hand
$$\varphi(h)-\frac{\varphi(\lambda h)}{\lambda}\to\varphi(h),\qquad\text{as}\quad\lambda\to 0.$$
In virtue of the closedness of the positive cone K we obtain $\varphi(h)\geq 0$. Of course, $\varphi(0)=0$. $\Diamond$

The main property of convex differentiable functions is contained in the following

Theorem 5. *If $f:C\to M$ is convex and differentiable at $x_0\in C$, then*
$$f(x_0+h)\geq f(x_0)+f'(x_0)h \tag{4}$$
for all $h\in L$ such that $x_0+h\in C$.

Proof. Take $h\in L$ such that $x_0+h\in C$. With respect to the differentiability of f at x_0 we may find a $\lambda_0\in(0,1)$ such that
$$f(x_0+\lambda h)=f(x_0)+f'(x_0)(\lambda h)+\varphi(\lambda h)$$
for $0<\lambda<\lambda_0$. Th. 4 states that λ can be chosen small enough to have also $\varphi(\lambda h)\geq 0$. Thus
$$f(x_0+\lambda h)\geq f(x_0)+f'(x_0)(\lambda h).$$
Since f is convex we have

$$f(x_0) + \lambda f'(x_0)(h) \leq (1-\lambda)f(x_0) + \lambda f(x_0+h),$$

whence

$$f(x_0) + f'(x_0)(h) \leq f(x_0+h). \qquad \Diamond$$

Our definition of differentiability (Def. 1) does not contain the requirement that the derivative $f'(x_0)$ of f is an additive transformation. However, if f is convex, then we can show that $f'(x_0)$ is convex.

Theorem 6. *If $f : C \to M$ is convex and differentiable at $x_0 \in C$, then*

(5) $$T((1-\lambda)h + \lambda k) \leq (1-\lambda)T(h) + \lambda T(k)$$

for all $h, k \in L$ and $\lambda \in (0,1)$, where $T = f'(x_0)$.

Proof. Take arbitrary $h, k \in L, \lambda \in (0,1)$ and $\delta > 0$ small enough to have $\varphi(h) \geq 0$ for all $h \in S(\delta)$. We can find a $\eta > 0$ such that th and tk belong to $S(\delta)$ for $0 \leq t < \eta$. Thus

(6) $$f(x_0 + th) = f(x_0) + T(th) + \varphi(th),$$

(7) $$f(x_0 + tk) = f(x_0) + T(tk) + \varphi(tk)$$

for $0 \leq t < \eta$. We have also

(8) $$f(x_0 + (1-\lambda)th + \lambda tk) = \\ = f(x_0) + T((1-\lambda)th + \lambda tk) + \varphi((1-\lambda)th + \lambda tk)$$

for the same t. Multiplying equality (6) by $1-\lambda$ and (7) by λ and adding them together we obtain

$$(1-\lambda)f(x_0+th) + \lambda f(x_0+tk) = \\ = f(x_0) + (1-\lambda)tT(h) + \lambda tT(k) + (1-\lambda)\varphi(th) + \lambda\varphi(tk).$$

Hence by the convexity of f and by (8)

$$f(x_0) + tT((1-\lambda)h + \lambda k) + \varphi((1-\lambda)th + \lambda tk) \leq \\ \leq f(x_0) + (1-\lambda)tT(h) + \lambda tT(k) + (1-\lambda)\varphi(th) + \lambda\varphi(tk).$$

Consequently

$$T((1-\lambda)h + \lambda k) + \frac{\varphi((1-\lambda)th + \lambda tk)}{t} \leq \\ \leq (1-\lambda)T(h) + \lambda T(k) + (1-\lambda)\frac{\varphi(th)}{t} + \lambda\frac{\varphi(th)}{t}$$

for $0 \leq t < \eta$. Letting $t \to 0+$ we obtain hence (5). $\Diamond$

To receive an inverse result to Th. 5 we assume additionally that the derivative $f'(x)$ is convex.

