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AUTOMORPHISMS OF FREE NILPOTENT ASSOCIATIVE ALGEBRAS

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Abstract: In this paper we study the automorphism group of the free nilpotent algebra $F_m(\mathbf{N}_c)$ of class c and m generators over a field K of characteristic 0. Under some restrictions on m and c, we have found three automorphisms which, together with the general linear group $GL_m(K)$ generate the whole group of automorphisms. A similar problem has been considered by Drensky and C.K. Gupta for free nilpotent Lie algebras and free nilpotent metabelian Lie algebras.

Let K be a field of characteristic 0. Drensky and C.K. Gupta [3] have shown that the automorphism group of the free metabelian and nilpotent of class c Lie algebra $L_m(\mathbf{N}_c \cap \mathbf{A}^2)$ of rank m is generated by

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the automorphisms induced by the action of the general linear group $GL_m(K)$ and by one more automorphism δ_1 defined by $\delta_1(x_1) = x_1 + [x_1, x_2]$, $\delta_1(x_j) = x_j$, $j > 1$. (Another proof of this result based on different techniques is given by Papistas [6].) As a consequence in [3] it has been established that for $m \geq c$ the automorphism group of the free nilpotent Lie algebra $L_m(\mathbf{N}_c)$ is generated by $GL_m(K)$ and the same automorphism δ_1. The approach in [3] is based on the representation theory of $GL_m(K)$. Using the same approach and the result of Anick [1] that the tame automorphisms of the polynomial algebra $K[x_1, \dots, x_m]$ form a dense subgroup of the whole automorphism group with respect to the formal power series topology, one can show that the automorphism group of the free nilpotent of class c commutative and associative algebra $F_m(\mathbf{A} \cap \mathbf{N}_c)$ is generated by $GL_m(K)$ and one more automorphism δ_2 defined by $\delta_2(x_1) = x_1 + x_1x_2$, $\delta_2(x_j) = x_j$, $j > 1$. In this paper we establish the following main result.

Theorem. *Let $F_m(\mathbf{N}_c)$ be the free nilpotent of class c associative algebra freely generated by $x_1, \dots, x_m$ over a field K of characteristic 0, $m \geq 2$. Let δ_i, $i = 1, 2, 3$, be the automorphisms of $F_m(\mathbf{N}_c)$ defined by*

$$\delta_1(x_1) = x_1 + [x_1, x_2], \delta_2(x_1) = x_1 + x_1x_2, \delta_3(x_1) =$$

$$= x_1 + x_1^2x_2, \delta_i(x_j) = x_j, j > 1.$$

If $c = 3$ and m is arbitrary, or if $m = 2$, $c \leq 7$ or if m and c satisfy the inequality $m(m-4) \geq c-2$, then $\operatorname{Aut} F_m(\mathbf{N}_c)$ is generated by $GL_m(K)$ and the automorphisms $\delta_1, \delta_2, \delta_3$.

Our proof is based on the same idea as in [3]. Some factors of the subgroup of $\operatorname{Aut} F_m(\mathbf{N}_c)$ generated by $GL_m(K)$ and $\delta_1, \delta_2, \delta_3$ have a natural structure of $GL_m(K)$-modules and we show that these factors are isomorphic to the corresponding factors of the whole automorphism group. The main difference of our approach from that in [3] is the following. The essential difficulty in [3] is to handle the case of the automorphisms of the free nilpotent metabelian Lie algebra. It turns out that the number of the irreducible $GL_m(K)$-submodules in all factors in consideration is bounded. In our case this number is not bounded and we have further developed the technique from [3].

The paper is organized as follows. In Section 1 we present necessary preliminaries, fix the notation and consider the simplest case of nilpotent of class 3 algebras. Section 2 is devoted to the automorphism group of the free nilpotent algebras in two generators and of class ≤ 7. Finally, in Section 3 we handle the case $m > 2$ and $m(m-4) \geq c-2$.

The results of our paper have been announced in [4]. Due to a technical error, in [4] we have imprecisely stated that $GL_m(K)$ and the automorphisms δ_j, $j = 1, 2, 3$, generate $\operatorname{Aut} F_m(\mathbf{N}_c)$ for $m^2 \geq c-1$. We still do not know whether $\operatorname{Aut} F_m(\mathbf{N}_c)$ is generated by $GL_m(K)$ and δ_j, $j = 1, 2, 3$, for any m and c.

1. Preliminaries

Throughout the paper we fix a field K of characteristic 0 and all algebras, vector spaces and tensor products are over K. We also fix the set $\{x_1, \dots, x_m\}$, $m \geq 2$, as a set of free generators of the free algebras.

We start with the necessary background on the representation theory of the general linear group $GL_m = GL_m(K)$. We follow the exposition from [3] and [2]. Let V_m be the vector space with basis $x_1, \dots, x_m$ and let GL_m act canonically from the left on V_m. We consider the elements from GL_m as invertible $m \times m$ matrices with entries from K. For an s-dimensional vector space W, also with a fixed basis, a homomorphism

$$\phi : GL_m \longrightarrow GL_s = GL(W)$$

is called a polynomial representation of GL_m if the entries $\phi_{pq}(g)$ of the $s \times s$ matrix $\phi(g)$ are polynomial functions of the entries a_{ij} of the matrix $g = (a_{ij}) \in GL_m$. Similarly, if $\phi_{pq}(g)$ are rational functions of a_{ij}, then ϕ is called a rational representation. If the functions ϕ_{pq} are homogeneous of the same degree, the representation ϕ and the corresponding GL_m-module W are called homogeneous. Let

$$D_m = \{d \in GL_m \mid d = (z_1, \dots, z_m) = z_1 e_{11} + \dots z_m e_{mm}\}$$

be the subgroup of the diagonal matrices of GL_m. For any degree sequence $\alpha = (\alpha_1, \dots, \alpha_m)$ of length m we define the α-homogeneous component of the GL_m-module W by

$$W^\alpha = \{w \in W \mid d(z_1, \dots, z_m)w = z_1^{\alpha_1} \dots z_m^{\alpha_m} w,\ d(z_1, \dots, z_m) \in D_m\}.$$

We shall use the following well known statements for the finite dimensional rational representation $\phi : GL_m \longrightarrow GL(W)$.

1. The GL_m-module W is completely reducible and is a direct sum of its homogeneous submodules.

2. As a K-vector space, W is a direct sum of its homogeneous components W^α.

3. The Hilbert series of W

$$H(W) = H(W, t_1, \dots, t_m) = \sum (\dim_K W^\alpha) t_1^{\alpha_1} \dots t_m^{\alpha_m}$$

is a symmetric function in $t_1, \dots, t_m$.

4. If $(\det)^n$, $n \in \mathbb{Z} \setminus \{0\}$ is the one-dimensional representation of GL_m defined by

$$(\det)^n(g) = (\det g)^n,$$

then every rational GL_m-module has the form W or $(\det)^{-n} \otimes W$, where $n \in \mathbb{N}$ and W is a polynomial GL_m-module; $(\det)^{-n} \otimes W$ is irreducible if and only if W is irreducible. The Hilbert series of $\det^n$ is

$$H(\det^n) = \left(\prod_{i=1}^{m} t_i\right)^n.$$

The irreducible polynomial representations of GL_m are described by partitions and Young diagrams (see e.g. [5, 7]). For a partition $\lambda = (\lambda_1, \dots, \lambda_m)$, $\lambda_1 \geq \dots \geq \lambda_m \geq 0$, $\lambda_1 + \dots + \lambda_m = n$, we consider the corresponding Young diagram $[\lambda]$ and the related GL_m-module $W_m(\lambda)$. A λ-tableau with content $\alpha = (\alpha_1, \dots, \alpha_m)$ is a diagram $[\lambda]$ whose boxes are filled in with α_1 numbers $1, \dots, \alpha_m$ numbers m. A tableau is semistandard if its entries do not decrease from left to right in the rows and increase from top to bottom in the columns. The Hilbert series of the module $W_m(\lambda)$ is equal to the Schur function $S_\lambda(t_1, \dots, t_m)$,

$$H(W_m(\lambda), t_1, \dots, t_m) = S_\lambda(t_1, \dots, t_m) =$$
$$= \sum \dim_K (W_m(\lambda))^\alpha t_1^{\alpha_1} \dots t_m^{\alpha_m},$$

where the coefficient $a_\alpha = \dim_K(W_m(\lambda))^\alpha$ is equal to the number of the semistandard λ-tableaux of content $\alpha = (\alpha_1, \dots, \alpha_m)$.

