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EXTREMAL DISCONNECTEDNESS MODULO DUAL FILTRATIONS

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Abstract: The aim of this paper is to continue the study of general topological properties via ideals. Extremal disconnectedness is considered from a more global point of view. Some already existing concepts are unified utilizing the notion of a topological ideal.

1. Extremally disconnected spaces: an introduction

In this preliminary section we make an attempt to cover the recent progress in the study of extremally disconnected spaces.

A topology τ on a set X is *extremally disconnected* (= ED) [47] if the τ-closure of every member of τ is also in τ. Extremally disconnected spaces exist in profusion: all hyperconnected, i.e. irreducible

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spaces and all partition spaces are extremally disconnected. Herrman [28, 29] showed that every S-closed weakly Hausdorff and every S-closed almost regular space is extremally disconnected. In 1978, Cameron [4] proved that every maximally S-closed space is extremally disconnected. In 1980, Noiri [38] showed that if a space X is locally S-closed and either weakly Hausdorff or almost regular, then X is extremally disconnected. Recently [16] it was shown that every locally countably S-closed km-perfect space is extremally disconnected.

In [25], Ganster and Andrijević proved that $SPO(X,\tau)=PO(X,\tau)$ if and only if (X,τ) is extremally disconnected, i.e. if and only if every set which is dense in some regular closed subspace is included in the interior of its closure. Several characterizations of extremally disconnected space were given by Noiri in [39].

In [37], Nagura considered extremally disconnected spaces in connection with the concept extension property. A property $\mathcal{P}$ is said to be an extension property if it is closed-hereditary and productive and each $\mathcal{P}$-regular space has a $\mathcal{P}$-regular compactification (see [37] for details. An extension theorem for extremally disconnected L-spaces is given by Kubiak in [34]. Long, Herrington and Janković [36], proved that if an invertible space X has a nonempty open subset U which is as a subspace extremally disconnected, then so is X.

By saying that a point x of a topological space X is discretely untouchable, one means that x cannot be obtained as a cluster point of a countable discrete subset of X. Having in view the conjecture that extremally disconnected compact spaces always have discretely untouchable points, Simon [44] proved that if additionally X is CCC, has the weight continuum and $\mathrm{cf}(g(CO(X))) > \omega$, then the conclusion of the conjecture holds. Here, $CO(X)$ stands for the Boolean algebra of all clopen subsets of X and $g(B)$ means the minimal cardinality of those C, $C \subset B$, for which B is the completion. Recently extremally disconnected spaces were studied by Dow and Vermeer [21] and by Błaszczyk and Vermeer [1]. Contrasequentialness, a weaker form of extremal disconnectedness in the class of Hausdorff spaces, was recently studied by Dow and Vaughan in [20].

There exists a way of generating extremally disconnected topologies from Hausdorff spaces, namely for every Hausdorff space (X,τ), there exists an associated topological space EX, called *the Iliadis absolute* of X, which is unique (up to homeomorphism) with respect to being ED (see [8, 41] for details). In [8], Császár derived some fundamental properties of the spaces EX such as: the full subcategory of θ-TOP whose objects are the regular extremally disconnected spaces is coreflective. For a systematic study of various ramifications of the construction of an Iliadis absolute, beyond the class of regular spaces, the reader may check another paper of Császár [9].

Extremally disconnectedness has been recently under consideration in connection with the space $\mathbf{Seq}(\xi)$ [6, 18, 32, 35]. Let $\mathbf{Seq}$ denote the set of all finite sequences on ω along with the empty sequence. If ξ is a free ultrafilter on ω, then $\mathbf{Seq}(\xi)$ denotes the topological space whose underlying set is $\mathbf{Seq}$ and whose topology is defined in the following way: U in $\mathbf{Seq}$ is open if and only if for every $s \in U$, the set $\{n \in \omega: s \frown n \in U\}$ is a member of ξ. In the definition of the topology on $\mathbf{Seq}$, $s \frown n$ denotes the concatenation of the two sequences s and $\{(1, n)\}$. Not only the space $\mathbf{Seq}(\xi)$ is extremally disconnected [18] but also the orbit of every point of the Stone-Čech compactification of $\mathbf{Seq}(\xi)$ is a homogeneous extremally disconnected countably compact space [6, 32]. Other recent contributions to the theory of extremally disconnected spaces are [2, 10, 11, 19, 26, 49, 50, 51].

A *topological ideal* is a nonempty collection of subsets of a topological space (X, τ), which is closed under heredity and finite additivity. Proper ideals are called *dual filters*. Throughout this paper we will be interested only in such ideals. Except the trivial ideals, the following collections of sets form important ideals on any topological space (X, τ): the finite sets $\mathcal{F}$, the countable sets $\mathcal{C}$, the closed and discrete sets $\mathcal{CD}$, the nowhere dense sets $\mathcal{N}$, the meager sets $\mathcal{M}$, the scattered sets $\mathcal{S}$ (only when X is T_0 [17]), the bounded sets $\mathcal{B}$, the relatively compact sets $\mathcal{R}$, the S-bounded sets $\mathcal{SB}$ [33] and the Lebesgue null sets $\mathcal{L}$.

