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TOLERANCES ON SEMILATTICES

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Abstract: The aim of this note is to prove that the tolerance lattice of semilattice is a p-algebra. An example shows that this p-algebra fails to be a relative p-algebra.

1. Preliminaries

A *p-algebra* (or pseudocomplemented lattice) is a universal algebra $(L; \vee, \wedge, {}^*, 0, 1)$ of type $(2, 2, 1, 0, 0)$ in which the deletion of the unary operation * yields a bounded lattice and * is the operation of pseudocomplementation that is

$$x \leq a^* \text{ if and only if } a \wedge x = 0.$$

The class $\mathcal{B}_\omega$ of all distributive p-algebras is equational. K. B. Lee [4] has shown that the lattice of all equational subclasses of $\mathcal{B}_\omega$ forms a chain

$$\mathcal{B}_{-1} \subset \mathcal{B}_0 \subset \mathcal{B}_1 \subset \ldots \subset \mathcal{B}_n \subset \ldots \subset \mathcal{B}_\omega,$$

of type $\omega + 1$ where $\mathcal{B}_{-1}$ denotes the class of all trivial p-algebras, $\mathcal{B}_0$ is the class of all Boolean algebras and for $n \geq 1$ the class $\mathcal{B}_n$ consists of all distributive p-algebras satisfying identity

$$(L_n) (x_1 \wedge x_2 \wedge \ldots \wedge x_n)^* \vee (x_1^* \wedge x_2 \wedge \ldots \wedge x_n)^* \vee \ldots \vee (x_1 \wedge x_2 \wedge \ldots \wedge x_n^*)^* = 1.$$

We call the elements of $\mathcal{B}_1$ the Stone algebras. For $n \geq 2$ the elements of $\mathcal{B}_n$ are called (L_n)-lattices. Distributive p-algebra in which for some $n \geq 1$ every subinterval is an (L_n)-lattice is called a *relative* (L_n)*-lattice.*

Proposition 1.1 ([1]; Th. 1). *Let L be a distributive lattice with 1. The following conditions are equivalent:*

(i) *L is a relative (L_n)-lattice,*

(ii) *for every $a \in L$, $[a,1]$ is an (L_n)-lattice.*

If we give up the distributivity we can study the following classes of p-algebras

$$\mathcal{P}_0 \subset \mathcal{P}_1 \subset \ldots \subset \mathcal{P}_n \subset \ldots \subset \mathcal{P}_\omega,$$

where $\mathcal{P}_\omega$ denotes the class of all p-algebras, $L \in \mathcal{P}_n$ if and only if L is a p-algebra satisfying the identity (L_n) for $1 \leq n < \omega$ and the elements of the class $\mathcal{P}_0$ are uniquely determined by the identity

$$(L_0) \qquad (x \wedge y)^* = x^* \vee y^*.$$

(For distributive p-algebras the identities (L_0) and (L_1) are equivalent.)

Let S be a $\wedge$-semilattice. A *tolerance* on a semilattice S is a reflexive and symmetric binary relation T on S which has the substitution property with respect to $\wedge$, i.e.

$$(a,b) \in T, \ (c,d) \in T \ \text{ implies } (a \wedge c, b \wedge d) \in T.$$

The set of all tolerances on S forms an algebraic lattice $\mathrm{Tol}(S)$ with respect to the set inclusion and with Δ, ∇ the least and greatest elements, respectively (see [2]). The meet in this lattice corresponds with the intersection, i.e.

$$A \wedge B = A \cap B$$

and

$$A \vee B = T(A \cup B),$$

for any tolerances A, B on S, where $T(M)$ denotes the least tolerance containing the set $M \subseteq S \times S$. It is called the *tolerance generated by* M. If $M = \{(a,b)\}$ then we denote $T(M) = T(a,b)$ and we call it a *principal* tolerance.

The following properties are easy to verify:

(1) Let $M \subseteq S \times S$ be arbitrary set. Then $(x,y) \in T(M)$ if and only if $x = x_1 \wedge x_2 \wedge \ldots \wedge x_r$, $y = y_1 \wedge y_2 \wedge \ldots \wedge y_r$ and $(x_i, y_i) \in M$ or $(y_i, x_i) \in M$ or $x_i = y_i$, for $i = 1, 2, \ldots, r$.