Let $\mathbf{N}_c$ be the variety of all nilpotent of class $\leq c$ associative algebras. By definition, this is the class of all algebras satisfying the polynomial identity $x_1 \dots x_c = 0$. We fix the notation

$$F = F_m(\mathbf{N}_c)$$

for the free nilpotent of class c algebra of rank m. If $A_m = K\langle x_1, \dots, x_m \rangle$ is the free associative nonunitary algebra of rank m, then $F \cong A_m / A_m^c$. Every map $\varepsilon : x_s \to F$, $s = 1, \dots, m$, is uniquely extended to an endomorphism $\phi \in \operatorname{End} F$ of the algebra F. Since F is nilpotent, ϕ is an automorphism if and only if it induces an invertible linear map on the vector space F/F^2. Therefore, an endomorphism $\phi \in \operatorname{End} F$ is an automorphism when there exists a $g \in GL_m$ (considered as an automorphism of F) such that $g^{-1} \circ \phi(x_s) = x_s + u_s$, $u_s \in F^2$, $s = 1, \dots, m$. We consider the descending series of normal subgroups of $\operatorname{Aut} F$

$$I_k A = \{\phi \in \operatorname{Aut} F \mid \phi(x_s) = x_s + u_s,\ u_s \in F^k,\ s = 1, \dots, m\}.$$

Clearly

$$I_2A > I_3A > \ldots > I_cA = \langle \mathrm{id} \rangle.$$

The factor groups $I_kA/I_{k+1}A$ are abelian and $\operatorname{Aut} F = GL_m.I_2A$, i.e. $\operatorname{Aut} F$ is a split extension of GL_m and I_2A. Let $F^{(k)}$ be the homogeneous component of degree k of the algebra F. The group GL_m acts diagonally on $F^{(k)}$ by

$$g(a(x_1, \ldots, x_m)) = a(g(x_1), \ldots, g(x_m)),\ g \in GL_m, a \in F^{(k)}.$$

We define a map

$$\theta_k : I_kA/I_{k+1}A \longrightarrow (F^{(k)})^{\oplus m} = F^{(k)} \oplus \ldots \oplus F^{(k)}$$

in the following way. If $\phi \in I_kA$ and $\bar{\phi} = \phi I_{k+1}A$, where $\phi(x_s) = x_s + a_s + b_s$, $a_s \in F^{(k)}$, $b_s \in F^{k+1}$, $s = 1, \ldots, m$, then

$$\tilde{\phi} = \theta_k(\bar{\phi}) = (a_1, \ldots, a_m) \in (F^{(k)})^{\oplus m}.$$

We do not write the index k of θ_k if it is clear from the context. By [3] θ is an isomorphism of the multiplicative group $I_kA/I_{k+1}A$ and the additive group $(F^{(k)})^{\oplus m}$ and, if $\bar{\phi} \in I_kA/I_{k+1}A$ we write $\bar{\phi}(x_s) = x_s + a_s$, $s = 1, \ldots, m$. The group $I_kA/I_{k+1}A$ admits also the multiplication with elements from the base field K. If $\bar{\phi}(x_s) = x_s + a_s$, $s = 1, \ldots, m$, and $\nu \in K$, then

$$(\nu\bar{\phi})(x_s) = x_s + \nu a_s,\ s = 1, \ldots, m,$$

and in this way I_kA/I_{k+1} has a natural structure of a K-vector space. The addition is replaced by the composition of automorphisms and the multiplication by scalars is as defined above. Since $\theta(\nu\bar{\phi}) = \nu(\theta(\bar{\phi}))$, the group isomorphism θ is also an isomorphism of vector spaces and very often we shall identify $I_kA/I_{k+1}A$ and $(F^{(k)})^{\oplus m}$ via θ. The groups I_kA are normal and they are invariant under the conjugation by elements from GL_m. It turns out that the action

$$g \circ \bar{\phi} = (g\phi g^{-1})I_{k+1}A,\ g \in GL_m, \phi \in I_kA,$$

equips the vector space $I_kA/I_{k+1}A$ with the structure of a GL_m-module. This action induces a GL_m-module structure also on $(F^{(k)})^{\oplus m}$ described as follows.

Proposition 1.1 [3]. *Let* $\bar{\phi} \in I_kA/I_{k+1}A$, $\theta(\bar{\phi}) = (a_1, \ldots, a_m) \in (F^{(k)})^{\oplus m}$. *Considering* $\theta(\bar{\phi})$ *and* $g \in GL_m$ *as a row vector and an* $m \times m$ *matrix, respectively, it holds*

$$g \circ (a_1, \ldots, a_m) = (g(a_1), \ldots, g(a_m))(b_{ij}),$$

where $(b_{ij}) = g^{-1}$.

Example 1.2. Let $u = x_{p_1} \ldots x_{p_k}$, $\deg_{x_j} u = \alpha_j$, $j = 1, \ldots, m$, and let $\phi \in I_kA$ be defined by $\phi(x_s) = x_s + u$, $\phi(x_i) = x_i$, $i \neq s$. If $d = d(z_1, \ldots, z_m)$ is a diagonal matrix from GL_m, then

$$d\circ\tilde{\phi}=z_1^{\alpha_1}\dots z_m^{\alpha_m}z_s^{-1}\tilde{\phi},$$

i.e. the element $\theta(\bar{\phi})$ of the GL_m-module $\theta(I_kA/I_{k+1}A)=(F^{(k)})^{\oplus m}$ is homogeneous of degree $(\alpha_1,\dots,\alpha_{s-1},\alpha_s-1,\alpha_{s+1},\dots,\alpha_m)$.

Similarly, the element $(f_1,\dots,f_m)\in(F^{(k)})^{\oplus m}$ is homogeneous of degree $\alpha=(\alpha_1,\dots,\alpha_m)$ if every polynomial $f_i\in F^{(k)}$, $i=1,\dots,m$, is homogeneous of degree α_i+1 in x_i and α_j in the other variables x_j, $j\neq i$. For example, for $m=k=3$, the elements $(x_1^2x_2,0,0)$ and $(x_1x_2x_1,x_1x_2^2,x_1x_3x_2)$ are homogeneous of degree $(1,1,0)$.

One of the key observations in [3, 2] is the GL_m-module isomorphism

$$I_kA/I_{k+1}A\cong\det{}^{-1}\otimes W_m(1^{m-1})\otimes F^{(k)}.$$

The GL_m-module decomposition of the homogeneous component of degree $k<c$ of the free nilpotent of class c algebra is the same as that of the homogeneous component of the free associative algebra (or of the k-th tensor power of V_m), i.e.

$$F^{(k)}\cong\sum d_\lambda W_m(\lambda),$$

where the summation runs on all partitions $\lambda=(\lambda_1,\dots,\lambda_m)$ of k. Since for $\lambda=(\lambda_1,\dots,\lambda_m)$, $\lambda_m\geq 1$,

$$\det{}^{-1}\otimes W_m(\lambda)\cong W_m(\lambda_1-1,\dots,\lambda_m-1),$$

as in [3] we can use the Young rule in order to calculate the multiplicities of the irreducible components of $I_kA/I_{k+1}A$. In particular, we obtain

Corollary 1.3. *The GL_m-module $I_kA/I_{k+1}A$ is isomorphic to a direct sum of irreducible GL_m-modules of the following kind:*

(i) $W_m(\mu)$, *where $\mu=(\mu_1,\dots,\mu_m)$ is a partition of $k-1$;*

(ii) $\det^{-1}\otimes W_m(\mu')$, *where $\mu'=(\mu_1',\dots,\mu_{m-1}',0)$ is a partition of $k+m-1$.*

Let δ_i, $i\in I$, be automorphisms of F, where I is a set of indices, and let

$$G=\langle GL_m,\delta_i\mid i\in I\rangle$$

be the subgroup of $\operatorname{Aut}F_m(\mathbf{N}_c)$ generated by GL_m and δ_i, $i\in I$.

Proposition 1.4 [3]. *Let $\tilde{G}\subset(F^{(c-1)})^{\oplus m}$ be the image of $(G\cap\cap I_{c-1}A))/(G\cap I_cA)$ ($\cong G\cap I_{c-1}A$ because $I_cA=\langle\mathrm{id}\rangle$) under the isomorphism θ. Then $\tilde{G}$ is a GL_m-submodule of $(F^{(c-1)})^{\oplus m}$.*

The following assertion follows easily by induction as in the proof of [3, Th. 3.7]. Since we consider nilpotent algebras, it is equivalent to [2, Cor. 2.9].

Corollary 1.5 [3, 2]. *Let $G=\langle GL_m,\delta_i\mid i\in I\rangle\subseteq\operatorname{Aut}F_m(\mathbf{N}_c)$ and let*

$$(G \cap I_k A)/(G \cap I_{k+1} A) = I_k A / I_{k+1} A, \ k = 2, \dots, c-1.$$

Then the groups G and $\mathrm{Aut}\, F_m(\mathbf{N}_c)$ *coincide.*

We shall use the following technical assertion which is a consequence of Prop. 1.4 and the fact that every rational GL_m-module is a direct sum of its homogeneous K-vector spaces.