2. $\mathcal{I}$-ED-spaces

Definition 1. A topological space $(X, \tau, \mathcal{I})$ is called an *$\mathcal{I}$-extremally disconnected space* (= $\mathcal{I}$-ED-space) if for each regular open set $R \subseteq X$, $\text{bd}(R) \in \mathcal{I}$.

Clearly, whatever ideal $\mathcal{I}$ we set on a topological space (X, τ), the extremal disconnectedness of (X, τ) always implies the validity of the newly defined condition $\mathcal{I}$-ED on the space $(X, \tau, \mathcal{I})$, since a space X is ED if and only if regular open sets have empty boundaries. Moreover, every space is $\mathcal{N}$-ED and hence also $\mathcal{M}$-ED, since $\mathcal{N}$ is contained in $\mathcal{M}$ and boundaries of regular open sets are nowhere dense.

Proposition 2.1. *If $\mathcal{I}$ and $\mathcal{J}$ are ideals with $\mathcal{I}$ contained in $\mathcal{J}$, then X is $\mathcal{I}$-ED implies that X is $\mathcal{J}$-ED.* $\Diamond$

Throughout this paper, by an ideal we will always mean a dual filter, since the fundamental definition of $\mathcal{I}$-ED-spaces is always valid when $\mathcal{I}$ is the maximal ideal. The following proposition shows that under different settings on $\mathcal{I}$, we obtain from the definition of $\mathcal{I}$-ED-spaces some already known classes of topological spaces.

Recall first that a topological space (X, τ) is called *almost extremally disconnected* [12] (resp. *ω_1-extremally disconnected* [15]) if the boundary of every regular open set is finite (resp. countable). By $\mathcal{RB}$, we denote the ideal of all R-bounded sets. A set A in a space X is *R-bounded* (cf. [33]) if every cover of X by regular semi-open sets in X has a finite subfamily covering A. A set A is called *regular semi-open* [5] if A lies between a regular open set and its closure.

Proposition 2.2. *Let (X, τ) be a topological space. Then:*

(i) *X is $\{\emptyset\}$-ED if and only if X is extremally disconnected.*
(ii) *X is $\mathcal{F}$-ED if and only if X is $\mathcal{RB}$-ED if and only if X is almost extremally disconnected.*
(iii) *X is $\mathcal{C}$-ED if and only if X is ω_1-extremally disconnected.* $\Diamond$

The *ideal Cantor-Bendixson derivative* $ID(X)$ of a topological space $(X, \tau, \mathcal{I})$ is the set of all non-isolated non-ideal points of X, where a point x is said to be an *ideal point* if $\{x\} \in \mathcal{I}$. The set of all isolated points of X will be denoted as usual by $I(X)$. Recall that a topological space (X, τ) is called *Bolzano-Weierstrass (compact)* [3] if every (countably) infinite subset of X has at least one cluster point or equivalently if every discrete and closed subset of X is finite. A point $x \in A \subseteq$ $\subseteq (X, \tau, \mathcal{I})$ is called *K-isolated* in A if there exists $U \in \tau$ such that the set $U \cap (X \setminus (ID(X) \cup I(X))$ is compact and $U \cap A = \{x\}$. A set A is called *K-dense-in-itself* if A has no K-isolated points. Note that every dense-in-itself set is K-dense-in-itself but not vice versa as the set of all integers in the space $(\mathbb{R}, \tau, \mathcal{N})$ shows, where $(\mathbb{R}, \tau)$ is the real line with the usual topology. In the case of the minimal ideal the two concepts coincide. By a T_3-space, we mean a regular T_1-space.

Theorem 2.3. *The ideal Cantor-Bendixson derivative of every Bolzano-Weierstrass, $\mathcal{I}$-extremally disconnected, T_3-space $(X, \tau, \mathcal{I})$ is K-dense-in-itself.*

Proof. Assume that $ID(X)$ is not K-dense-in-itself, i.e., let $U \in \tau$ and $x \in ID(X)$ such that $U \cap ID(X) = \{x\}$ and $K = U \cap (X \setminus (ID(X) \cup \cup I(X))$ is compact. Set $U' = U \setminus K$. Since X is Hausdorff and K is compact, then K is closed in X and thus U' is an open neighborhood of x. Since X is regular, then there exist a regular open set V such that $x \in V \subseteq \mathrm{cl}(V) \subseteq U'$. Since every point of the boundary of V must be isolated in X, then $\mathrm{bd}(V)$ is empty and thus V is clopen. Note that since $x \in D(X)$, $D(X)$ being the (usual) Cantor-Bendixson derivative of X, and since X is a T_1-space, then every neighborhood of x is infinite, in particular, V is infinite too. Let $V = A \cup B$, where both A and B are infinite and disjoint. We can assume that $x \in B$. If $x \notin \mathrm{cl}(A)$, then A must be closed. Since $A \subseteq I(X)$, so A is discrete, and since X is Bolzano-Weierstrass, then by contradiction $x \in \mathrm{cl}(A)$. Set $W = = \mathrm{cl}(A) = A \cup \{x\}$. If $x \notin \mathrm{int}(W)$, then $\{x\} = \mathrm{bd}(\mathrm{int}(W)) \in \mathcal{I}$, since $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-ED. By contradiction ($\{x\} \notin \mathcal{I}$), $x \in \mathrm{int}(W)$, and thus W is open and hence clopen. Now, since finite intersection of clopen sets is clopen, $B \setminus \{x\} = V \cap (X \setminus W)$ is clopen and since it is discrete by being a subset of $I(X)$, then we have been able to construct an infinite closed and discrete subset of the Bolzano-Weierstrass space X. By contradiction, x is not a K-isolated point of $ID(X)$, i.e. the ideal Cantor-Bendixson derivative of $(X, \tau, \mathcal{I})$ is K-dense-in-itself. $\Diamond$

Corollary 2.4.