Theorem 7. *Assume that $f : C \to M$ is differentiable throughout C and*

(9) $$f(x+h) \geq f(x) + f'(x)(h)$$

for each $x \in C$ *and* $h \in L$ *such that* $x + h \in C$. *If* $f'(x)$ *is convex on* L *for all* $x \in C$, *then* f *is convex.*

Proof. Fix arbitrarily $y, z \in C$ and $\lambda \in (0,1)$ and write $x := (1-\lambda)y + \lambda z$. By (9)

$$f(y) \geq f(x) + f'(x)(y-x), \quad f(z) \geq f(x) + f'(x)(z-x).$$

Hence, since $f'(x)$ is convex we have

$$(1-\lambda)f(y) + \lambda f(z) \geq f(x) + (1-\lambda)f'(x)(y-x) + \lambda f'(x)(z-x) \geq$$
$$\geq f(x) + f'(x)((1-\lambda)(y-x) + \lambda(z-x)) =$$
$$= f(x) + f'(x)(0) = f(x). \qquad \Diamond$$

Analogous results to Ths. 4–7 for concave functions can be obtained too (if $f : C \to M$ is concave then $-f$ is convex).

4. In this part of the paper we shall introduce a suitable definition of differentiability of set-valued functions. Let Y be a reflexive Banach space. The symbol $\mathcal{B}(Y) = \mathcal{B}$ will be used to denote the family of all non-empty, closed, bounded and convex subsets of Y. $\mathcal{B}$ with the addition defined by formula

$$A + B = \{a + b \in Y : a \in A, b \in B\}$$

is an Abelian semigroup with zero element $0 := \{0\}$ in which the cancellation law holds true, i.e., if

$$A + B = C + B, \qquad \text{then} \quad A = C \quad \text{for all} \quad A, B, C \in \mathcal{B}$$

(cf. [4]).

We define also multiplication αA of a nonnegative number α and a subset A of Y by

$$\alpha A := \{\alpha a : a \in A\}.$$

This multiplication has the following properties:

$$\alpha(\beta A) = (\alpha\beta)A, \ 1 \cdot A = A, \ \alpha(A + B) = \alpha A + \alpha B$$

and

(10) $$(\alpha + \beta)A = \alpha A + \beta A$$

for all $\alpha, \beta \geq 0$ and $A, B \in \mathcal{B}$. The convexity of the elements of $\mathcal{B}$ is used both in the proof of (10) and in the proof of cancellation law. The reflexivity of Y is used to show closedness the sum $A + B$ whenever $A, B \in \mathcal{B}$.

The Hausdorff distance d_H between $A, B \in \mathcal{B}$ is defined by relation

$$d_H(A,B) = \inf\{t > 0 : A \subset B + t\overline{S},\ B \subset A + t\overline{S}\},$$

where $\overline{S} = \{x \in Y : \|x\| \leq 1\}$. Since Y is complete, $(\mathcal{B}(Y), d_H)$ is also complete (see e.g., [2]).

The assumption that Y is reflexive in our considerations can be replaced by the requirement that Y is a normed linear space but then we have to take the subfamily $\mathcal{K}(Y)$ consisting of compact elements of $\mathcal{B}(Y)$.

Rådström's embedding theorem (see [4]) states that there exists a real normed space $\mathcal{M} = \mathcal{M}(Y)$ and the isometry $\pi : \mathcal{B} \to \mathcal{M}$ such that $\pi(\mathcal{B})$ is a convex cone in $\mathcal{M}$. Moreover addition and multiplication in $\mathcal{M}$ induce the coresponding operations in $\mathcal{B}$.

Now we shortly remind the definition of $\mathcal{M}$ (cf. [4]). An equivalence relation $\sim$ can be defined on $\mathcal{B}^2 = \mathcal{B} \times \mathcal{B}$:

$$(A,B) \sim (C,D) \Leftrightarrow A + D = B + C.$$

The equivalence class determined by (A, B) shall be denoted by $[A, B]$. The space $\mathcal{M}$ is the quotient space $\mathcal{B}^2/\sim$. In this space we define addition by

$$[A,B] + [C,D] = [A + C, B + D] \tag{11}$$

and multiplication by

$$\lambda[A,B] = \begin{cases} [\lambda A, \lambda B] & \text{if } \lambda \geq 0, \\ [|\lambda| B, |\lambda| A] & \text{if } \lambda < 0. \end{cases} \tag{12}$$