Corollary 1.6. *In the notation of* Prop. 1.4 *let $\tilde{\phi} = \tilde{\phi}_1 + \ldots + \tilde{\phi}_s \in \tilde{G}$, where $\tilde{\phi}_j$, $j = 1, \dots, s$, are the homogeneous components of $\tilde{\phi}$ in the GL_m-module $\tilde{G}$. Then $\tilde{\phi} \in \tilde{G}$, $j = 1, \dots, s$.*

Definition 1.7. Let W_i, $i \in I$, be the irreducible components of the rational GL_m-module W. We call the *content* $\alpha = (\alpha_1, \dots, \alpha_m)$ *essential* if all homogeneous components W_i^α, $i \in I$, are nonzero.

Clearly, if α is an essential content for the GL_m-module W and W_1 is a submodule of W, then the equality

$$\dim_K W_1^\alpha = \dim_K W^\alpha$$

implies that $W_1 = W$.

Lemma 1.8. *Let $\lambda = (\lambda_1, \dots, \lambda_m)$ be a partition of s, $s = pm + r$, $0 \leq r < m$. Then the content $\alpha = (p + \varepsilon_1, \dots, p + \varepsilon_m)$ is essential for the GL_m-module $W_m(\lambda)$, where $\varepsilon_1 = \ldots = \varepsilon_r = 1$, $\varepsilon_{r+1} = \ldots = \varepsilon_m = 0$.*

Proof. It is sufficient to show that there exists a semistandard λ-tableau of content α. We use induction on s and m. The base of the induction $m = 1$, s any and $s = 1$ and m any is obviously true. We assume that the statement holds for all $m_0 \leq m$ and $s_0 < s$.

Case 1. Let $\lambda_m \neq 0$. If T is a semistandard λ-tableau, then each of the first λ_m columns of the diagram $[\lambda]$ is filled in with the integers $1, \dots, m$. We delete these columns and reduce the problem to the existence of a semistandard ν-tableau of content β, where

$$\nu = (\lambda_1 - \lambda_m, \dots, \lambda_{m-1} - \lambda_m, 0), \ \beta = (\alpha_1 - \lambda_m, \dots, \alpha_m - \lambda_m).$$

Since ν is a partition of $s - m\lambda_m < s$, we use the inductive assumption and obtain that α is an essential content for $W_m(\lambda)$.

Case 2. Let $\lambda_m = 0$. We consider all the boxes of the diagram $[\lambda]$ which are in the bottoms of the columns of $[\lambda]$. The number of these boxes is $\lambda_1 \geq p$ and we fill in p of them with m. We delete these p boxes and obtain a new diagram $[\nu]$. In order to make the inductive step we have to construct a semistandard ν-diagram of content $\beta = (\alpha_1, \dots, \alpha_{m-1}, 0)$. Since $\nu = (\nu_1, \dots, \nu_{m-1}, 0)$ is a partition of $s - p < s$, and $s - p = p(m-1) + r$ if $r < m - 1$, or $s - p = (p+1)(m-1)$ if $r = m - 1$, by the inductive arguments the content $(\alpha_1, \dots, \alpha_{m-1})$ is essential for the GL_{m-1} module $W_{m-1}(\nu_1, \dots, \nu_{m-1})$. $\Diamond$

Corollary 1.9. *Let $k-1=pm+r$, $0\le r<m$. Then the content $\alpha=(p+\varepsilon_1,\dots,p+\varepsilon_m)$ is essential for the GL_m-module $I_kA/I_{k+1}A$, where $\varepsilon_1=\ldots=\varepsilon_r=1$, $\varepsilon_{r+1}=\ldots=\varepsilon_m=0$.*

Proof. By Cor. 1.3 the GL_m-module $I_kA/I_{k+1}A$ is a direct sum of irreducible submodules $W_m(\mu)$, where $\mu=(\mu_1,\dots,\mu_m)$ is a partition of $k-1$, and $\det^{-1}\otimes W_m(\mu')$, where $\mu'=(\mu'_1,\dots,\mu'_{m-1},0)$ is a partition of $k+m-1$. The submodules $W_m(\mu)$ are covered directly by Lemma 1.8. In order to prove that the homogeneous component of degree α of $W=\det^{-1}\otimes W_m(\mu')$ is not zero, it is sufficient to show that the coefficient a_α in the Hilbert series of W

$$H(W)=\sum a_\gamma t_1^{\gamma_1}\dots t_m^{\gamma_m}$$

is different from 0. If

$$H(W_m(\mu'))=\sum b_\beta t_1^{\beta_1}\dots t_m^{\beta_m}$$

is the Hilbert series of $W(\mu')$, then

$$H(W)=(t_1\dots t_m))^{-1}H(W_m(\mu')).$$

Therefore, $a_\alpha=b_{\alpha'}$, where $\alpha'=(\alpha_1+1,\dots,\alpha_m+1)$. Again, the proof follows directly from Lemma 1.8. ◊

Remark 1.10. Let $\sigma\in S_m$ be an element from the symmetric group S_m of degree m and let $g_\sigma\in GL_m$ be defined by $g_\sigma(x_i)=x_{\sigma(i)}$, $i==1,\dots,m$. If $w\in W=(F^{(k)})^{\oplus m}$ and $w=(f_1,\dots,f_m)\in W^\alpha$, then direct verification shows that $g_\sigma\circ w\in W^\beta$, where $\beta=(\beta_1,\dots,\beta_m)$ is obtained by permuting of $\alpha=(\alpha_1,\dots,\alpha_m)$ by σ^{-1}. We shall use this statement only in the case when σ is a transposition.

Proposition 1.11. *The automorphism group of the free nilpotent of class 3 algebra $F_m(\mathbf{N}_3)$ of any rank m is generated by GL_m and the automorphisms δ_1 and δ_2 defined by*

$$\delta_1(x_1)=x_1+x_1^2,\qquad \delta_2(x_1)=x_1+[x_1,x_2],$$
$$\delta_j(x_i)=x_i,\ i=2,\dots,m,\quad j=1,2.$$

Proof. Let G be the subgroup of $\operatorname{Aut}F_m(\mathbf{N}_3)$ generated by GL_m, δ_1 and δ_2, and let $\tilde G$ be the image under θ of $(G\cap I_2A)/(G\cap I_3A)\cong\cong G\cap I_2A$. If $W=\theta(I_2A/I_3A)=(F^{(2)})^{\oplus m}$, by Corollaries 1.5 and 1.9, it is sufficient to show that $\tilde G^\alpha=W^\alpha$ for $\alpha=(1,0,\dots,0)$. The homogeneous component W^α has a basis

$$\{(x_1^2,0,\dots,0),(0,\dots,0,x_1x_i,0,\dots,0),$$
$$(0,\dots,0,x_ix_1,0,\dots,0)\mid i=2,\dots,m\},$$

where x_1x_i and x_ix_1 are in the i-th position. We shall complete the proof in two steps.

Step 1. Clearly, $\tilde{\delta}_1 \in \tilde{G}$. Let $g_1 \in GL_m$ be defined by

$$g_1(x_1) = x_1 + x_i,\ g_1(x_s) = x_s,\ s \neq 1.$$

Then $g_1^{-1}(x_1) = x_1 - x_i$, $g_1^{-1}(x_s) = x_s$, $s \neq 1$, and direct calculations show that

$$(g_1 \circ \delta_1)(x_1) = g_1\delta_1 g_1^{-1}(x_1) = x_1 + (x_1 + x_i)^2,\ (g_1 \circ \delta_1)(x_s) = x_s,\ s \neq 1.$$

By Cor. 1.6 the homogeneous components of $g_1 \circ \tilde{\delta}_1$ also belong to $\tilde{G}$. Considering the component of degree $(0, \dots, 0, 1, 0, \dots, 0)$, where 1 is in the i-th position, we obtain that $\tilde{\psi} = (x_1 \circ x_i, , \dots, 0) \in \tilde{G}$, where by definition $x_1 \circ x_i = x_1x_i + x_ix_1$. Let $\sigma = (1i) \in S_m$ and let $g_\sigma \in GL_m$ be as in Remark 1.10. Then

$$\tilde{\eta}_i = g_\sigma \circ \tilde{\psi} = (0, \dots, 0, x_1 \circ x_i, 0, \dots, 0) \in \tilde{G}.$$

Step 2. We start with $\tilde{\delta}_2 = ([x_1, x_2], 0, \dots, 0) \in \tilde{G}$. For $\tau = (12) \in \in S_m$, $g_\tau \in GL_m$, we obtain

$$g_\tau \circ \delta_2(x_2) = x_2 - [x_1, x_2],\ g_\tau \circ \delta_2(x_s) = x_s,\ s \neq 2,$$

$$\tilde{\pi} = g_\tau \circ \tilde{\delta}_2 = -(0, [x_1, x_2], 0, \dots, 0) \in \tilde{G}.$$

For $\rho = (2i) \in S_m$, $g_\rho \in GL_m$ we calculate

$$\tilde{\zeta}_i = g_\rho \circ \tilde{\pi} = -(0, \dots, 0, [x_1, x_i], 0, \dots, 0) \in \tilde{G}.$$

Since $\tilde{\delta}_1$, $\tilde{\eta}_i$ and $\tilde{\zeta}_i$, $i = 2, \dots, m$, belong to $\tilde{G}$ and form a basis for the whole homogeneous component of degree $\alpha = (1, 0, \dots, 0)$ of W, we obtain that $\tilde{G}^\alpha = W^\alpha$ and this implies that $\tilde{G} = W$. $\Diamond$

The following technical lemma is useful in the further computations. It is a restatement of [2, Lemma 2.3, Remark 2.4].