(2) $(x,y) \in T(a,b)$ if and only if $x = y$ or $x = a \wedge r$, $= y = b \wedge r$ or $x = b \wedge r$, $y = a \wedge r$ for some $r \in S$.

(3) $A \vee B = A \cup B \cup \{(x_1 \wedge x_2, y_1 \wedge y_2) : \ (x_1, y_1) \in A, \ (x_2, y_2) \in B\}$, for any $A, B \in \mathrm{Tol}(S)$.

From these properties we immediately obtain next simple statement.

Lemma 1.2. *Let S be a $\wedge$-semilattice, $a, b \in S$, $a \neq b$ and $T \in$ Tol $=$ $= (S)$. Then $T \wedge T(a,b) = \Delta$ if and only if $a \wedge r = c$, $b \wedge r = d$ implies $a \wedge r = b \wedge r$, for any $r \in S$ and $(c,d) \in T$.*

In particularly if $a \neq b$, $c \neq d$ then $T(a,b) \wedge T(c,d) = \Delta$ if and only if $a \wedge r = c \wedge s$, $b \wedge r = d \wedge s$ or $b \wedge r = c \wedge s$, $a \wedge r = d \wedge s$ implies $a \wedge r = b \wedge r$, for any $r, s \in S$.

2. Tolerance distributive semilattices

The following theorem is a connection of [6; Cor. 1.1] and [3; Th. 7].

Theorem 2.1. *Let S be a $\wedge$-semilattice. The following conditions are equivalent:*

(a) Tol(S) *is modular,*

(b) Tol(S) *is distributive,*

(c) *S is a chain or S contains a maximal chain S_0 and an element $z \in S_0$ such that each element of $S \setminus S_0$ covers z.*

Since Tol(S) is an algebraic lattice the condition (c) characterizes all $\wedge$-semilatttices whose tolerance lattices are distributive relative p-algebras. From [7] follows that Tol$(S) \in \mathcal{B}_0$ if and only if S is a trivial semilattice or a two-element chain. In this section we will prove that for tolerance-distributive semilattice S the tolerance lattice Tol(S) is a relative (L_2)-lattice.

Let S be a tolerance distributive semilattice and $T, U \in$ Tol(S), $T \leq U$. We denote $U * T$ the relative pseudocomplement of U in $[T, \nabla]$. It is easy computation to verify that

$$U * T = T \vee \bigvee \{T(a,b) : \ (T(a,b) \vee T) \wedge U = T\}.$$

Lemma 2.2. *Let S be a $\wedge$-semilattice. If S is a chain then* Tol(S) *is a relative Stone algebra.*

Proof. Take arbitrary $T, U \in$ Tol(S), $T \leq U$. We will prove that $U * T \cup (U * T) * T = \nabla$. On the contrary suppose that $(a,b) \notin U * * T \cup (U * T) * T$ for some $a, b \in S$, $a < b$. It follows that $(a,c) \in U$, $(a,c) \notin T$ and $(a,d) \in U * T$, $(a,d) \notin T$ for some $c, d \in S$, $a < c, d \leq \leq b$. Hence $(a, c \wedge d) \in U \wedge U * T = T$ which is a cotradiction with $(a,c), (a,d) \notin T$. Therefore $U * T \vee (U * T) * T \supseteq U * T \cup (U * T) * T = = S \times S = \nabla$ and $[T, \nabla] \in \mathcal{B}_1$. From Prop. 1.1 we obtain that Tol(S) is a relative Stone algebra. $\Diamond$

Lemma 2.3. *Let S be a tolerance-distributive $\wedge$-semilattice and S is not a chain. Then* $\mathrm{Tol}(S)$ *is a relative* (L_2)*-lattice but it is not a Stone algebra.*

Proof. Let us denote S_0 the maximal chain in S and $z \in S_0$ the element which is covered with every element from $S \setminus S_0$. Firstly we will show that $\mathrm{Tol}(S)$ is not a Stone algebra.