$\mathcal{M}$ is a linear space with addition given by (11) and multiplication given by (12). The embedding map $\pi : \mathcal{B} \to \mathcal{M}$ is given by

$$\pi(A) = [A, 0], \quad A \in \mathcal{B}.$$

The element $[0,0] = [A, A]$ is zero in $\mathcal{M}$. In the sequel we will use the abbreviation $\hat{A} = \pi(A)$. In the linear space $\mathcal{M}$ a metric ∂_H is defined by

$$\partial_H([A,B],[C,D]) = d_H(A + D, B + C).$$

The Hausdorff metric d_H is positively homogeneous and translation invariant, so the formula

$$\|[A,B]\| = \partial_H([A,B],[0,0])$$

defines the norm in $\mathcal{M}$ such that

$$d_H(A,B) = \partial_H(\pi(A), \pi(B)) = \|[A,B]\|.$$

Let X be a normed space and let C be an open set in X. Consider a set-valued map $F : C \to \mathcal{B}$. H.T. Banks and M.Q. Jacobs [1] have introduced the following definition of differentiability of F at

$x_0 \in C$. F is said to be *$\pi-$differentiable at x_0* if the function $\hat{F} : C \to \to \mathcal{M}, x \mapsto \hat{F}(x)$, where $\hat{F}(x) = \pi(F(x)) = [F(x), 0]$ is differentiable at x_0 in Fréchet sense. It means that there exists a continuous linear transformation $\hat{F}'(x_0) : X \to \mathcal{M}$ such that

$$\lim_{x \to x_0} \frac{\hat{F}(x) - \hat{F}(x_0) - \hat{F}'(x_0)(x - x_0)}{\|x - x_0\|} = 0. \tag{13}$$

For our purposes it suffices to adopt a weaker definition of differentiability of set-valued functions. Our notion of differentiability will be connected with Def. 1.

Definition 4. A set-valued function $F : C \to \mathcal{B}$ is said to be *differentiable at $x_0 \in C$* if the function $\hat{F} : C \to \mathcal{M}$ is differentiable at x_0 with respect to Def. 1, i.e., there exists a homogeneous function $\hat{F}'(x_0) : X \to \mathcal{M}$ such that (13) holds.

Continuity and additivity of the transformation $\hat{F}'(x_0)$ are omitted in Def. 4.

Write $\hat{F}'(x_0)(h) =: [A(h), B(h)]$. There is a $\delta > 0$ such that

$$\hat{F}(x_0 + h) - \hat{F}(x_0) - \hat{F}'(x_0)(h) =$$
$$= \hat{F}(x_0 + h) - \hat{F}(x_0) - [A(h), B(h)] =: [P(h), R(h)],$$

where $P(h), R(h) \in \mathcal{B}$ for $h \in S(\delta)$. Hence it follows by (13) that

$$F(x_0 + h) + B(h) + R(h) = F(x_0) + A(h) + P(h) \tag{14}$$

and

$$\frac{\|[P(h), R(h)]\|}{\|h\|} = \frac{d_H(P(h), R(h))}{\|h\|} \to 0 \tag{15}$$

as $h \to 0$. Since $\hat{F}'(x_0)$ is homogeneous,

$$A(th) + tB(h) = B(th) + tA(h) \quad \text{for} \quad t \geq 0 \tag{16}$$

and

$$A(th) + (-t)A(h) = B(th) + (-t)B(h) \quad \text{for} \quad t < 0 \tag{17}$$

for all $h \in X$. On the other hand, if (14)-(17) hold, then F is differentiable at x_0. Thus we can formulate

Theorem 8. *A set-valued function $F : C \to \mathcal{B}$ is differentiable at $x_0 \in \in C$ if and only if there exist a $\delta > 0$, set-valued functions $A, B : X \to \mathcal{B}$ and $P, R : S(\delta) \to \mathcal{B}$ such that* (14)-(17) *hold.*