Lemma 1.12. *Let $\phi \in I_kA$ and $\psi \in I_lA$ be automorphisms of $F = = F_m(\mathbf{N}_c)$, $k + l \leq c$. Then the group commutator $(\phi, \psi) = \phi^{-1}\psi^{-1}\phi\psi$ is in $I_{k+l-1}A$. If*

$$\tilde{\phi} = (f_1, \dots, f_m) \in (F^{(k)})^{\oplus m},\ \tilde{\psi} = (g_1, \dots, g_m) \in (F^{(l)})^{\oplus m},$$

then $\theta((\phi, \psi)I_{k+l}A) = (h_1, \dots, h_m) \in (F^{(k+l-1)})^{\oplus m}$ satisfies

$$h_i = g_i(x_1 + f_1, \dots, x_m + f_m)_{(k+l-1)} + f_i(x_1 - g_1, \dots, x_m - g_m)_{(k+l-1)},$$

where we denote by $u_{(s)}$ the homogeneous component of degree s of $u \in \in F$.

For simplicity of notation, when k and l for $\phi \in I_kA$ and $\psi \in \in I_lA$ are clear from the context, we shall write $(\tilde{\phi}, \tilde{\psi})$ instead of $\theta_{k+l-1}((\phi, \psi)I_{k+l}A)$.

Remark 1.13. In the notation of Lemma 1.12, let Δ_ϕ be the derivation of F defined by $\Delta_\phi x_i = f_i$, $i = 1, \dots, m$; we define Δ_ψ in a similar way. Then

$$(\tilde{\phi},\tilde{\psi})=(\Delta_\phi g_1-\Delta_\psi f_1,\dots,\Delta_\phi g_m-\Delta_\psi f_m).$$

Example 1.14. Let $m=2$, $c>3$, and let $\phi,\psi\in I_2A$ be such that

$$\tilde{\phi}=(x_1^2,0),\tilde{\psi}=(x_1\circ x_2,0)\in F^{(2)}\oplus F^{(2)}.$$

Then, modulo F^4,

$$(\phi,\psi)(x_1)=x_1-((x_1\circ x_2)x_1+x_1(x_1\circ x_2))+x_1^2\circ x_2=$$

$$=x_1-2x_1x_2x_1,\ (\phi,\psi)(x_2)=x_2.$$

Hence

$$(\tilde{\phi},\tilde{\psi})=(-2x_1x_2x_1,0)\in F^{(3)}\oplus F^{(3)}.$$

Similarly, for $\tilde{\phi}=(x_2^2,0)$, $\tilde{\psi}=(0,x_1^2)$,

$$(\tilde{\phi},\tilde{\psi})=(x_2\circ x_1^2,-x_1\circ x_2^2).$$

2. Nilpotent algebras with two generators

Till the end of the paper we fix the notation $\delta_1,\delta_2,\delta_3$ for the automorphisms of the algebra $F=F_m(\mathbf{N}_c)$ defined by

$$\delta_1(x_1)=x_1+x_1^2,\qquad \delta_3(x_1)=x_1+x_1^2x_2,\qquad \delta_2(x_1)=x_1+[x_1,x_2],$$

$$\delta_j(x_i)=x_i,\ i\neq 1,\ j=1,2,3.$$

Especially in this section we denote by $G=\langle GL_2,\delta_1,\delta_2,\delta_3\rangle$ the subgroup of $\operatorname{Aut}F$ generated by GL_2 and δ_j, $j=1,2,3$. Here $F=F_2(\mathbf{N}_c)$, $c=4,5,6,7$.

Proposition 2.1. *The group* $\operatorname{Aut}F_2(\mathbf{N}_4)$ *is generated by* GL_2 *and* δ_j, $j=1,2,3$.

Proof. Let $W=\theta_3(I_3A/I_4A)=F^{(3)}\oplus F^{(3)}$ and let $\tilde{G}=\theta_3(G\cap I_3A)/(G\cap I_4A)\subseteq W$. In virtue of Cor. 1.9 the proof of the proposition will be completed if we establish that $\tilde{G}^\alpha=W^\alpha$ for the essential content $\alpha=(1,1)$. The following elements form a K-basis for the homogeneous component $W^{(1,1)}$

$$e_1=(x_1x_2x_1,0),\quad e_2=(x_1^2x_2,0),\quad e_3=(x_2x_1^2,0),$$

$$e_4=(0,x_2x_1x_2,\quad e_5=(0,x_2^2x_1),\quad e_6=(0,x_1x_2^2).$$

If $\sigma=(12)\in S_2$, then the element $g_\sigma\in GL_2$ defined in Remark 1.10 satisfies

$$g_\sigma\circ e_i=e_{i+3},\ i=1,2,3.$$

We shall show that $e_1,e_2,e_3\in\tilde{G}$ in several steps.

Step 1. Let $\tilde{\phi}=(x_1^2,0),\tilde{\psi}=(x_1\circ x_2,0)\in F^{(2)}\oplus F^{(2)}$. By Prop. 1.11 there exist automorphisms ϕ and ψ in G such that their images in $\theta:I_2A/I_3A\longrightarrow F^{(2)}\oplus F^{(2)}$ coincide respectively with $\tilde{\phi}$

and $\tilde{\psi}$. In Ex. 1.14 we have seen that the image $(\tilde{\phi}, \tilde{\psi})$ in W of group commutator $(\phi, \psi$ is $2(x_1x_2x_1, 0)$, i.e. $e_1 \in \tilde{G}$. Therefore, e_4 also belong to $\tilde{G}$.

Step 2. Let $g \in GL_2$, $g(x_1) = -2x_1$, $g(x_2) = x_2$. We calculate for $\varepsilon = g \circ \delta_1$ that

$$\varepsilon(x_1) = x_1 - 2x_1^2,\ \delta_1^2(x_1) = x_1 + 2x_1^2 + 2x_1^3,\ \varepsilon(x_2) = x_2,\ \delta_1^2(x_2) = x_2,$$

$$(\varepsilon \circ \delta_1^2)(x_1) = x_1 - 6x_1^3,\ (\varepsilon \circ \delta_1^2)(x_2) = x_2,$$

i.e. $(x_1^3, 0) \in \tilde{G}$.

Now, let $g_1 \in GL_2$, $g_1(x_1) = x_1 + x_2$, $g_1(x_2) = x_2$. Then

$$g_1 \circ (x_1^3, 0) = ((x_1 + x_2)^3, 0) \in \tilde{G}.$$

Considering the $(1,1)$-homogeneous component of this element, which is also in $\tilde{G}$, we obtain $e_1 + e_2 + e_3 \in \tilde{G}$. Since $e_1, e_2 \in \tilde{G}$, this gives that $e_3 \in \tilde{G}$. In this way, all basis elements e_j, $j = 1, \ldots, 6$, of the K-vector space $W^{(1,1)}$ are in $\tilde{G}^{(1,1)}$ and this implies that $\tilde{G} = W$. $\Diamond$

In the following considerations m and c are arbitrary and $F = F_m(\mathbf{N}_c)$. Let $*$ be the involution of F given by

$$(x_{i_1} \ldots x_{i_k})^* = x_{i_k} \ldots x_{i_1}.$$

For an automorphism ϕ of F we denote by ϕ^* the automorphism defined by $\phi^*(x_j) = (\phi(x_j))^*$. Clearly, $*$ induces a K-vector space automorphism of $F^{(k)}$, $k = 1, 2, \ldots, c-1$, and if $\phi \in I_kA$, then $\phi^* \in I_kA$.

Lemma 2.2. *If $\phi \in I_kA$, $\psi \in I_lA$, then for the group commutators of ϕ and ψ it holds*

$$\theta_{k+l-1}((\phi, \psi)^*) = \theta_{k+l-1}(\phi^*, \psi^*).$$

Proof. Let $\zeta = (\phi, \psi) \in I_{k+l-1}A$,

$$\tilde{\phi} = \theta_k(\phi) = (f_1, \ldots, f_m) \in (F^{(k)})^{\oplus m},$$

$$\tilde{\psi} = \theta_l(\phi) = (g_1, \ldots, g_m) \in (F^{(l)})^{\oplus m}.$$

By Lemma 1.12, the coordinates h_i of

$$\tilde{\zeta} = \theta_{k+l-1}(\zeta) = (h_1, \ldots, h_m) \in (F^{(k+l-1})^{\oplus m}$$

are equal to the homogeneous components of degree $k + l - 1$ of

$$g_i(x_1 + f_1, \ldots, x_m + f_m) + f_i(x_1 - g_1, \ldots, x_m - g_m).$$

Since $*$ is an involution of F, we obtain

$$g_i^*((x_1 + f_1)^*, \ldots, (x_m + f_m)^*) + f_i^*((x_1 - g_1)^*, \ldots, (x_m - g_m)^*) = h_i^*,$$

and this completes the proof. $\Diamond$

Proposition 2.3. *The group* $\operatorname{Aut} F_2(\mathbf{N}_5)$ *is generated by* GL_2 *and* δ_j, $j = 1, 2, 3$.