Let $x, y \in S$ and $x \| y$. Then $x \wedge y = z$ and we can assume that $y \in S \setminus S_0$. Let $T = T(x, z)$. Clearly $T(y, z) = \{(y, z), (z, y)\} \cup \Delta$ and $T \wedge T(y, z) = \Delta$. Obviously $(y, z) \in T(y, z) \subseteq T^*$ and $(x, z) \in T^{**}$. Hence $(x, y) \notin T^* \cup T^{**}$. Since x, y are both $\wedge$-irreducible elements $(x, y) \notin T^* \vee T^{**}$ and $\mathrm{Tol}(S) \notin \mathcal{B}_1$. It remains to prove that $[T, \nabla] \in \mathcal{B}_2$ for arbitrary $T \in \mathrm{Tol}(S)$.

Let $U, V \in [T, \nabla]$. We denote $T_1 = U \wedge V, T_2 = U * T \wedge V$, $T_3 = U \wedge V * T$. Clearly $T_i \wedge T_j = T$ for $i, j \in \{1, 2, 3\}$, $i \neq j$.

Let $(x, y) \in S \times S$, $(x, y) \notin T$. Three possibilities can occure.

(a) $x, y \in S_0$, $x < y$. Then $T(x, y) = \{(x, t), (t, x) : x < t \leq y\} \cup \Delta$. Since $T_i \wedge T_j = T$ for $i \neq j$ there exists $j \in \{1, 2, 3\}$ such that T_j contains no element (x, u) such that $x < u$ and $(x, u) \notin T$. Therefore $(T \vee T(x, y)) \wedge T_j = (T \wedge T_j) \vee (T(x, y) \wedge T_j) = T$ and $(x, y) \in T_j * T$.

(b) $x, y \in S \setminus S_0$. Then $T(x, y) = \{(x, y), (y, x), (x, z), (z, x), (y, z), (z, y)\} \cup \Delta$. Since $T_i \wedge T_j = T$ for $i \neq j$ we can find again $j \in \{1, 2, 3\}$ such that T_j does not contain neither (x, z) neither (y, z) if $(x, z), (y, z) \notin T$. Again $(T \vee T(x, y)) \wedge T_j = T$ and $(x, y) \in T_j * T$.

(c) $x \in S_0$, $y \in S \setminus S_0$. In this case $T(x, y) = \{(x, y), (y, x), (x \wedge y, y), (y, x \wedge y), (x\wedge = y, t), (t, x \wedge y) : x \wedge y < t \leq x\}$. Repeating similar considerations as in (a) and (b) one can easily verify that there exists $j \in \{1, 2, 3\}$ for which the tolerance T_j does not contain neither any element $(x \wedge y, s)$ such that $x \wedge y < s$ and $(x \wedge y, s) \notin T$ neither element $(x \wedge y, y)$ if $(x \wedge y, y) \notin T$. So again $(T \vee T(x, y)) \wedge T_j = (T \wedge T_j) \vee (T(x, y) \wedge T_j) = T$ and $(x, y) \in T_j * T$

We can conclude that $T_1 * T \vee T_2 * T \vee T_3 * T \supseteq T_1 * T \cup T_2 * T \cup T_3 * T = S \times S = \nabla$. It means that $[T, \nabla] \in \mathcal{B}_2$ and $\mathrm{Tol}(S)$ is a relative (L_2)-lattice. $\Diamond$

3. Non-distributive case

Our aim in this section is to prove that Tol(S) is a p-algebra even for tolerance non-distributive semilattices. The following lemma plays the key role in our next considerations.

Lemma 3.1. *Let S be a $\wedge$-semilattice, $a, b, c_i, d_i \in S$, $a \neq b, c_i \neq d_i$, $i = 1, 2$. If $T(c_i, d_i) \wedge T(a, b) = \Delta$, $i = 1, 2$ then $(T(c_1, d_1) \vee T(c_2, d_2)) \wedge \wedge T(a, b) = \Delta$.*