(i) [42, Semadeni] *The Cantor-Bendixson derivative of every countably compact, extremally disconnected, T_3-space is dense-in-itself.*

(ii) *Every Bolzano-Weierstrass, scattered, extremally disconnected T_3-space (X, τ) is discrete and hence finite.* $\Diamond$

Recall that a topological space (X, τ) is called *nearly compact* [45] if every cover of X by regular open sets has a finite subcover. A topological space (X, τ) is called *quasi-H-closed* or *almost compact* [40] if every open cover of X has a finite proximate subcover, i.e. finite subfamily the closures of whose members cover X.

Proposition 2.5. *Every almost compact $\mathcal{F}$-ED space is nearly compact.*

Proof. Let X be almost compact. Then for every open cover there is a finite subcollection $\mathcal{V}$ whose union is dense. If X is also $\mathcal{F}$-ED, then $\mathrm{cl}(V) \setminus \mathrm{int}(\mathrm{cl}(V))$ is finite for each V in the subcollection $\mathcal{V}$. Thus, $X \setminus \cup_{\mathcal{V}}(\mathrm{int}(\mathrm{cl}(V))$ is finite. But this finite set requires only finitely many more open sets from the original open cover to be covered. Therefore, the open cover has a finite subcollection such that the union of the int(cl(V)) from this subcollection covers X. $\Diamond$

Corollary 2.6. *Every H-closed $\mathcal{F}$-ED-space is nearly compact.* $\Diamond$

Corollary 2.7. *Every minimal Hausdorff $\mathcal{F}$-ED-space is compact.*

Proof. If X is minimal Hausdorff, then X is a semiregular H-closed space. $\Diamond$

Recall that a subset A of a topological space (X, τ) is called *locally dense* [7] (= preopen) if $A \subseteq \text{int}(\text{cl}(A))$. Note that every open and every dense set is locally dense but not vice versa. Recall also [14] that a subset S of a topological space $(X, \tau, \mathcal{I})$ is a topological space with an ideal $\mathcal{I}_S = \{I \in \mathcal{I}: I \subseteq S\} = \{I \cap S: I \in \mathcal{I}\}$ on S. The families of all regular open (resp. regular closed) subsets of a space (X, τ) will be denoted by $RO(X)$ (resp. $RC(X)$).

Theorem 2.8.

(i) *Every locally dense subset of an $\mathcal{I}$-ED-space is $\mathcal{I}$-ED.*

(ii) *Every regular closed subset of an $\mathcal{I}$-ED-space is $\mathcal{I}$-ED.*

Proof. (i) Let $(X, \tau, \mathcal{I})$ be $\mathcal{I}$-ED and let $S \subseteq X$ be locally dense. If R is a regular open subset of $(S, \tau_S, \mathcal{I}_S)$, then $R = S \cap T$, where T is regular open in (X, τ) [13 Lemma 1.1]. Since $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-ED, then $\text{cl}(T) \setminus T \in \mathcal{I}$. Hence $(\text{cl}(T) \cap S) \setminus (S \cap T) = (\text{cl}(T) \cap S) \setminus R \in \mathcal{I}_S$. Note that $\text{cl}_S(R) \subseteq \text{cl}(R) \subseteq \text{cl}(T)$. Thus, $\text{cl}_S(R) \setminus R \subseteq (\text{cl}(T) \cap S) \setminus R \in$ $\in \mathcal{I}_S$. This shows that $(S, \tau_S, \mathcal{I}_S)$ is $\mathcal{I}$-ED, more precisely, $(S, \tau_S, \mathcal{I}_S)$ is $\mathcal{I}_S$-ED.

(ii) Let $S \in RC(X)$, where $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-ED. Clearly, $S = S_1 \cup$ $\cup S_2$, where S_1 is regular open in (X, τ), $S_2 \in \mathcal{I}$ and $S_1 \cap S_2 = \emptyset$. Let $U \in RO(S)$. Note that $\text{bd}_S(U) = \text{cl}_S(U) \setminus U = ((\text{cl}_S(U) \cap S_1) \cup$ $\cup (\text{cl}_S(U) \cap S_2)) \setminus U$. Since $\text{cl}_S(U) \cap S_2 \in \mathcal{I}$ due to the heredity of $\mathcal{I}$, then because of the finite additivity of $\mathcal{I}$, we only need to show that $(\text{cl}_S(U) \cap S_1) \setminus U \in \mathcal{I}$. By [13 Lemma 1.1], $U \cap S_1$ is a regular open subset of S_1. Since $S_1 \in \tau$, then $\text{bd}_{S_1}(U \cap S_1) = (\text{cl}_{S_1}(U \cap S_1)) \setminus U \in$ $\in \mathcal{I}_{S_1} \subseteq \mathcal{I}$. Since $(\text{cl}_S(U) \cap S_1) \setminus U \subseteq (\text{cl}_{S_1}(U \cap S_1)) \setminus U$, then the proof is complete. $\Diamond$