Proof. We repeat the scheme and the notation of the proof of Prop. 2.1. By Cor. 1.9, the content $\alpha = (2,1)$ is essential for the GL_2-module $W = \theta_4(I_4A/I_5A) = F^{(4)} \oplus F^{(4)}$ and it is sufficient to show that $\tilde{G}^\alpha = W^\alpha$, where $\tilde{G}$ is the image of G in W. The basis of the K-vector space W^α consists of

$$e_1 = (x_1^2x_2^2, 0), \quad e_2 = (x_1x_2^2x_1, 0), \quad e_3 = (x_2^2x_1^2, 0),$$
$$e_4 = ((x_1x_2)^2, 0), \quad e_5 = ((x_2x_1)^2, 0), \quad e_6 = (x_2x_1^2x_2, 0),$$
$$e_7 = (0, x_1x_2^3), \quad e_8 = (0, x_2x_1x_2^2), \quad e_9 = (0, x_2^2x_1x_2),$$
$$e_{10} = (0, x_2^3x_1).$$

We make use of Prop. 2.1 and assume that for any $(f_1, f_2) \in F^{(k)} \oplus F^{(k)}$, $k = 2, 3$, there exists a $\phi \in G$ such that $\theta_k(\phi I_k A) = (f_1, f_2)$. We shall complete the proof in several steps. In the following calculations $\phi, \psi \in G$.

Step 1. Let $\tilde{\phi} = (0, x_2^2)$, $\tilde{\psi} = (x_1^2x_2, 0)$. Then we calculate that

$$(\tilde{\phi}, \tilde{\psi}) = (x_1^2x_2^2, 0) = e_1 \in \tilde{G}.$$

By Lemma 2.2 we obtain that $e_3 \in \tilde{G}$.

Step 2. Let $\tilde{\phi} = (x_2x_1, 0)$, $\tilde{\psi} = (x_1^2x_2, 0)$. Then

$$(\tilde{\phi}, \tilde{\psi}) = ((x_1x_2)^2, 0) = e_4 \in \tilde{G},\ e_4^* = e_5 \in \tilde{G}.$$

Step 3. Let $\tilde{\phi} = (0, x_2^2)$, $\tilde{\psi} = (x_1x_2x_1, 0)$. Then

$$(\tilde{\phi}, \tilde{\psi}) = (x_1x_2^2x_1, 0),\ e_2 \in \tilde{G}.$$

Step 4. Let $\tilde{\phi} = (x_1^2, 0)$, $\tilde{\psi} = (x_2x_1x_2, 0)$. Then

$$(\tilde{\phi}, \tilde{\psi}) = (x_2x_1^2x_2 - (x_1x_2)^2 - (x_2x_1)^2, 0) = e_6 - e_4 - e_5 \in \tilde{G},\ e_6 \in \tilde{G}.$$

In this way all vectors $(f, 0) \in W^\alpha$ are in $\tilde{G}^\alpha$ and in the rest of the calculations we are not interested in the first coordinates of the elements of $\tilde{G}^\alpha$: if $(f_1, f_2) \in \tilde{G}^\alpha$, then $(0, f_2) \in \tilde{G}^\alpha$.

Step 5. If $\tilde{\phi} = (x_1x_2, 0)$, $\tilde{\psi} = (0, x_1x_2^2)$, then for some $f \in F^{(2,2)}$,

$$(\tilde{\phi}, \tilde{\psi}) = (f, x_1x_2^3) \in \tilde{G},\ e_7, e_{10} = e_7^* \in \tilde{G}.$$

Step 6. If $\tilde{\phi} = (x_2x_1, 0)$, $\tilde{\psi} = (0, x_1x_2^2)$, then for some $f \in F^{(2,2)}$,

$$(\tilde{\phi}, \tilde{\psi}) = (f, x_2x_1x_2^2) \in \tilde{G},\ e_8, e_9 = e_8^* \in \tilde{G}.$$

In this way, all $e_i \in \tilde{G}$, $i = 1, \dots, 10$. ◊

Proposition 2.4. *The group* $\operatorname{Aut} F_2(\mathbf{N}_6)$ *is generated by* GL_2 *and* δ_j, $j = 1, 2, 3$.

Proof. We repeat the arguments from the proof of Prop. 2.3. The content $\alpha = (2,2)$ is essential for the GL_2-module $W = \theta_5(I_5A/I_6A)$. The subspace of W^α, consisting of all vectors $(f, 0)$ has a basis

$$e_1=(x_1^2x_2x_1x_2,0),\quad e_2=(x_1x_2x_1^2x_2,0),\quad e_3=((x_1x_2)^2x_1,0),$$
$$e_4=(x_2x_1^3x_2,0),\quad e_5=(x_2x_1^2x_2x_1,0),\quad e_6=(x_2x_1x_2x_1^2,0),$$
$$e_7=(x_2^2x_1^3,0),\quad e_8=(x_1x_2^2x_1^2,0),\quad e_9=(x_1^2x_2^2x_1,0),$$
$$e_{10}=(x_1^3x_2^2,0).$$

The basis of the subspace of W^α consisting of all vectors $(0,f)$ has a basis

$$\{g_\sigma\circ e_k \mid k=1,\dots,10\},$$

where $\sigma=(12)\in S_2$ and $g_\sigma\in GL_2$ was defined in Remark 1.10. It is sufficient to show that $e_i\in\tilde{G}$, $i=1,\dots,10$.

Step 1. The elements $e_7,\dots,e_{10}$ are of the form $(x_1^{a_1}x_2x_2^{a_2}x_1^{a_3},0)$. Starting with $\phi,\psi\in G$ such that

$$\tilde{\phi}=\theta_k(\phi I_{k+1}A)=(0,x_2^{a_2}),\ \tilde{\psi}=\theta_l(\psi I_{l+1}A)=(x_1^{a_1}x_2x_1^{a_3},0),$$

$k=a_2$, $l=a_1+a_3+1$, we obtain

$$(\tilde{\phi},\tilde{\psi})=\theta_{k+l-1}((\phi,\psi)I_{k+l}A)=(x_1^{a_1}x_2x_2^{a_2}x_1^{a_3},0)\in\tilde{G}$$

and $e_7,\dots,e_{10}\in\tilde{G}$.

Step 2. For $\tilde{\phi}=(x_1x_2,0)$, $\tilde{\psi}=(x_1x_2x_1^2,0)$,

$$(\tilde{\phi},\tilde{\psi})=(x_1x_2^2x_1^2+(x_1x_2)^2x_1,0)=e_3+e_8\in\tilde{G},\ e_3\in\tilde{G}.$$

Step 3. For $\tilde{\phi}=(x_1^2,0)$, $\tilde{\psi}=((x_1x_2)^2,0)$,

$$(\tilde{\phi},\tilde{\psi})=(x_1x_2x_1^2x_2-(x_1x_2)^2x_1,0)=e_2-e_3\in\tilde{G}.$$

Hence $e_2\in\tilde{G}$ and by Lemma 2.2, $e_5\in\tilde{G}$.

Step 4. For $\tilde{\phi}=(x_1^2x_2,0)$, $\tilde{\psi}=(x_1x_2x_1,0)$,

$$(\tilde{\phi},\tilde{\psi})=(x_1^2x_2^2x_1-x_1^2x_2x_1x_2,0)=e_9-e_1\in\tilde{G}.$$

Hence $e_1\in\tilde{G}$ and by Lemma 2.2, $e_6\in\tilde{G}$.

Step 5. For $\tilde{\phi}=(x_1^2,0)$, $\tilde{\psi}=(x_2x_1^2x_2,0)$,

$$(\tilde{\phi},\tilde{\psi})=(2x_2x_1^3x_2-x_1x_2x_1^2x_2-x_2x_1^2x_2x_1,0)=2e_4-e_2-e_5\in\tilde{G},\ e_4\in\tilde{G}.$$

In this way $\tilde{G}^\alpha=W^\alpha$ and this gives that $\operatorname{Aut}F=G=\langle GL_2,\delta_1,\delta_2,\delta_3\rangle$ and this completes the proof. ◊

Proposition 2.5. *The group* $\operatorname{Aut}F_2(\mathbf{N}_7)$ *is generated by* GL_2 *and* δ_j, $j=1,2,3$.