Proof. Let $T = T(c_1, d_1) \vee T(c_2, d_2)$. From (3) we obtain

$$T = T(c_1, d_1) \cup T(c_2, d_2) \cup \{(c_1 \wedge c_2 \wedge r, d_1 \wedge d_2 \wedge r), (d_1 \wedge d_2 \wedge r, c_1 \wedge c_2 \wedge r), (c_1 \wedge d_2 \wedge r, d_1 \wedge c_2 \wedge r), (d_1 \wedge c_2 \wedge r, c_1 \wedge d_2 \wedge r) : r \in S\}.$$

Assume that $T \wedge T(a, b) \neq \Delta$, i.e. $c_1 \wedge c_2 \wedge r = a \wedge s$ and $d_1 \wedge d_2 \wedge r = = b \wedge s$ for some $r, s \in S$ and $a \wedge s \neq b \wedge s$. (Next three possibilities can be solved the same way only interchanging the letters c_i, d_j.) Then $(a \wedge s, b \wedge s) \wedge (c_1 \wedge c_2 \wedge d_2) = (c_1 \wedge c_2 \wedge r, d_1 \wedge d_2 \wedge r) \wedge (c_1 \wedge c_2 \wedge \wedge d_2) = (c_1 \wedge c_2 \wedge d_2 \wedge r, c_1 \wedge c_2 \wedge d_1 \wedge d_2 \wedge r) \in T(a, b)$. But since $(c_1 \wedge r, d_1 \wedge r) \wedge (c_1 \wedge c_2 \wedge d_2) = (c_1 \wedge c_2 \wedge d_2 \wedge r, c_1 \wedge c_2 \wedge d_1 \wedge r = = d_2 \wedge r) \in T(c_1, d_1)$, and $T(a, b) \wedge T(c_1, d_1) = \Delta$, we obtain that $c_1 \wedge c_2 \wedge d_2 \wedge r = c_1 \wedge c_2 \wedge d_1 \wedge r = d_2 \wedge r$. Clearly $(c_2 \wedge r, d_2 \wedge r) \wedge \wedge (c_1 \wedge c_2) = (c_1 \wedge c_2 \wedge r, c_1 \wedge c_2 \wedge d_2 \wedge r) \in T(c_2, d_2)$. But again $(a \wedge s, b \wedge s) \wedge (c_1 \wedge c_2) = (c_1 \wedge c_2 \wedge r, d_1 \wedge d_2 \wedge r) \wedge (c_1 \wedge c_2) = (c_1 \wedge c_2 \wedge \wedge r, c_1 \wedge c_2 \wedge d_1 \wedge d_2 \wedge r) \in T(a, b)$. Since $T(a, b) \wedge T(c_2, d_2) = \Delta$, we obtain $c_1 \wedge c_2 \wedge r = c_1 \wedge c_2 \wedge d_1 \wedge d_2 \wedge r$. The same way can be proved that also $d_1 \wedge d_2 \wedge r = c_1 \wedge c_2 \wedge d_1 \wedge d_2 \wedge r$. But this is a contradiction with the assumption $a \wedge s = c_1 \wedge c_2 \wedge r \neq d_1 \wedge d_2 \wedge r = b \wedge s$. Therefore $(T(c_1, d_1) \vee T(c_2, d_2)) \wedge T(a, b) = \Delta$. $\Diamond$

The property (1) enables us to generalize the previous statement for arbitrary set of principal tolerances disjoint with $T(a, b)$.

Lemma 3.2. *Let S be a $\wedge$-semilattice, $a, b, c_i, d_i \in S$ for $i \in I$ and $a \neq b$. Let $T(c_i, d_i) \wedge T(a, b) = \Delta$ for $i \in I$. Then*

$$\bigvee_{i \in I} (T(c_i, d_i)) \wedge T(a, b) = \Delta.$$

Proof. Let $(e, f) \in \bigvee_{i \in I} (T(c_i, d_i)) \wedge T(a, b)$. From (1) follows that $(e, f) \in \bigvee_{i \in J} (T(c_i, d_i)) \wedge T(a, b)$, for some finite $J \subseteq I$. So it is enough to prove our statement only for finite index set I.