Recall next that a subset A of a space $(X, \tau, \mathcal{I})$ is said to be *$\mathcal{I}$-open* [31] if $A \subset \text{int}(A^*)$, where A^* is the *local function* of A with respect to $\mathcal{I}$ and τ, i.e. $A^*(\mathcal{I}) = \{x \in X : U \cap A \notin \mathcal{I} \text{ for every } U \in \tau(x)\}$, written simply as A^* when there is no chance for confusion (here $\tau(x)$ denotes the open neighborhood system at x).

Corollary 2.9.

(i) *Every open and $\mathcal{I}$-open subset of an $\mathcal{I}$-ED-space $(X,\tau,\mathcal{I})$ is an $\mathcal{I}$-ED-space.*

(ii) *Every β-open subset of an $\mathcal{I}$-ED-space $(X,\tau,\mathcal{I})$ is $\mathcal{I}$-ED, in particular β-open, i.e. semi-preopen subspaces of extremally disconnected spaces are extremally disconnected.*

Proof. (i) Note that every $\mathcal{I}$-open and every open set is locally dense.

(ii) If B is β-open in $(X,\tau,\mathcal{I})$, then B is dense in some regular closed subset R of (X,τ). Since R is $\mathcal{I}$-ED by Th. 2.8 (ii), then B is $\mathcal{I}$-ED by Th. 2.8 (i). ◊

Recall that the ideal defined on the topological sum $X=\sum_{\alpha\in\Omega}X_\alpha$ of the family of spaces $(X_\alpha,\tau_\alpha,\mathcal{I}_\alpha)_{\alpha\in\Omega}$ is $\mathcal{I}=\bigvee_{\alpha\in\Omega}\mathcal{I}_\alpha=\{\cup_{\alpha\in\Omega}I_\alpha: I_\alpha\in \in \mathcal{I}_\alpha\}$ [14]. For a set U, $\mathrm{cl}(U)$ will denote the closure of U in the topological sum X.

Corollary 2.10. *Let $(X_\alpha,\tau_\alpha,\mathcal{I}_\alpha)_{\alpha\in\Omega}$ be a family of topological spaces. For the topological sum $X=\sum_{\alpha\in\Omega}X_\alpha$ the following conditions are equivalent:*

(1) *X is a $\mathcal{I}$-ED.*

(2) *Each X_α is $\mathcal{I}$-ED.*

Proof. (1) $\Rightarrow$ (2) follows from Cor. 2.9.

(2) $\Rightarrow$ (1) Let $R\subseteq X$ be regular open. Set $R_\alpha=R\cap X_\alpha$. Since $\{R_\alpha:\alpha\in\Omega\}$ is a locally finite family of sets in (X,τ), then $\mathrm{cl}(R)= =\mathrm{cl}(\cup_{\alpha\in\Omega}R_\alpha)=\cup_{\alpha\in\Omega}\mathrm{cl}(R_\alpha)=\cup_{\alpha\in\Omega}\mathrm{cl}_{X_\alpha}(R_\alpha)$. By (2), for each $\alpha\in \in\Omega$, $\mathrm{cl}_{X_\alpha}(R_\alpha)\setminus R_\alpha\in\mathcal{I}_\alpha$. Thus, $\mathrm{cl}(R)\setminus R=\cup_{\alpha\in\Omega}(\mathrm{cl}_{X_\alpha}(R_\alpha))\setminus R_\alpha\in\mathcal{I}$. This shows that X is a $\mathcal{I}$-ED. ◊

Remark 2.11. Let $(X,\tau_i)_{i\in I}$ be a family of topological spaces. If we set on each one of those spaces a 'same type of ideal', then the ideal formed on the topological sum of those spaces, need not be from the 'same type'. More precisely, the topological sum of $\mathcal{F}$-ED-space, i.e., almost extremally disconnected spaces, need not always be almost extremally disconnected. Let X be the set of all real numbers. Set $X_n=X$, while let τ_n be the point excluded topology for each $n\in\omega$. Then each (X_i,τ_i) is almost extremally disconnected but one can easily check that the topological sum $\sum_{i\in\omega}(X_i,\tau_i)$ is not almost extremally disconnected. However, if each one of the ideals set on a collection of spaces is the minimal ideal, then the ideal formed on their topological sum is also the minimal ideal. Thus we have the following corollary.

Corollary 2.12 [22, Engelking]. *The topological sum of extremally disconnected spaces is also extremally disconnected.* $\Diamond$

A topological space (X, τ) is called a *P_2-space* [24] (originally to satisfy condition (P2)) if every nonempty open set contains a nonempty closed set.

Theorem 2.13. *For a P_2-space $(X, \tau, \mathcal{I})$, the following conditions are equivalent:*

(1) *$(X, \tau, \mathcal{I})$ is $\mathcal{I}$-ED.*

(2) *Every proper open subspace of X is $\mathcal{I}$-ED.*

Proof. (1) $\Rightarrow$ (2) is in Cor. 2.9 (i).