In connection with Th. 1 we get the following

Corollary 1. *Let a set-valued function $F : C \to \mathcal{B}$ be differentiable at $x_0 \in C$ with $\hat{F}'(x_0)(h) = [A(h), B(h)]$, $h \in X$. Then F is continuous* *at x_0 with respect to Hausdorff metric if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$d_H(A(h), B(h)) < \varepsilon \quad \textit{for} \quad h \in S(\delta).$$

According to Th. 2 we have

Corollary 2. *Let C be an open and connected set in X. A set-valued function $F : C \to \mathcal{B}$ is constant if and only if it is differentiable throughout C and $A_x(h) = B_x(h)$ for all $x \in C$ and $h \in X$, where $[A_x(h), B_x(h)] = \hat{F}'(x)(h)$, $x \in C$, $h \in X$.*

5. Example. Let $f, g : (a, b) \to \mathbb{R}$ and let $f \leq g$ on (a, b). Assume that f, g are differentiable at $x_0 \in (a, b)$. Then the set-valued function $F(x) := \langle f(x), g(x)\rangle$ ($\langle f(x), g(x)\rangle$ denotes the closed interval of the line with endpoints $f(x)$ and $g(x)$), $x \in (a, b)$ is $\pi-$differentiable at x_0 (see Cor. 3.1 in [1]).

Now suppose that F is differentiable at $x_0 \in (a, b)$. Then there exists a $\delta > 0$ compact intervals $A(h) = \langle a(h), c(h)\rangle$, $B(h) = \langle b(h), d(h)\rangle$ for $h \in \mathbb{R}$ and compact intervals $P(h) = \langle p(h), q(h)\rangle$, $R(h) = \langle r(h), s(h)\rangle$ for $|h| < \delta$ such that

$$f(x_0 + h) + b(h) + r(h) = f(x_0) + a(h) + p(h),$$
$$g(x_0 + h) + d(h) + s(h) = g(x_0) + c(h) + q(h)$$

for $|h| < \delta$ as well as

$$\lim_{h\to 0} \frac{\max\{|r(h) - p(h)|,\ |s(h) - q(h)|\}}{|h|} = 0. \tag{18}$$

Furthermore in virtue of homogeneity of $\hat{F}'(x_0)$ we have

$$a(th) + tb(h) = b(th) + ta(h), \quad c(th) + td(h) = d(th) + tc(h)$$

for all $h, t \in \mathbb{R}$, whence

$$\frac{a(h) - b(h)}{h} = a(1) - b(1), \quad \frac{c(h) - d(h)}{h} = c(1) - d(1).$$

Consequently

$$\frac{f(x_0 + h) - f(x_0)}{h} = a(1) - b(1) + \frac{p(h) - r(h)}{h}$$

and

$$\frac{g(x_0 + h) - g(x_0)}{h} = c(1) - d(1) + \frac{q(h) - s(h)}{h}$$

so the differentiability of f and g results from (18).

In general, each set-valued function defined on an interval is differentiable with respect to Def. 4 if and only if it is π−differentiable. The set-valued function F, $F(x_1, x_2) = \langle\sqrt[3]{x_1^3 + x_2^3}, \sqrt[3]{x_1^3 + x_2^3 + 1}\rangle$, $(x_1, x_2) \in \mathbb{R}^2$, is differentiable at $(0,0)$ with respect to Def. 4 but it is not π -differentiable.

6. In this part of the paper we transfer results of the Part 3 to convex set-valued functions.

Definition 5. (cf., e.g., [3]). Let X, Y be linear spaces and let C be a convex subset of X. A set-valued function F defined on C with nonempty values in Y is said to be *convex (concave)* if

$$(1-t)F(x) + tF(y) \subset F((1-t)x + ty)$$

$$\Big(F((1-t)x + ty) \subset (1-t)F(x) + tF(y)\Big)$$

for all $x, y \in C$ and $t \in (0,1)$.