Let $T(c_i, d_i) \wedge T(a, b) = \Delta$ for $i = 1, 2, \dots n$ and $(e, f) \in \bigvee_{i=1}^{n} (T(c_i, d_i)) \wedge$

$\wedge T(a,b)$. The previous lemma implies that $e = f$ for $n = 2$. Assume that our statement is true for arbitrary $n \leq k$ and that $(e,f) \in \bigvee_{i=1}^{k+1} (T(c_i,d_i)) \wedge T(a,b)$. From (1) we obtain that

$$e = x_1 \wedge x_2 \wedge \ldots \wedge x_m \wedge r, \ f = y_1 \wedge y_2 \wedge \ldots \wedge y_m \wedge r,$$

for $r \in S$ and $x_i = c_{j_i}, y_i = d_{j_i}$ or $x_i = d_{j_i}, y_i = c_{j_i}$ for $i = 1,2,\ldots,m$. If $j_i \leq k$ for all $i = 1,2\ldots m$ then $(e,f) \in \bigvee_{i=1}^{k} (T(c_i,d_i)) \wedge T(a,b)$ and $e = f$. Assume that

$$e = x_1 \wedge x_2 \wedge \ldots \wedge x_{m-1} \wedge c_{k+1} \wedge r$$

and

$$f = y_1 \wedge y_2 \wedge \ldots \wedge y_{m-1} \wedge d_{k+1} \wedge r.$$

Then $(e,f) \in T(x_1 \wedge x_2 \wedge \ldots \wedge x_{m-1}, y_1 \wedge y_2 \wedge \ldots \wedge y_{m-1}) \vee T(c_{k+1},d_{k+1})$ and $(e,f) \in T(a,b)$. But $T(x_1 \wedge x_2 \wedge \ldots \wedge x_{m-1}, y_1 \wedge y_2 \wedge \ldots \wedge y_{m-1}) \subseteq \bigvee_{i=1}^{k} (T(c_i,d_i))$ and $\bigvee_{i=1}^{k} (T(c_i,d_i)) \wedge T(a,b) = T(c_{k+1},d_{k+1}) \wedge T(a,b) = \Delta$. Using Lemma 2.1 we obtain that $\bigvee_{i=1}^{n} (T(c_i,d_i)) \wedge T(a,b) = \Delta$. $\Diamond$

Lemma 3.3. *Let S be a $\wedge$-semilattice. Let $a,b \in S$ and $a \neq b$. Then*

$$T^*(a,b) = \bigvee(T(c,d) : \ T(c,d) \wedge T(a,b) = \Delta).$$

Proof. Let us denote the righ-hand tolerance T, i.e. $T = \bigvee(T(c,d) : T(c,d) \wedge T(a,b) = \Delta)$. We have already proved that $T \wedge T(a,b) = \Delta$. Let $U \in \mathrm{Tol}(S)$ and $U \wedge T(a,b) = \Delta$. Clearly $T(e,f) \subseteq U$ and $T(e,f) \wedge T(a,b) = \Delta$ for every $(e,f) \in U$. Therefore $U = \bigvee(T(e,f) : \ (e,f) \in U) \subseteq \bigvee(T(c,d) : \ T(c,d) \wedge T(a,b) = \Delta) = T$ and $T^*(a,b) = T$. $\Diamond$

Theorem 3.4. *Let S be a $\wedge$-semilattice. The lattice* Tol(S) *of tolerances on S is a p-algebra. More precisely*

$$T^* = \bigwedge(\ T^*(c,d) : \ (c,d) \in T \ )$$

for arbitrary tolerance $T \in$ Tol(S).

Proof. First we will prove that $T \wedge \bigwedge(T^*(c,d) : \ (c,d) \in T) = \Delta$. Let $(e,f) \in T \wedge \bigwedge(T^*(c,d) : \ (c,d) \in T)$. Then $T(e,f) \subseteq T$ and $(e,f) \in T(e,f) \wedge \bigwedge(T^*(c,d) : \ (c,d) \in T) \subseteq T(e,f) \wedge T^*(e,f) = \Delta$. Suppose that $U \in \mathrm{Tol}(S)$ and $T \wedge U = \Delta$. Let $(c,d) \in T$. Then $U \wedge T(c,d) \subseteq U \wedge T = \Delta$, i.e. $U \subseteq T^*(c,d)$ for any $(c,d) \in T$. Since Tol(S) is an algebraic lattice $U \subseteq \bigwedge(T^*(c,d) : \ (c,d) \in T)$ and $\bigwedge(T^*(c,d) : \ (c,d) \in T) = T^*$. $\Diamond$

The previous result reminds of results of Dona Papert. She proved [5] that congruences on semilattice form a p-algebra. Moreover she showed that for any two comparable congruences θ, φ on S such that $\theta \leq \varphi$ we can define a congruence $\varphi * \theta$ for which $\varphi \wedge (\varphi * \theta) = \theta$ and which is the greatest congruence satisfying this equation.