(2) $\Rightarrow$ (1) Let $U \subseteq X$ be regular open. We can assume that U is not dense in (X, τ), since otherwise $U = X$ and we are done (the void set is a member of every ideal). Then clearly $V = X \setminus \mathrm{cl}(U)$ is a nonempty member of τ. Since X is P_2, then there exists a nonempty closed subset W such that $W \subseteq V$. Hence $S = X \setminus W$ is a proper open subset of X. Since U is open in S, then by (2), $\mathrm{cl}_S(U) \setminus U = (\mathrm{cl}(U) \cap \cap S) \setminus U = \mathrm{cl}(U) \setminus U \in \mathcal{I}_S$. Since $\mathcal{I}_S \subseteq \mathcal{I}$, then $\mathrm{cl}(U) \setminus U \in \mathcal{I}$. Thus $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-ED. $\Diamond$

Corollary 2.14 [30, Isiwata]. *A Tychonoff space (X, τ) is extremally disconnected if and only if every proper open subspace of X is extremally disconnected.*

Proof. Set $\mathcal{I} = \{\emptyset\}$ in Th. 2.13 and note that every Tychonoff space is a P_2-space. $\Diamond$

Remark 2.15. Isiwata [30] proves that a Tychonoff space is stonian if and only if it is extremally disconnected. A Tychonoff topological space (X, τ) is called *stonian* [30] if every bounded continuous real-valued function defined on a set $U \in \tau$ can be extended over X.

Proposition 2.16. *Let (X, τ, σ) be a bitopological space such that $RO(X, \tau) = RO(X, \sigma)$, i.e. both topologies support the same regular open sets. Then (X, τ) is $\mathcal{I}$-ED if and only if (X, σ) is $\mathcal{I}$-ED.*

Proof. Let U be a τ-regular open subset of X. Then the τ-boundary of U is equal to the σ-boundary of U. Clearly, $\mathrm{bd}_\tau(U) = \mathrm{cl}_\tau(U) \setminus U$ and $\mathrm{bd}_\sigma(U) = \mathrm{cl}_\sigma(U) \setminus U$. But, $RO(X, \tau) = RO(X, \sigma)$ implies that $\mathrm{cl}_\tau(U) = \mathrm{cl}_\sigma(U)$. For if x is an element of $\mathrm{cl}_\tau(U) \setminus \mathrm{cl}_\sigma(U)$, and if V is any σ-open neighborhood of x which misses U, then $\mathrm{cl}_\sigma(V)$ misses U and $W = \mathrm{int}_\sigma(\mathrm{cl}_\sigma(U))$ misses U. But, W is also a τ-regular open set containing x so that $W \cap U \neq \emptyset$. Contradiction! By symmetry of argument, the result follows. $\Diamond$

Remark 2.17. It follows that infinite space with cofinite topology or uncountable space with cocountable topology, etc. is extremally disconnected since it has the same regular open sets as the indiscrete topology, i.e., the two trivial regular open sets, the empty set and the entire space.

Corollary 2.18. *If $RO(X,\tau) = RO(X,\sigma)$ and $RO(X,\sigma)$ is the family of clopen subsets of (X,σ), then $RO(X,\tau)$ is the family of clopen subsets of (X,τ), and in particular, (X,τ) is ED.* ◊

Corollary 2.19. *If τ_s is the semiregularization of τ, then (X,τ) is $\mathcal{I}$-ED if and only if (X,τ_s) is $\mathcal{I}$-ED.* ◊

Corollary 2.20. *If $\mathcal{I}$ is τ-codense, i.e., $\mathcal{I}$ intersected with τ contains only the empty set, then (X,τ) is $\mathcal{I}$-ED if and only if $(X,\tau^*(\mathcal{I}))$ is $\mathcal{I}$-ED, where $\tau^*(\mathcal{I}) = \tau[\mathcal{I}]$ is the topology having basic open sets of the form $U \setminus N$, where $U \in \tau$ and $N \in \mathcal{I}$, i.e., $\tau[\mathcal{I}]$ is the smallest topology expansion of τ in which members of $\mathcal{I}$ are closed.*

Proof. Since $\mathcal{I}$ is τ-codense, then $\tau^*(\mathcal{I})_s = \tau_s$. ◊

The next two results are consequences of Cor. 2.19 and Cor. 2.20.

Corollary 2.21. *Every minimal $\mathcal{I}$-ED topology is semiregular.* ◊

In Cor. 2.21 minimality may be taken relative to the $\mathcal{I}$-ED property alone or may be taken relative to the $\mathcal{I}$-ED property in the class of spaces having the same regular open subsets. In this last sense we also have the following.

Corollary 2.22. *Let $\mathcal{I}$ be τ-codense. If the space (X,τ) is maximal $\mathcal{I}$-ED (in the class of spaces having the same regular open sets as (X,τ)), then X is $\mathcal{CD}$-ED.* ◊

Recall that a topological space (X,τ) is called an *α-space* if τ coincides with its α-topology $\tau^*(\mathcal{N})$, i.e. if $\tau = \tau^*(\mathcal{N})$. A topological space (X,τ) is called *N-scattered* [17] if every nowhere dense subset of X is scattered. Another way of relating the classes of α-spaces and N-scattered spaces to the theory of topological ideals is as follows:

Theorem 2.23.

