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A STABILITY THEOREM FOR THE AREA SUM OF A CONVEX POLYGON AND ITS POLAR RECIPROCAL

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Abstract: Let P be a plane convex polygon inscribed in the unit circle K, and let P^* be the polar reciprocal of P with respect to K. J. Aczél and L. Fuchs [1] showed that the area sum of P and P^*, denoted by $S(P)$, is greater than or equal to 6, with equality only if P is a square. In this paper we prove a corresponding stability theorem: assuming that $S(P)$ is not much greater than 6, we give upper bounds for a special deviation of P from a square as well as for the Hausdorff distance. The Handbook article of H. Groemer [4] is a survey of stability results for geometric inequalities.

Let K be the unit circle centred at the origin O, and let P be a convex polygon inscribed in K. We denote by $P^\star$ the polar reciprocal of P with respect to K, that is, the circumscribed polygon whose points of contact with K are the vertices of P. J. Aczél and L. Fuchs [1] proved that

$$S(P) \equiv a(P) + a(P^\star) \geq 6, \tag{1}$$

where $a(X)$ denotes the area of the set X. Equality holds if and only if P is a square. The papers [2], [3], [5] and [6] contain alternative proofs and various extensions of (1).

In the present paper we deal with the following question: if $S(P)$ is not much greater than 6, what can be said about the deviation of P from a square? The deviation of P from a square can be measured, e.g., by

(2) $$\rho^H(P) = \min \rho^H(P, Q),$$

where qρ^H indicates Hausdorff distance, and the minimum extends over all squares Q inscribed in K. Still it appears difficult to establish a connection between $S(P)$ and $\rho^H(P)$.

Without loss of generality we may assume that the origin is an interior point of P, since otherwise $a(P^\star) = \infty$. Throughout this paper we consider only such polygons which are inscribed in K and contain the origin as an interior point. This assumption will not be mentioned explicitly, where there is no danger of misunderstanding. Let us denote the central angles spanned by the sides of P by $2x_1, \ldots, 2x_n$, $n \geq 3$, where

(3) $$0 < x_k < \pi/2 \quad (k = 1, \ldots, n), \quad x_1 + \ldots + x_n = \pi.$$

We shall refer to $x_1, \ldots, x_n$ as *the parameters of* P. Occasionally, it will be convenient to introduce some additonal $x_k = 0$.

Four of the parameters corresponding to a square are $\pi/4$, the others being 0. Thus, for any $x_k \in [0, \pi/2)$,

$$\min\left(x_k, \left|\frac{\pi}{4} - x_k\right|\right)$$

may be viewed as the deviation of x_k from the parameters of a square. This suggests an alternative concept of deviation of the polygon P (with parameters $x_1, \ldots, x_n$) from a square.

Definition. The *deviation* of the polygon P from a square is defined by

(4) $$\rho(P) = \sum_{k=1}^{n} \min\left(x_k, \left|\frac{\pi}{4} - x_k\right|\right).$$

Observe that rotation of P about O or permutation of its sides leaves $\rho(P)$ unchanged.

From (4) it follows that

(5) $$0 \leq \rho(P) \leq \pi$$

holds for any polygon P, with $\rho(P) = 0$ only if P is a square, and $\rho(P) = \pi$ only if $\max x_k \leq \pi/8$. If T is a triangle, then $\pi/4 \leq \rho(T) \leq \pi/2$. For a regular n-gon $\overline{P}_n$ we have $\rho(\overline{P}_3) = \rho(\overline{P}_5) = \pi/4$, $\rho(\overline{P}_6) = \pi/2$, $\rho(\overline{P}_7) = 3\pi/4$ and $\rho(\overline{P}_n) = \pi$ when $n \geq 8$.

We now prove that ρ *is a continuous function of* P *in the Hausdorff metric.*

Let $R = A_1A_2 \dots A_m$ be a fixed convex m-gon inscribed in K. If r is a positive number, we denote by R_r the outer parallel domain of R at distance r. Let r be subject to the following three conditions:

(i) Every line segment of the boundary of R_r intersects the interior of K. Then $R_r \cap bdK$ consists of m mutually disjoint arcs $a_1, \dots, a_m$ with central angles $2\alpha_1, \dots, 2\alpha_m$;

(ii) let $\alpha_k \leq \pi/8$, for $k = 1, \dots, m$;

(iii) the distance of A_k from the convex hull of $\bigcup_{i \neq k} a_i$ is greater than r, for $k = 1, \dots, m$.

These conditions are clearly satisfied if r is sufficiently small.