Since tolerance is a generalization of congruence a natural question arises whether we can analogously define a tolerance $U * T$ for any two comparable tolerances $T \leq U$. The following example shows that this is not possible in general.

Example 1. Let S be a semilattice shown in Fig.1.

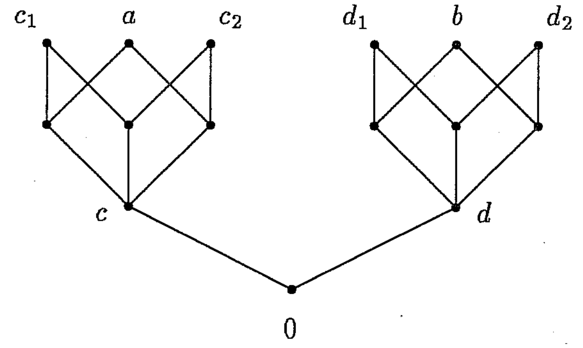

Fig. 1

Let $T = T(a,b)$ and $U = T(\{(a,b),(c,d)\})$. Clearly $T \subseteq U$. We will show that tolerance $U * T$ does not exist in Tol(S). On the contrary suppose that $U*T$ exists. Then undoubtly $U*T \supseteq \bigvee(T(c,d) : : (T(c,d) \vee T) \wedge U = T) \vee T$. It does not take a long time to verify that $T(c_i, d_i) = \{(c_i, d_i), (d_i, c_i), (c_i \wedge a, 0), (0, c_i \wedge a), (d_i \wedge b, 0), (0, d_i \wedge b), (c_1 \wedge \wedge c_2, 0), (0, c_1 \wedge c_2), (d_1 \wedge d_2, 0), (0, d_1 \wedge d_2), (c, 0), (0, c), (d, 0), (0, d)\} \cup \Delta$ and $T \vee T(c_i, d_i) = T \cup T(c_i, d_i) \cup \{(a \wedge c_i, b \wedge d_i), (b \wedge d_i, a \wedge c_i)\}$ for $i = 1, 2$.

Therefore $(T \vee T(c_i, d_i)) \wedge U = T$, for $i = 1, 2$. But $T \vee T(c_1, d_1) \vee \vee T(c_2, d_2) \supseteq T \cup T(c_1, d_1) \cup T(c_2, d_2) \cup \{(a \wedge c_1 \wedge c_2, b \wedge d_1 \wedge d_2), (b \wedge d_1 \wedge d_2, a \wedge c_1 \wedge c_2)\} = T \cup T(c_1, d_1) \cup T(c_2, d_2) \cup \{(c,d), (d,c)\}$ and so $(T \vee T(c_1, d_1) \vee T(c_2, d_2)) \wedge U = U \supsetneq T$ which is a contradiction. So we can conclude that $U * T$ does not exist.

In Section 2. we proved that the identity (L_2) is satisfied in Tol(S) for every tolerance distributive semilattice. Asking which is the smallest n for which the identity (L_n) is satisfied in a tolerance non-distributive semilattice we obtain a much more motley answer.

Lemma 3.5. *For arbitrary* $n = 1, 2, 3, \ldots$ *there exists a finite* $\wedge$*-semilattice* S_n *such that* $\mathrm{Tol}(S_n) \in \mathcal{P}_{n+1} \setminus \mathcal{P}_n$.
Proof. Let S_1 denotes the $\wedge$-semilattice from Fig.2.

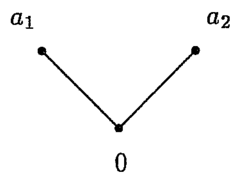

Fig. 2.