(i) *Every α-space is a $\mathcal{CD}$-ED-space.*

(ii) *Every N-scattered space is an $\mathcal{S}$-ED-space.*

Proof. (i) It is not difficult to verify that a topological space (X,τ) is an α-space if and only if the boundary of every open set is closed and discrete.

(ii) It is proved in [17], that a space is N-scattered if and only if the boundary of every open set is scattered. ◊

Example 2.24. (i) Every infratopology $\{\emptyset, A, \mathbb{R}\}$ on the real line $\mathbb{R}$ (such that $|\mathbb{R} \setminus A| \neq 1$) shows that $\mathcal{CD}$-ED-spaces, even hyperconnected spaces, need not be α-spaces.

(ii) The left ray topology on the real line shows that T_0- $\mathcal{S}$-ED-spaces, even hyperconnected T_0-spaces, need not be N-scattered.

Definition 2. A topological space $(X, \tau, \mathcal{I})$ is called *basically $\mathcal{I}$-ED* if there is an open base for the topology τ such that for each basic open V, $\text{cl}(V) \setminus \text{int}(\text{cl}(V))$ is a member of $\mathcal{I}$.

For example, every T_0 rim-scattered space would be basically $\mathcal{S}$-ED. So, N-scattered (strictly) implies rim-scattered [17] which implies basically $\mathcal{S}$-ED. Our next example will show that this last implication is strict.

Example 2.25. Let X denote the real line $\mathbb{R}$ and let τ be a topology on X whose non-trivial members are generated by the intervals $(-n, -\frac{1}{n})$ and $(\frac{1}{n}, n)$, where $n = 2, 3, \ldots$. Note that for each such interval U, $\text{cl}(U) \setminus \text{int}(\text{cl}(U)) = \{0\}$, so clearly (X, τ) is basically $\mathcal{S}$-ED. However, since for each such interval U, $\text{cl}(U) \setminus U$ contains the non-scattered ray (x, ∞) for some $x \in X$, then (X, τ) is not rim-scattered. Note additionally that X is a T_0-space.

Definition 3. A topological space $(X, \tau, \mathcal{I})$ is called *$\mathcal{I}$-totally disconnected* if for each two different points x and y, there exists a regular open set R containing x such that $\text{bd}(R) \in \mathcal{I}$ and $y \notin \text{cl}(R)$.

Recall that a topological space (X, τ) is called *totally disconnected* [22, 43] if the quasi-component of each point of X is the point itself. Clearly a space (X, τ) is totally disconnected if and only if it is $\{\emptyset\}$-totally disconnected. In [47], totally disconnected spaces are called ultrahausdorff. We note that some authors use the term totally disconnected for spaces whose components (not quasi-components) are singletons.

Theorem 2.26. *Every Hausdorff $\mathcal{I}$-ED-space $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-totally disconnected.*

Proof. Let $x, y \in X$ such that $x \neq y$. Since X is Hausdorff, then x and y can be separated by disjoint regular open sets U and V. Since X is $\mathcal{I}$-ED, then $\text{bd}(U) \in \mathcal{I}$. Moreover, $y \notin \text{cl}(U)$. This shows that $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-totally disconnected. $\Diamond$

Corollary 2.27. *Every extremally disconnected Hausdorff space is totally disconnected and hence functionally Hausdorff.* $\Diamond$

Theorem 2.28. *Every locally dense, and hence every open and $\mathcal{I}$-open* *subset of an $\mathcal{I}$-totally disconnected space is $\mathcal{I}$-totally disconnected.*

Proof. Let $(X, \tau, \mathcal{I})$ be $\mathcal{I}$-totally disconnected and let $A \subseteq X$ be a locally dense subspace of (X, τ). Let $x, y \in A$ with $x \neq y$. By assumption, there exists $U \in RO(X, \tau)$ such that $x \in U$, $\mathrm{bd}(U) \in \mathcal{I}$ and $y \notin \mathrm{cl}(U)$. By [13 Lemma 1.1], $R = U \cap A \in RO(A, \tau_A)$. Observe that $y \notin \mathrm{cl}_A(R)$. Additionally, $\mathrm{bd}_A(R) = \mathrm{cl}_A(R) \setminus R = \mathrm{cl}_A(R) \setminus U \subseteq$ $\subseteq \mathrm{cl}(U) \setminus U \in \mathcal{I}$. Hence $\mathrm{bd}_A(R) \in \mathcal{I}_A$. Thus $(A, \tau_A, \mathcal{I}_A)$ is $\mathcal{I}_A$-totally disconnected. ◊

Corollary 2.29. *Let $(X_\alpha, \tau_\alpha, \mathcal{I}_\alpha)_{\alpha \in \Omega}$ be a family of pairwise disjoint topological spaces. For the topological sum $X = \sum_{\alpha \in \Omega} X_\alpha$ the following conditions are equivalent:*

(1) *X is a $\mathcal{I}$-totally disconnected.*
(2) *Each X_α is $\mathcal{I}$-totally disconnected.*

Proof. (1) ⇒ (2) follows from Th. 2.28.