Let $P = B_1B_2 \dots B_n$ be a convex n-gon inscribed in K such that

$$\rho^H(P, R) = r. \tag{6}$$

By (6) and (i), every B_j lies on some a_k, and by (6) and (iii) every a_k contains some B_j. Let $2x_k$ and $2y_j$ denote the central angles corresponding to A_kA_{k+1} and B_jB_{j+1}, respectively. We distinguish two cases:

(a) If B_j and B_{j+1} lie on the same arc a_k, then (ii) implies that $\min\left(y_j, \left|\frac{\pi}{4} - y_j\right|\right) = y_j$.

(b) If the vertex B_j lies on a_k, and B_{j+1} on a_{k+1}, we make use of the inequality

$$\left| \min\left(y_j, \left|\frac{\pi}{4} - y_j\right|\right) - \min\left(x_k, \left|\frac{\pi}{4} - x_k\right|\right) \right| \leq |y_j - x_k|, \tag{7}$$

which holds for any $x_k \in [0, \pi/2)$ and any $y_j \in [0, \pi/2)$ and which can be proved by straightforward calculation. Alternatively, (7) follows from the fact that the graph of the function $\min\left(x, \left|\frac{\pi}{4} - x\right|\right)$ consists of three segments, each of which forms with the positive x-axis either the angle $\pi/4$ or $-\pi/4$.

Thus we finally obtain

$$|\rho(P) - \rho(R)| \leq \sum_{k=1}^{m} \sum_{j} y_j + \sum_{k=1}^{m} |y_j - x_k|, \tag{8}$$

where the first term and the second term correspond to the cases (a) and (b), respectively. The right side of (8) attains its maximum when both endpoints of every a_k are vertices of P. This shows that

$$|\rho(P) - \rho(R)| \leq 2(\alpha_1 + \dots + \alpha_m). \tag{9}$$

But $\sum_{k=1}^{m} \alpha_k$ tends to 0 when $r \to 0$, as required. ◊

We shall state our main result (Theorem 1) in terms of the function ρ. The following lemma establishes the existence of two positive numerical constants confining the ratio $\rho^H(P)/\rho(P)$ for every polygon P different from a square.

Lemma. *For any convex polygon P inscribed in K and containing O in its interior we have*

$$\frac{1}{\pi}\left(\cos\frac{\pi}{8}-\cos\frac{\pi}{4}\right)\rho(P)\leq\rho^H(P)\leq 2\rho(P). \tag{10}$$

Proof. (a) Let Q be a square and put $\rho^H(P,Q)=r$. Since $\rho(P)\leq\pi$, we may assume that

$$r\leq\cos\frac{\pi}{8}-\cos\frac{\pi}{4}. \tag{11}$$

If we maintain the above notation with $R=Q$ and write α for $\alpha_1 = = \alpha_2=\alpha_3=\alpha_4$, then by (11)

$$\alpha=\frac{\pi}{4}-\arccos\left(\frac{1}{\sqrt{2}}+r\right)\leq\frac{\pi}{8}.$$

The distance of the vertex A_k of Q from the convex hull of $\bigcup_{i\neq k} a_i$ is $1-\sin\alpha>r$. Thus the conditions (i), (ii), (iii) are satisfied. Since $\rho(Q)=0$, inequality (9) implies that $\rho(P)\leq 8\alpha$ or

$$\frac{\rho(P)}{r}\leq 8\frac{\frac{\pi}{4}-\arccos\left(\frac{1}{\sqrt{2}}+r\right)}{r}. \tag{12}$$

The right side of (12) being an increasing function of r, the first inequality (10) is a consequence of (11) and (12).

If we take $\alpha=\pi/8$, then the endpoints of a_1,a_2,a_3,a_4 are the vertices of a regular octagon P with $\rho^H(P,Q)=\cos\pi/8-\cos\pi/4$. As $\rho(P)=\pi$, we have equality in the first inequality (10). This holds also for the polygon P' =conv($P\cup M$), where M is any finite subset of $\bigcup_{i=1}^{4} a_i$.

(b)If P is any convex polygon inscribed in K containing O in its interior, and if Q is any inscribed square, then $\rho^H(P,Q)<1$. Therefore, for the proof of the second inequality (10), we can assume that

$$0<\rho(P)\leq\frac{\pi}{6}, \tag{13}$$

and write $\rho(P)=\rho$.