Let X be a normed space and let Y be a reflexive Banach space. Assume that C is an open and convex subset of X. We introduce an order in the normed space $\mathcal{M} = \mathcal{M}(Y) = \mathcal{B}^2/\sim$ as follows:

$$[A,B] \leq [D,E] \Leftrightarrow B + D \subset A + E$$

One can easy check that the order satisfies the two first conditions of Def. 2. Write

$$\mathcal{K} := \{[A,B] \in \mathcal{M} : [A,B] \geq 0\}.$$

It is clear that $[A,B] \geq 0 \Leftrightarrow A \subset B$. To prove the last condition of Def. 2 we take the sequence $\{[A_n, B_n]\}$ converging to $[A,B]$ with terms belonging to $\mathcal{K}$. Then $A_n \subset B_n$ for all $n \in \mathbb{N}$ and $d_H(A_n + B, B_n + A) \to 0$ as $n \to \infty$. Let us fix an $\varepsilon > 0$. There is an $n \in \mathbb{N}$ such that

$$B_n + A \subset A_n + B + \varepsilon\overline{S},$$

whence

$$B_n + A \subset B_n + B + \varepsilon\overline{S}.$$

Canceling B_n (see [4], Lemma 1) we get

$$A \subset B + \varepsilon\overline{S}.$$

Since the set B is closed the relation $A \subset B$, i.e., $[A,B] \geq 0$ follows in view of the unrestricted choice of $\varepsilon > 0$.

It has been shown that $(\mathcal{M}, \leq)$ is the ordered normed space. Observe that the map $F : C \to \mathcal{B}$ is convex (concave) if and only if the map $C \ni x \mapsto \hat{F}(x) = [F(x), 0] \in \mathcal{M}$ is convex (concave). Therefore in virtue of considerations Parts 3 and 4 we may obtain the following theorems which characterize differentiable convex set-valued functions.

Theorem 4′. *Let $F : C \to \mathcal{B}$ be convex and differentiable at $x_0 \in C$. Then there exists a $\delta > 0$ such that $P(h) \subset R(h)$ for $h \in S(\delta)$, where set-valued functions P and R are given in* Th. 8.

Theorem 5′. *If $F : C \to \mathcal{B}$ is convex and differentiable at $x_0 \in C$ and $\hat{F}'(x_0)(h) = [A(h), B(h)]$, $h \in X$, then $F(x_0+h)+B(h) \subset F(x_0)+A(h)$ for $h \in X$ such that $x_0 + h \in C$.*

Theorem 6′. *If $F : C \to \mathcal{B}$ is convex and differentiable at $x_0 \in C$ and $\hat{F}'(x_0)(h) = [A(h), B(h)]$, $h \in X$, then*

$$B((1-\lambda)h + \lambda k) + (1-\lambda)A(h) + \lambda A(k) \subset \subset A((1-\lambda)h + \lambda k) + (1-\lambda)B(h) + \lambda B(k) \tag{19}$$

for $h, k \in X$ and $\lambda \in (0,1)$.

Theorem 7′. *Assume that $F : C \to \mathcal{B}$ is differentiable throughout C and the derivative $\hat{F}'(x), \hat{F}'(x)(h) = [A_x(h), B_x(h)]$ satisfies*

$$F(x+h) + B_x(h) \subset F(x) + A_x(h) \tag{20}$$

for each $x \in C$ and $h \in X$ such that $x + h \in C$, as well as inclusion (19) holds for every $h, k \in X$, $\lambda \in (0,1)$ and $x \in C$. Then F is convex.

Analogous theorems hold true for concave functions. In this case sign of "$\subset$" should be replaced by "$\supset$".

Example. Take $F(x) = \langle x^2, \sqrt{x} \rangle$, for $x \in (0,1)$. We can set

$$A_x(h) = \begin{cases} \langle 2x, 2 + \frac{1}{2\sqrt{x}} \rangle h & \text{for } h \geq 0 \\ \langle -2, 0 \rangle h & \text{for } h < 0, \end{cases}$$

$$B_x(h) = \begin{cases} \langle 0, 2 \rangle h & \text{for } h \geq 0 \\ \langle -2 - \frac{1}{2\sqrt{x}}, -2x \rangle h & \text{for } h < 0 \end{cases}$$

and $P_x(h) = \langle 0, \sqrt{x+h} \rangle$, $R_x(h) = \langle -h^2, \sqrt{x} + \frac{h}{2\sqrt{x}} \rangle$ for all $x \in (0,1)$ and $h \in \mathbb{R}$ such that $x + h \in (0,1)$. It is easy to check that inclusions (19) and (20) hold.

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