(2) ⇒ (1) Let x and y be two different points of X. If $x \in X_\xi$ and $y \in X_\mu$, such that $\xi, \mu \in \Omega$ and $X_\xi \neq X_\mu$, then X_ξ is the required regular open subset of X. If $x, y \in X_\alpha$ for some $\alpha \in \Omega$, then find (due to condition (2)) a regular open subset U of (X_α, τ_α) such that $x \in U$, $\mathrm{bd}_\alpha(U) \in \mathcal{I}_\alpha$ and $y \notin \mathrm{cl}_\alpha(U)$. Observe easily that $U \in RO(X)$ and $y \notin \mathrm{cl}(U)$. Additionally, $\mathrm{bd}(U) = \mathrm{cl}(U) \setminus U = \mathrm{cl}_{X_\alpha}(U) \setminus U \in \mathcal{I}_\alpha \subseteq \mathcal{I}$. All this shows that X is $\mathcal{I}$-totally disconnected. ◊

Extremal disconnectedness has several other aspects. In that connection we have the following:

Question. Could some of the properties of ED-spaces mentioned in the introduction as well as the following major properties of ED-spaces be generalized somehow to $\mathcal{I}$-ED-spaces:

- [22, Engelking] Every hereditarily normal ED-space is hereditarily ED.
- [23, Frolík] If (U, V) is a pair of separated countable sets of a regular ED-space (X, τ), then U and V are functionally separated.
- [27, Gleason] Every convergent sequence $(x_1, x_2, \dots)$ of a Hausdorff ED-space (X, τ) is stationary.
- [48, Strauss] If (X, τ) is Hausdorff and (Y, σ) is Hausdorff and ED, then every irreducible closed surjective mapping $f\colon (X, \tau) \to$ $\to (Y, \sigma)$ is a homeomorphism.

References

[1] BłASZCZYK, A. and VERMEER, J.: Some old and some new results on combinatorial properties of fixed-point free maps, *Annals of the New York Academy of Sciences*, Papers on General Topology and Applications, Vol. **767** (1995), 1–16.

[2] BłASZCZYK, A., RAJAGOPALAN, M. and SZYMANSKI, A.: Spaces which are hereditarily extremally disconnected, *J. Ramanujan Math. Soc.* **8**/1-2 (1993), 81–94.

[3] CAMERON, D.: Maximal and minimal topologies, *Trans. Amer. Math. Soc.* **160** (1971), 229–248.

[4] CAMERON, D.: Properties of S-closed spaces, *Proc. Oklahoma Topology Conf.,* Oklahoma, 1978.

[5] CAMERON, D.E.: Properties of S-closed spaces, *Proc. Amer. Math. Soc.* **72** (1978), 581–586.

[6] COMFORT, W.W. and van MILL, J.: A homogeneous extremally disconnected countably compact space, *Topology Appl.* **25** (1987), 65–73.

[7] CORSON, H.H. and MICHAEL, E.: Metrizability of certain countable unions, *Illinois J. Math.* **8** (1964), 351–360.

[8] CSÁSZÁR, Á.: Iliadis absolutes for arbitrary spaces, *Acta Math. Hungar.* **57**/1-2 (1991), 111–120.

[9] CSÁSZÁR, Á.: General absolutes of topological spaces, *Acta Math. Hungar.* **58**/1-2 (1991), 141–151.

[10] DANG, S., BEHERA, A. and NANDA, S.: Fuzzy extremally disconnected spaces, *J. Fuzzy Math.* **1**/4 (1993), 723–737.

[11] DIMOV, G.D.: On the commutability between hyperspaces and subspaces, and on Schmidt's conjecture, Proceedings of the Eleventh International Conference of Topology (Trieste, 1993), *Rend. Istit. Mat. Univ. Trieste* **25**/1-2 (1993), 175–194.

[12] DLASKA, K., ERGUN, N. and GANSTER, M.: On the topology generated by semi-regular sets, *Indian J. Pure Appl. Math.* **25**/11 (1995), 1163–1170.

[13] DLASKA, K. and GANSTER, M.: Almost rc-Lindelöf spaces, *Bull. Malaysia Math. Soc.* **18** (1995), 51–56.

[14] DONTCHEV, J.: On Hausdorff spaces via topological ideals and $\mathcal{I}$-irresolute functions, *Annals of the New York Academy of Sciences*, Papers on General Topology and Applications, Vol. **767** (1995), 28–38.

[15] DONTCHEV, J. and GANSTER, J.: On covering spaces with semi-regular sets, *Ricerche di Matematica* **45**/1 (1996), 219–235.

[16] DONTCHEV, J., GANSTER, M. and KONSTADILAKI, Ch.: On locally countably S-closed and locally rc-Lindelöf spaces, preprint.

[17] DONTCHEV, J. and ROSE, D.: On spaces whose nowhere dense subsets are scattered, preprint.

[18] DOW, A., GUBBI, A. and SZYMANSKI, A.: Rigid Stone Spaces within ZFC, *Proc. Amer. Math. Soc.* **102** (1988), 745–748.