Then $\mathrm{Tol}(S_1)$ is a five-element lattice depicted in Fig.3. and obviously $\mathrm{Tol}(S_1) \in \mathcal{P}_2 \setminus \mathcal{P}_1$.

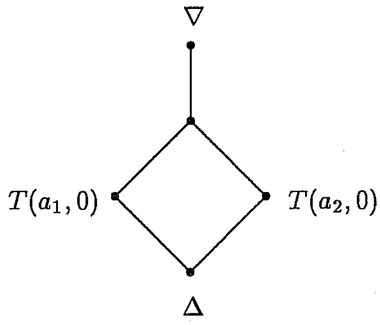

Fig.3.

For $n \geq 2$ we denote S_n the $\wedge$-semilattice from Fig.4.

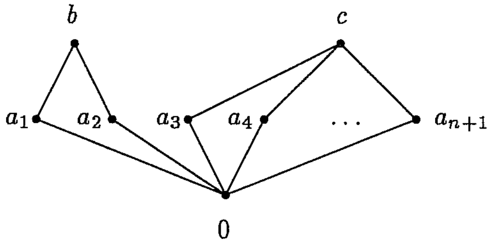

Fig. 4

Let T_j be a tolerance generated by the set $\{(a_i, 0) : i \neq j\}$, i.e. $T_j = \{(a_i, 0), (0, a_i) : i \neq j\} \cup \Delta$, $j = 1, 2, \ldots, n$. Hence $T_j^* \supseteq T(a_j, 0)$, $j = 1, 2, \ldots, n$ and

$$T_1 \wedge T_2 \wedge \ldots \wedge T_n = T(a_{n+1}, 0),$$
$$T_1^* \wedge T_2 \wedge \ldots \wedge T_n = T(a_1, 0),$$
$$\ldots$$
$$T_1 \wedge T_2 \wedge \ldots \wedge T_n^* = T(a_n, 0).$$

It yields that $(b, c) \notin T^*(a_i, 0)$ for $i = 1, 2, \ldots, n+1$ and since b, c are both maximal elements, $(b, c) \notin T^*(a_1, 0) \vee T^*(a_2, 0) \vee \ldots \vee T^*(a_{n+1}, 0)$. Therefore
$(T_1 \wedge T_2 \wedge \ldots \wedge T_n)^* \vee (T^*{}_1 \wedge T_2 \wedge \ldots \wedge T_n)^* \vee \ldots \vee (T_1 \wedge T_2 \wedge \ldots \wedge T_n^*)^* \neq \nabla$ and $\mathrm{Tol}(S_n) \notin \mathcal{P}_n$.

Now we wish to prove that $\mathrm{Tol}(S_n) \in \mathcal{P}_{n+1}$. Let $T_1, T_2 \ldots T_{n+1}$ be arbitrary tolerances on S_n and $U_1 = T_1 \wedge T_2 \wedge \ldots \wedge T_{n+1}, U_2 = T_1^* \wedge T_2 \wedge \wedge \ldots \wedge T_{n+1}, \ldots, U_{n+2} = T_1 \wedge T_2 \wedge \ldots \wedge T_{n+1}^*$. Since $U_1, U_2 \ldots U_{n+2}$ are $n+2$ pairwise disjoint tolerances there exists $j \in \{1, 2 \ldots n+2\}$ such that $(a_i, 0) \notin U_j$ for $i = 1, 2 \ldots n+1$. Two possibilities can occure:

(i) If $n > 2$ then $U_j = \Delta$ and trivially $U_1^* \vee U_2^* \vee \ldots \vee U_{n+2}^* = \nabla$.

(ii) If $n = 2$ then $U_j = \Delta$ or $U_j = T(a_3, c)$.

In the second case $U_j^* = (S \times S) \setminus \{(a_3, c), (c, a_3)\}$. Since for any tolerance U such that $(a_3, c) \notin U$ is $(a_3, c) \in U^*$ we obtain $U_1^* \vee U_2^* \vee \vee U_3^* \supseteq U_1^* \cup U_2^* \cup U_3^* = S \times S = \nabla$. $\Diamond$

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