Let $x_1,\ldots,x_n$ be the parameters of P. We shall frequently refer to the *arc* $2x_k$ instead of the respective central angle. If $x_k\leq\pi/8$, we continue to denote this by x_k and call $2x_k$ a *small arc.* If $x_k>\pi/8$, we shall write y_k in place of x_k and call $2y_k$ a *large arc.* By (3) and (4), we have

$$\sum x_k + \sum y_k = \pi\,, \tag{14}$$
$$\sum x_k + \sum \left|\frac{\pi}{4} - y_k\right| = \rho\,.$$

For any polygon, the number of large arcs is at most seven. On the assumption (13), this number is exactly four. To show this, set $|\pi/4 - - y_k| = d_k$, whence

$$\frac{\pi}{4} - d_k \le y_k \le \frac{\pi}{4} + d_k\,. \tag{15}$$

Suppose that there are five large arcs, $2y_1, \dots, 2y_5$. Then, by (15), (14) and (13),

$$y_1 + \ldots + y_5 \ge \frac{5\pi}{4} - (d_1 + \ldots + d_5) \ge \frac{5\pi}{4} - \rho \ge \frac{5\pi}{4} - \frac{\pi}{6} > \pi\,.$$

Next, suppose that there are at most three large arcs $2y_k$. Then, by (15), (14) and (13),

$$\sum x_k + \sum y_k \le \sum x_k + \frac{3\pi}{4} + \sum d_k = \frac{3\pi}{4} + \rho \le \frac{3\pi}{4} + \frac{\pi}{6} < \pi\,,$$

which leads to a contradiction in either case.

We shall prove the second inequality (10) by showing that there is a square Q such that

$$\rho^H(P,Q) \le 2\sin\rho\,. \tag{16}$$

Let A be any vertex of P and let $Q = ABCD$ be a square. We progress on the boundary of K in the direction $A \to B \to C \to D \to \to A$ covering the large arcs $2y_1, 2y_2, 2y_3, 2y_4$ as well as the intermediate small arcs. By (14), the endpoint of every small arc between A and $2y_1$ has from A a circular distance $\le 2\sum x_k \le 2\rho$. The endpoint of $2y_1$ has from B a circular distance $\le 2d_1 + 2\sum x_k \le 2\rho$, and this holds also for the endpoints of the small arcs between $2y_1$ and $2y_2$. The endpoint of $2y_2$ has from C a circular distance $\le 2d_1 + 2d_2 + 2\sum x_k \le 2\rho$, and this holds also for the endpoints of the small arcs between $2y_2$ and $2y_3$, and so on. Therefore, every vertex of P has an Euclidean distance $\le 2\sin\rho$ from some vertex of Q, and vice versa. This completes the proof of the lemma. ◊

Remarks. i) If a sequence of polygons (P_m) is H-convergent to a polygon P, then $\rho(P_m)$ tends to 0 if and only if P is a square. However, the mere assumption $\rho(P_m) \to 0$ does not imply the convergence of (P_m). Still, for any square Q there exists a sequence of rotations about O, say (r_m), such that $r_m P_m \to Q$.

(ii) It may be that in (10) the upper bound $2\rho(P)$ can be replaced by $c\rho(P)$ with a constant $c < 2$. The example of the equilateral triangle shows that $c \geq 4\pi^{-1}\left(\cos\frac{\pi}{12} - \frac{1}{2}\right) = 0.59323\ldots$.

Returning to our stability problem, let us consider a convex n-gon P with the parameters $x_1, \ldots, x_n$. The area sum of P and P^* defined by (1) can be expressed by the formula

$$S(P) = \sum_{k=1}^{n} f(x_k)\,, \tag{17}$$

where

$$f(x) = \sin x \cos x + \tan x. \tag{18}$$

We can now state our main result.

Theorem 1. *Let P be a convex polygon inscribed in K and let ε be a given number with $0 \leq \varepsilon \leq \varepsilon_0 \equiv \pi/12$. If $\rho(P) \geq 2\varepsilon$, then*

$$S(P) \geq f(\varepsilon) + 4f\left(\frac{\pi-\varepsilon}{4}\right) \equiv S(\varepsilon),$$

with equality if and only if $\rho(P) = 2\varepsilon$ and P is a pentagon with the parameters $x_1 = \varepsilon$, $x_2 = x_3 = x_4 = x_5 = (\pi-\varepsilon)/4$.

Proof. First we show that the function S is strictly increasing in $0 \leq \leq \varepsilon \leq \varepsilon_0$ and observe that

$$\varepsilon < \frac{\pi-\varepsilon}{4}. \tag{19}$$

From

$$f'(x) = 2\cos^2 x - 1 + \frac{1}{\cos^2 x} \tag{20}$$

and

$$f''(x) = 2\frac{\sin x}{\cos^3 x}(1 - 2\cos^4 x) \tag{21}$$

we see that f is (i) strictly increasing in $0 \leq x < \pi/2$, (ii) strictly concave in $0 \leq x \leq x_0$, and (iii) strictly convex in $x_0 \leq x < \pi/2$, where

$$x_0 = \arccos\frac{1}{\sqrt[4]{2}} = 32.765\ldots^\circ.$$

To examine the sign of

$$S'(\varepsilon) = f'(\varepsilon) - f'\left(\frac{\pi-\varepsilon}{4}\right),$$

we set $\varepsilon = x$ and $(\pi-\varepsilon)/4 = y$ and obtain from (20)

$$f'(x) - f'(y) = (\cos^2 x - \cos^2 y)\left(2 - \frac{1}{\cos^2 x \cos^2 y}\right).$$