[19] DOW, A. and van MILL, J.: Dense extremally disconnected subspaces, *Proc. Amer. Math. Soc.* **121** (3) (1994), 931–936.

[20] DOW, A. and VAUGHAN, J.E.: Accessible and biaccessible points in contrasequential spaces, *Annals of the New York Academy of Sciences*, Papers on General Topology and Applications, Vol. **704** (1995), 92–102.

[21] DOW, A. and VERMEER, J.: Extremally disconnected spaces, subspaces and retracts, *Topology Appl.* **50**/3 (1993), 263–282.

[22] ENGELKING, R.: General Topology, Warszawa, 1977.

[23] FROLÍK, Z.: Maps of extremally disconnected spaces, theory of types, and applications, *General Topology and its Relations to Modern Analysis and Algebra III*, Proc. Conf. Kanpur, 1968; pp. 131–142 (Academia, Prague, 1971).

[24] GANSTER, M.: A note on extensions generated by regular closed sets, *Tamkang J. Math.* **26**/2 (1995), 125–129.

[25] GANSTER, M. and ANDRIJEVIĆ, D.: On some questions concerning semipreopen sets, *J. Inst. Math. Comput. Sci. Math. Ser.* **1** (2) (1988), 65–75.

[26] GHOSH, B.: Fuzzy extremally disconnected spaces, *Fuzzy Sets and Systems* **46** (2) (1992), 245–250.

[27] GLEASON, A.M.: Projective topological spaces, *Illinois J. Math.* **2** (1958), 482–489.

[28] HERRMAN, R.A.: A nonstandard approach to S-closed spaces, *Notices Amer. Math. Soc.* **25** (1978), A–364.

[29] HERRMAN, R.A.: RC-convergence, *Proc. Amer. Math. Soc.* **75** (1979), 311–317.

[30] ISIWATA, T.: On stonian spaces, *Sci. Rep. Tokyo Kyoiku Daigaku* **A 6** (1958), 147–176.

[31] JANKOVIĆ, D. and HAMLETT, T.R.: Compatible extensions of ideals, *Bollettino U.M.I.* (7) **6**–B (1992), 453–465.

[32] KATO, A.: A new construction of extremally disconnected topologies, *Topology Appl.* **58**/1 (1994), 1–16.

[33] KONSTADILAKI, Ch.: S-bounded and countably S-bounded sets, *Math. Japonica* **41**/2 (1995), 453–461.

[34] KUBIAK, T.: Extending continuous L-real functions, *Math. Japonica* **31**/6 (1986), 875–887.

[35] LINDGREN, W.F. and SZYMANSKI, A.: On some homogeneous extremally disconnected spaces, *Annals of the New York Academy of Sciences*, Papers on General Topology and Applications, Vol. **767** (1995), 108–114.

[36] LONG, P.E., HERRINGTON, L.L. and JANKOVIĆ, D.S.: Almost-invertible spaces, *Bull. Korean Math. Soc.* **23** (2) (1986), 91–102.

[37] NAGURA, T. Extension properties and absolutes, *Math. Japon.* **35** (2) (1990), 315–319.

[38] NOIRI, T.: A note on extremally disconnected spaces, *Proc. Amer. Math. Soc.* **79** (2) (1980), 327–330.

[39] NOIRI, T.: Characterizations of extremally disconnected spaces, *Indian J. Pure Appl. Math.* **19**/4 (1988), 325–329.

[40] PORTER, J. and THOMAS, J.: On H-closed and minimal Hausdorff spaces, *Trans. Amer. Math. Soc.* **138** (1969), 159–170.

[41] PORTER, J.R. and WOODS, R.G.: Extensions and Absolutes of Hausdorff Spaces, Springer–Verlag, New York 1988.

[42] SEMADENI, Z.: Sur les ensembles clairsemés, *Rozprawy Mat.* **19** (1959).

[43] SIERPIŃSKI, W.: Sur les ensembles connexes et non connexes, *Fund. Math.* **2** (1921), 81–95.

[44] SIMON, P.: Points in extremally disconnected compact spaces, Fourth Conference on Topology (Italian) (Sorrento, 1988), *Rend. Circ. Mat. Palermo (2) Suppl.* **24** (1990), 203–213.

[45] SINGAL, M.K. and MATHUR, A.: On nearly-compact spaces, *Bollettino U.M.I.* **4**/2 (1969), 702–710.

[46] STAUM, R.: The algebra of bounded continuous functions into a nonarchimedean field, *Pacific J. Math.*, **50** (1974), 169–185.

[47] STONE, H.M.: Algebraic characterizations of special Boolean rings, *Fund. Math.* **29** (1937), 223–302.

[48] STRAUSS, D.P.: Extremally disconected spaces, *Proc. Amer. Math. Soc.* **18** (1967), 305–309.

[49] SZAMBIEN, H.: On homogeneous extremally disconnected spaces, *Results Math.* **22**/1-2 (1992), 619–621.

[50] THUMMEL, T.: The fixed point set of open mappings on extremally disconnected spaces, *Comment. Math. Univ. Carolin.* **35**/4 (1994), 789–794.

[51] VERMEER, J.: Continuous functions on extremally disconnected spaces, *Trans. Amer. Math. Soc.* **347**/9 (1995), 3263–3285.