Proof. Again, we repeat the arguments from the proof of Prop. 2.3. The content $\alpha=(3,2)$ is essential for the GL_2-module $W=\theta_6(I_6A/$ $/I_7A)$. The subspace of W^α consisting of all vectors $(f,0)$ has a basis

$$e_1=(x_1^3x_2x_1x_2,0),\quad e_2=(x_1^2x_2x_1^2x_2,0),\quad e_3=(x_1^2(x_2x_1)^2,0),$$
$$e_4=(x_1x_2x_1^2x_2x_1,0),\quad e_5=(x_1x_2x_1^3x_2,0),\quad e_6=((x_1x_2)^2x_1^2,0),$$
$$e_7=(x_2x_1^4x_2,0),\quad e_8=(x_2x_1^3x_2x_1,0),\quad e_9=(x_2x_1^2x_2x_1^2,0),$$
$$e_{10}=(x_2x_1x_2x_1^3,0),\quad e_{11}=(x_2^2x_1^4,0),\quad e_{12}=(x_1x_2^2x_1^3,0),$$
$$e_{13}=(x_1^2x_2^2x_1^2,0),\quad e_{14}=(x_1^3x_2^2x_1,0),\quad e_{15}=(x_1^4x_2^2,0).$$

First we shall prove that all $(f,0)\in W^\alpha$ belong to $\tilde{G}$. The elements $e_{11},\dots,e_{15}$ are of the form $(x_1^{a_1}x_2x_2^{a_2}x_1^{a_3},0)$. As in the first step of the proof of Prop. 2.4 these basis elements are in $\tilde{G}$. The next calculations show that all elements $(f,0)\in W^\alpha$ are also in $\tilde{G}^\alpha$.

For $\phi,\psi\in G$ such that $\tilde{\phi}=(x_1x_2x_1,0)$, $\tilde{\psi}=(x_1^2x_2x_1,0)$ we obtain that

$$(\tilde{\phi},\tilde{\psi})=(x_1^2x_2x_1x_2x_1,0)=e_3\in\tilde{G}.$$

Hence, by Lemma 2.2, $e_6\in\tilde{G}$. Similarly, we choose properly $\phi,\psi\in G$ and compute $(\tilde{\phi},\tilde{\psi})$.

$\tilde{\phi}=(x_1^2x_2,0)$, $\tilde{\psi}=(x_1^3x_2,0)$, $(\tilde{\phi},\tilde{\psi})=(x_1^2x_2x_1^2x_2,0)=e_2\in\tilde{G}$; $e_9\in\tilde{G}$.

$\tilde{\phi}=(x_1^2,0)$, $\tilde{\psi}=(x_1^2x_2x_1x_2,0)$, $(\tilde{\phi},\tilde{\psi})=e_1+e_2-e_3\in\tilde{G}$; $e_1,e_{10}\in\tilde{G}$.

$\tilde{\phi}=(x_1x_2,0)$, $\tilde{\psi}=(x_1^3x_2x_1,0)$, $(\tilde{\phi},\tilde{\psi})=e_3+e_4+e_{14}\in\tilde{G}$; $e_4\in\tilde{G}$.

$\tilde{\phi}=(x_1^2,0)$, $\tilde{\psi}=(x_1x_2x_1^2x_2,0)$, $(\tilde{\phi},\tilde{\psi})=-e_4+2e_5\in\tilde{G}$; $e_5,e_8\in\tilde{G}$.

$\tilde{\phi}=(x_1^2,0)$, $\tilde{\psi}=(x_2x_1^3x_2,0)$, $(\tilde{\phi},\tilde{\psi})=-e_5+3e_7-e_8\in\tilde{G}$; $e_7\in\tilde{G}$.

Now we shall establish that all $(f,0)\in W^\beta$ are also in $\tilde{G}^\beta$ for $\beta=(2,3)$. Then, by Remark 1.10 for $\sigma=(12)\in S_2$, this will imply that all $(0,f)\in W^\alpha$ are in $\tilde{G}^\alpha$ and this will complete the proof. All the polynomials f for the vector space $\{(f,0)\in W^\beta\}$ are homogeneous and $\deg_{x_1}f=\deg_{x_2}f=3$. Hence this vector space is of dimension $\binom{6}{3}=20$. It has a basis $e_i=(f_i,0)$, $i=1,\dots,20$, where

$$f_1=x_1^3x_2^3,\quad f_2=x_1^2x_2x_1x_2^2,\quad f_3=x_1^2x_2^2x_1x_2,\quad f_4=x_1^2x_2^3x_1,$$
$$f_5=x_1x_2x_1^2x_2^2,\quad f_6=(x_1x_2)^3,\quad f_7=x_1x_2x_1x_2^2x_1,\quad f_8=x_1x_2^2x_1^2x_2,$$
$$f_9=f_7^*,\quad f_{10}=f_4^*,\quad f_{11}=x_2x_1^3x_2^2,\quad f_{12}=x_2x_1^2x_2x_1x_2,$$
$$f_{13}=f_8^*,\quad f_{14}=f_{12}^*,\quad f_{15}=f_6^*,\quad f_{16}=f_3^*,$$
$$f_{17}=f_{11}^*,\quad f_{18}=f_5^*,\quad f_{19}=f_2^*,\quad f_{20}=f_1^*,$$

where the involution $*$ is defined above Lemma 2.2. Four of the basis elements, e_1, e_4, e_{10}, e_{20} belong to $\tilde{G}$ because they are of the form $(x_1^{a_1}x_2x_2^{a_2}x_1^{a_3}, 0)$. For the other elements we work with $\phi, \psi \in G$ and compute $(\tilde{\phi}, \tilde{\psi})$:

$$\tilde{\phi}=(x_1x_2,0),\ \tilde{\psi}=(x_1x_2^2x_1^2,0),\quad (\tilde{\phi},\tilde{\psi})=e_9+e_{10}\in\tilde{G};\ e_9,e_7\in\tilde{G}.$$
$$\tilde{\phi}=(x_1^2,0),\ \tilde{\psi}=(x_1x_2^2x_1x_2,0),\quad (\tilde{\phi},\tilde{\psi})=e_8-e_9\in\tilde{G};\ e_8,e_{13}\in\tilde{G}.$$
$$\tilde{\phi}=(x_1^2,0),\ \tilde{\psi}=(x_1x_2x_1x_2^2,0),\quad (\tilde{\phi},\tilde{\psi})=e_5-e_7\in\tilde{G};\ e_5,e_{18}\in\tilde{G}.$$
$$\tilde{\phi}=(x_2x_1,0),\ \tilde{\psi}=(x_1^3x_2^2,0),\quad (\tilde{\phi},\tilde{\psi})=e_2+e_5\in\tilde{G};\ e_2,e_{19}\in\tilde{G}.$$
$$\tilde{\phi}=(x_1x_2,0),\ \tilde{\psi}=(x_1x_2x_1^2x_2,0),\ (\tilde{\phi},\tilde{\psi})=e_6+e_8\in\tilde{G};\ e_6,e_{15}\in\tilde{G}.$$
$$\tilde{\phi}=(x_1x_2,0),\ \tilde{\psi}=(x_1^2x_2x_1x_2,0),\ (\tilde{\phi},\tilde{\psi})=e_3+e_6\in\tilde{G};\ e_3,e_{16}\in\tilde{G}.$$
$$\tilde{\phi}=(x_1^2,0),\ \tilde{\psi}=(x_2x_1^2x_2^2,0),\quad (\tilde{\phi},\tilde{\psi})=-e_5+2e_{11}+e_{13}\in\tilde{G};$$
$$e_{11},e_{17}\in\tilde{G}.$$
$$\tilde{\phi}=(x_1^2x_2x_1,0),\ \tilde{\psi}=(x_2x_1x_2,0),\ (\tilde{\phi},\tilde{\psi})=-e_7+e_{12}-e_{15}\in\tilde{G};$$
$$e_{12},e_{14}\in\tilde{G}.$$

In this way all $(f,0)\in W^\beta$ are also in $\tilde{G}^\beta$. $\Diamond$

3. Nilpotent algebras with more than two generators

In this section we prove our theorem in the general case. We fix positive integers m and c satisfying the condition $m(m-4)\geq c-2$. Since it is sufficient to consider the case $c>3$, this implies that $m\geq$ ≥ 5. As in Section 2 we consider the automorphisms $\delta_1,\delta_2,\delta_3$ of the algebra $F=F_m(\mathbf{N}_c)$ and denote by $G=\langle GL_2,\delta_1,\delta_2,\delta_3\rangle$ the subgroup of $\operatorname{Aut}F$ generated by GL_m and δ_j, $j=1,2,3$. We also use the other conventions from Section 2. For example, if $\phi\in G\cap I_kA$ and $\psi\in G\cap$ $\cap I_lA$, and the values of k and l are clear from the context, we denote $\theta_k(\phi I_{k+1}A)$ and $\theta_l(\phi I_{l+1}A)$, respectively by $\tilde{\phi}$ and $\tilde{\psi}$ and by $(\tilde{\phi},\tilde{\psi})$ the image $\theta_{k+l-1}((\phi,\psi)I_{k+l}A$ of the group commutator (ϕ,ψ). The main result of Section 3 is the following.

Theorem 3.1. *For $m(m-4)\geq c-2$*
$$\operatorname{Aut}F_m(\mathbf{N}_c)=G=\langle GL_m,\delta_1,\delta_2,\delta_3\rangle.$$

We shall prove this theorem by induction on c. The base of the induction $c=3$ is considered in Section 1 and we assume that the statement of Th. 3.1 holds for all algebras $F_m(\mathbf{N}_k)$, $k < c$. Using Cor. 1.5, we shall establish the case $k = c$. By Cor. 1.9, it is sufficient to show that the homogeneous component of degree $\alpha = (p + \varepsilon_1, \ldots , p + \varepsilon_m)$ of the GL_m-module $W = (F^{(c-1)})^{\oplus m}$ coincides with the homogeneous component of $\tilde{G} = \theta_{c-1}(G \cap I_{c-1}A)/(G \cap I_cA)$, where $c - 2 = pm + r$, $\leq r < m$ and $\varepsilon_1 = \ldots = \varepsilon_r = 1$, $\varepsilon_{r+1} = \ldots = \varepsilon_m = 0$. Till the end of the section we fix α and the integers p and r as above. Finally, let $f \in F^{(k)}$, $k \leq 6$, and let f depend on x_1, x_2 only. By the results of Section 2, the automorphism ϕ of $F_2(\mathbf{N}_{k+1})$ defined by

$$\phi(x_1) = x_1 + f,\ \phi(x_2) = x_2,$$

belongs to the group generated by GL_2 and $\delta_1, \delta_2, \delta_3$. If we consider

$$\langle GL_2, \delta_1, \delta_2, \delta_3\rangle \subseteq \operatorname{Aut} F_2(\mathbf{N}_{k+1})$$

canonically embedded into $\operatorname{Aut} F_m(\mathbf{N}_{k+1})$ (fixing the other variables $x_3, \ldots , x_m$) we obtain that $\phi \in G \subseteq \operatorname{Aut} F_m(\mathbf{N}_{k+1})$.

We start the proof of Th. 3.1 with several lemmas.

Lemma 3.2. *Let $f = v_1uv_2$ be a monomial from $F^{(c-1)}$ such that $\deg u \geq 2$, $\deg v_1 + \deg v_2 \geq 1$, u does not contain the variable x_1 and the subwords v_1 and v_2 of the word f do not contain x_k for some $k > 1$. Then $(f, 0, \ldots , 0) \in \tilde{G}$.*

Proof. By inductive arguments, there exist automorphisms $\phi, \psi \in G$ such that

$$\tilde{\phi} = (0, \ldots , 0, u, 0, \ldots , 0),\ \tilde{\psi} = (v_1x_kv_2, 0, \ldots , 0),$$

(u is the k-th coordinate of $\tilde{\phi}$). Then by Lemma 1.12 we compute that

$$(\tilde{\phi}, \tilde{\psi}) = (f, 0, \ldots , 0) \in \tilde{G}. \qquad \diamondsuit$$

Lemma 3.3. *Let $f = f(x_1, \ldots , x_m) \in F^{(c-1)}$ be a monomial of degree λ_i with respect to x_i, where $\lambda_i \geq 0$ and $|\lambda_i - \lambda_j| \leq 2$, $1 \leq i, j \leq m$. If $\lambda_m = 1$ and $\lambda_t = 0$ for some $t < m$, then $(f, 0, \ldots , 0) \in \tilde{G}$.*

Proof. If $f = f(x_2, \ldots , x_m)$, then the statement is true by Lemma 3.2. If $f = f(x_1, x_m)$, then $\deg f \leq 4$ and we may work in $\operatorname{Aut} F_2(\mathbf{N}_c)$ for $c \leq 5$ applying the results of Section 2. Therefore, without loss of generality we may assume that f depends on x_2 and does not depend on x_{m-1}. If f contains a subword $u = x_lx_m$ or $u = x_mx_l$, $l > 1$, we apply Lemma 3.2 for $k = m$. Hence the only cases to consider are $f = v_1x_mx_1v_2$ and $f = v_1x_1x_mv_2$. Let $f = v_1x_mx_1v_2$ and let

$$\tilde{\phi} = (0, \ldots , 0, x_mx_1, 0),\ \tilde{\psi} = (v_1x_{m-1}v_2, 0, \ldots , 0).$$

We obtain

$$(\tilde{\phi}, \tilde{\psi}) = (f, 0, \ldots , 0) - b_{m-1} \in \tilde{G},\ b_{m-1} = (0, \ldots , 0, x_mv_1x_{m-1}v_2, 0).$$

For $\sigma = (1, m-1)$, we define the linear automorphism g_σ and obtain

$$g_\sigma \circ b_{m-1} = b_1 = (x_m \bar{v}_1 x_1 \bar{v}_2, 0, \dots, 0),$$

where x_1 does not participate in $\bar{v}_1$ and $\bar{v}_2$. Applying Lemma 3.2, it is sufficient to consider the case $b_1 = (x_m x_1 x_2, 0, \dots, 0)$. Let

$$\tilde{\phi} = (0, \dots, 0, x_m x_1),\ \tilde{\psi} = (x_m x_2, 0, \dots, 0).$$

Direct calculations show that

$$(\tilde{\phi}, \tilde{\psi}) = b_1 + (0, \dots, 0, x_m^2 x_2) \in \tilde{G},$$

$$(0, \dots, 0, x_m^2 x_2) = g_\sigma \circ \tilde{\delta}_3 \in \tilde{G},\ \sigma = (1m) \in S_m,$$

and $(f, 0, \dots, 0) \in \tilde{G}$. The case $f = v_1 x_1 x_m v_2$ follows applying Lemma 2.2 to f^*. $\Diamond$

Now we generalize Lemma 3.3 in the following way.

Lemma 3.4. *Let $f = f(x_1, \dots, x_m) \in F^{(c-1)}$ be a monomial of degree λ_i with respect to x_i, where $\lambda_i \geq 0$ and $|\lambda_i - \lambda_j| \leq 2$, $1 \leq i, j \leq m$. If $\lambda_k = 1$, $k \neq 1$, then $(f, 0, \dots, 0) \in \tilde{G}$.*

Proof. Without loss of generality we may assume that $k = m$. Clearly, λ_1 may be equal to 0, 1, 2 or 3 and, by Lemma 3.3 it is sufficient to consider the case when $\lambda_i > 0$ for all $i = 1, \dots, m$. We consider several cases.

Case 1. The monomial f is of the form $f = v_1 x_m x_l v_2$ or $f = v_1 x_l x_m v_2$ and $l \neq 1$. Then we apply Lemma 3.2.

Case 2. f is of the form $f = u_1 x_1^\gamma u_2$, where u_1 and u_2 do not depend on x_1. Excluding the monomials covered by Case 1 we may consider the symmetric cases $f = x_m x_1^\gamma u_2$ and $f = u_1 x_1^\gamma x_m$. Since $m \geq 5$, we apply twice Lemma 3.2 in order to obtain consequently

$$(x_m x_1^\gamma u_2, 0, \dots, 0), (u_1 x_1^\gamma x_m, 0, \dots, 0) \in \tilde{G},\ (f, 0, \dots, 0) \in \tilde{G}.$$

Case 3. Let $f = u_1 x_1^{\gamma_1} u_2 x_1^{\gamma_2} u_3$, where u_1, u_2 and u_3 do not contain x_1. Considering the monomials not covered by the previous cases we assume that x_m coincides with one of u_1, u_2, u_3. For example, let $f = u_1 x_1^{\gamma_1} x_m x_1^{\gamma_2} u_3$, the other cases are analogous. Since $m \geq 5$ and f depends on all the variables, $\deg u_1 + \deg u_3 \geq 3$. Let some of the variables x_2 or x_3 participate in one of the monomials u_1 and u_3 only, for example x_2 is in u_1. If $u_1 = x_2$, then we apply Lemma 3.2 for x_3 and u_3, because $\deg u_3 > 1$. If $\deg u_1 > 1$, then we apply Lemma 3.2 for x_2 and u_1. Now, let both x_2 and x_3 participate in u_1 and u_3. Let $u_1 = w_1 x_{i_1} w_2$, $i_1 \in \{2, 3\}$ and let w_1 do not contain x_2, x_3. We consider

$$\tilde{\phi} = (0, x_2w_2, 0, \dots, 0), \quad \text{if} \quad i_1 = 2, \quad \text{and}$$
$$\tilde{\phi} = (0, 0, x_3w_2, 0, \dots, 0), \quad \text{if} \quad i_1 = 3,$$
$$\tilde{\psi} = (w_1x_{i_1}x_1^{\gamma_1}x_mx_1^{\gamma_2}u_3, 0, \dots, 0),$$
$$(\tilde{\phi}, \tilde{\psi}) = (f, 0, \dots, 0) + (w_1x_{i_1}x_1^{\gamma_1}x_mx_1^{\gamma_2}\bar{u}_3, 0, \dots, 0) \in \tilde{G},$$

where $\bar{u}_3 = \Delta u_3$ and Δ is the derivation of F defined by $\Delta x_{i_1} = x_{i_1}w_2$, $\Delta x_j = 0$, $j \neq i_1$. The variable $x_{i_2} = x_2, x_3$, $x_{i_2} \neq x_{i_1}$, participates in $\bar{u}_3$ only and we apply Lemma 3.2. In this way, $(f, 0, \dots, 0) \in \tilde{G}$.

Case 4. Let $f = u_1x_1u_2x_1u_3x_1u_4$, where x_1 does not participate in the monomials u_i, $i = 1, 2, 3, 4$. Since the other possibilities are covered by the previous considerations, it is sufficient to assume that x_m coincides with some of the monomials u_i and the monomials u_2, u_3 are different from 1. We have to handle two subcases:

(i) $u_1 = x_m$ (or, by symmetry, $u_4 = x_m$), i.e. $f = x_mx_1u_2x_1u_3x_1u_4$. Let

$$\tilde{\phi} = (0, \dots, 0, x_mx_1), \ \tilde{\psi} = (x_mu_2x_1u_3x_1u_4, 0, \dots, 0).$$

We calculate that

$$(\tilde{\phi}, \tilde{\psi}) = (f, 0, \dots, 0) - b_m \in \tilde{G}, \ b_m = (0, \dots, 0, x_m^2x_1u_2x_1u_3x_1u_4).$$

By conjugation of b_m by g_σ, $\sigma = (1m)$ we obtain

$$b_1 = g_\sigma \circ g_m = (x_1^2x_m\bar{u}_2x_m\bar{u}_3x_m\bar{u}_4, 0, \dots, 0)$$

which belongs to $\tilde{G}$ by Lemma 3.2.

(ii) $u_2 = x_m$ (or $u_3 = x_m$), i.e. $f = u_1x_1x_mx_1u_3x_1u_4$, x_1 does not participate in the monomials u_i and $u_3 \neq 1$. For

$$\tilde{\phi} = (x_1x_m, 0, \dots, 0), \ \tilde{\psi} = (u_1x_1^2u_3x_1u_4, 0, \dots, 0)$$

we obtain

$$(\tilde{\phi}, \tilde{\psi}) = (f, 0, \dots, 0) + v \in \tilde{G},$$

where v is a sum of elements of W considered in Case 3. This completes all the possible cases and gives the proof of the lemma. $\Diamond$

Lemma 3.5. *Let $f = f(x_1, \dots, x_m) \in F^{(c-1)}$ be a monomial of degree λ_i with respect to x_i, where $\lambda_i \geq 0$ and $|\lambda_i - \lambda_j| \leq 2$, $1 \leq i, j \leq m$ and f be presented in the form*

$$f = w_0x_1^{\gamma_1}w_1x_1^{\gamma_2} \dots x_1^{\gamma_s}w_s,$$

where $\gamma_i > 0$, $i = 1, \dots, s$, w_i, $i = 0, 1, \dots, s$, are monomials which do not depend on x_1 and $\gamma_i > 0$, $i = 1, \dots, s$. Let there exist two sets with the same number of elements

$$M_1 = \{x_{i_1}, \dots , x_{i_t}\},\ M_2 = \{j_1, \dots , j_t\}$$

with the following property: In the above presentation of f the variables $x_{i_k} \in M_1$ participate only in the monomials w_{j_l} with indices j_l from the set M_2. Then $(f, 0, \dots , 0) \in \tilde{G}$.

Proof. We shall prove the statement by induction of the number t of the elements of M_1 and M_2. If $t = 1$, then there exists a variable x_k, $k \neq 1$, which participate in one monomial w_i only. Then $f = v_1 w_i v_2$, where v_1, v_2 do not depend on x_k. If $\lambda_k = \deg_{x_k} f = 1$, we apply Lemma 3.4. If $\lambda_k > 1$, since $\deg v_1 + \deg v_2 \geq 1$ and $\deg w_j \geq 2$, we apply Lemma 3.2. In this way we complete the case $t = 1$. Now we assume that the statement is true for sets M_1 and M_2 with less than t elements and shall prove it for $|M_1| = |M_2| = t$. Without loss of generality we may assume that $M_1 = \{x_2, \dots , x_{t+1}\}$. Let j_1 be the smallest index in M_2 with the property that w_{j_1} does depend on some x_k from M_1. Let, for example, $k = 2$. If $w_{j_1} = u_1 x_2 u_2$ and u_1, u_2 do not involve variables from M_1, then the other variables $x_3, \dots , x_{t+1}$ participate in the monomials $w_{j_2}, \dots , w_{j_t}$ and we apply the inductive arguments for $M_1 = \{x_3, \dots , x_{t+1}\}$ and $M_2 = \{j_2, \dots , j_t\}$. Let $w_{j_1} = u_1 x_2 u_2 x_l u_3$, where $x_l \in M_1$ and u_1, u_3 do not depend on elements from M_1. We consider

$\tilde{\phi} = (0, x_2 u_2 x_l, 0, \dots , 0)$, $\tilde{\psi} = (w_0 \dots x_1^{\gamma_{j_1}} u_1 x_2 u_3 x_1^{\gamma_{j_1+1}} \dots w_s, 0, \dots , 0)$

and calculate

$$(\tilde{\phi}, \tilde{\psi}) = (f, 0, \dots , 0) + (g, 0, \dots , 0) \in \tilde{G},$$

$$g = w_0 \dots x_1^{\gamma_{j_1}} u_1 x_2 u_3 \Delta(x_1^{\gamma_{j_1+1}} \dots w_s),$$

where Δ is the derivation of F defined by

$$\Delta x_2 = x_2 u_2 x_l,\ \Delta x_i = 0, i \neq 2.$$

If we write g as a linear combination of monomials we obtain that each summand is of the form

$$g_i = z_0 x_1^{\mu_1} z_1 x_1^{\mu_2} \dots z_{s-1} x_1^{\mu_s} z_s,$$

and the variables $x_3, \dots , x_{t+1}$ participate in $z_{j_2}, \dots , z_{j_t}$ only. Again, we apply the inductive assumption and obtain that $(g_i, 0, \dots , 0) \in \tilde{G}$. Hence $(f, 0, \dots , 0)$ also belongs to $\tilde{G}$. $\Diamond$

Now we are ready to prove Th. 3.1 and, in this way to complete the proof of the main result of our paper.

Proof of Theorem 3.1. It is sufficient to establish that the homogeneous component W^α is contained in $\tilde{G}$, where $\alpha = (p+\varepsilon_1, \dots , p+\varepsilon_m)$, $c - 2 = pm + r$, $0 \leq r < m$ and $\varepsilon_1 = \dots = \varepsilon_r = 1$, $\varepsilon_{r+1} = \dots = \varepsilon_m = 0$. First we consider the elements $(f, 0, \dots , 0) \in W^\alpha$. As in Ex. 1.2, $\deg_{x_1} f = p + 1$, if $r = 0$ and $\deg_{x_1} f = p + 2$, if $r > 0$. Let

$$f = w_0 x_1^{\gamma_1} w_1 \dots x_1^{\gamma_s} w_s,$$

where the w_i's do not depend on x_1. The condition $m(m-4) \geq c - 2$ gives that $p \leq m-4$, i.e. $p+3 \leq m-1$. The number of the words w_i does not exceed $p+3 \leq m-1$. Applying Lemma 3.5 for $M_1 = \{x_2, \dots, x_m\}$, we obtain that $(f, 0, \dots, 0) \in \tilde{G}$. Now, let $b_k = (0, \dots, 0, g, 0, \dots, 0) \in W^\alpha$, where g is the k-th coordinate of b_k, $k > 1$. Clearly, $\deg_{x_k} g = \alpha_k + 1$. Conjugating b_k with g_σ for $\sigma = (1k)$, we obtain that $b_1 = g_\sigma \circ b_k$ is of the form $(f, 0, \dots, 0)$. If $\varepsilon_1 = \varepsilon_k$, then $b_1 \in W^\alpha$. We have already shown that $b_1 \in \tilde{G}$, and hence b_k also belongs to $\tilde{G}$. If $\varepsilon_1 = 1$, $\varepsilon_k = 0$, then the monomial f which is the first coordinate of b_1 is of degree $\alpha_k + 1 = p + 1$ with respect to x_1. Again we can apply Lemma 3.5 and obtain that $b_1 \in \tilde{G}$. This shows that $W^\alpha \subset \tilde{G}$ and completes the proof of the theorem. $\Diamond$

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