By (19), the first factor is positive. The function g defined by

$$g(x) \equiv 2\cos x \cos y = \cos(y+x) + \cos(y-x)$$

is strictly concave in $0 \leq x \leq \pi/12$, since

$$g''(x) = -\frac{9}{16}\cos(y+x) - \frac{25}{16}\cos(y-x) < 0,$$

and takes at the boundary points the values $g(0) = \sqrt{2}$ and $g(\pi/12) = = 1.452\ldots > \sqrt{2}$. Hence $g(x) > \sqrt{2}$ for $0 < x \leq \pi/12$, which implies $f'(x) - f'(y) > 0$ and

(22) $$S'(\varepsilon) > 0 \quad \text{if} \quad 0 < \varepsilon \leq \varepsilon_0\,.$$

Observe that

(23) $$S(0) = 6, \qquad S(\varepsilon_0) = 6.00874\ldots.$$

In view of (22) it is sufficient to prove Thm. 1 for $0 < \varepsilon \leq \varepsilon_0$.

We now describe a procedure that can be used to simplify the proof (see [2]). Let us assume that two of the parameters of P lie between 0 and x_0:

$$0 < x_1 \leq x_2 < x_0\,.$$

We replace x_1 and x_2 by x_1' and x_2' such that

$$0 \leq x_1' < x_1 \leq x_2 < x_2' \leq x_0\,,$$

$$x_1' + x_2' = x_1 + x_2$$

and $x_1' = 0$ or $x_2' = x_0$, or both. The parameters $x_k \geq x_0$ remain unchanged. Since f is strictly concave in $[0, x_0]$, this process decreases the sum S. We shall refer to the (possibly repeated) application of this process as reduction. After a finite number of steps we obtain a finite set of real numbers, again denoted by $\{x_1, \ldots, x_n\}$, satisfying

(24) $$0 \leq x_1 \leq x_0 \leq x_2 \leq \ldots \leq x_n < \frac{\pi}{2}\,, \quad \sum_{k=1}^{n} x_k = \pi\,.$$

In the rest of the proof we shall only consider the polygon which corresponds to this set $\{x_1, \ldots, x_n\}$ and denote it again by P. It should be noted that reduction may diminish $\rho(P)$. This point will require special attention. Because $x_0 > \pi/6$, it follows from (24) that

$$n \leq 6.$$

In the following we shall make repeated use of the strict convexity of f in $[x_0, \pi/2)$ and Jensens's inequality. It will be convenient to distinguish the cases $n = 3, 4, 5, 6$.

n = 3. For $0 < x_1 < x_0$ we have

$$S(P) = \sum_{k=1}^{3} f(x_k) \geq f(x_1) + 2f\left(\frac{\pi - x_1}{2}\right) \equiv S_3(x_1)$$

and, by [2], (8),

$$S_3(x_1) > S_3(x_0) > 3f\left(\frac{\pi}{3}\right) = \frac{15}{4}\sqrt{3} = 6.495\ldots > S(\varepsilon_0) \geq S(\varepsilon)$$

for $0 < \varepsilon \leq \varepsilon_0$.

n = 4. For $0 \le x_1 < x_0$ we have

$$S(P) = \sum_{k=1}^{4} f(x_k) \ge f(x_1) + 3f\left(\frac{\pi - x_1}{3}\right) \equiv S_4(x_1)$$

and, by [2], (8),

$$S_4(x_1) > S_4(x_0) = 6.044\ldots > S(\varepsilon_0) \ge S(\varepsilon)$$

for $0 < \varepsilon \le \varepsilon_0$.

n = 5. For $0 \le x_1 \le x_0$ we have

$$S(P) = \sum_{k=1}^{5} f(x_k) \ge f(x_1) + 4f\left(\frac{\pi - x_1}{4}\right) \equiv S_5(x_1).$$

As shown in [2], p. 80, the derivative S_5' passes from positive to negative values. Therefore, S_5 attains its minimum in any subinterval of $[0, x_0]$, e.g. in $[\varepsilon, x_0]$, only at one of the endpoints. From

$$S_5(\varepsilon) = f(\varepsilon) + 4f\left(\frac{\pi - \varepsilon}{4}\right) = S(\varepsilon) \le S(\varepsilon_0) = 6.00874\ldots$$

and

$$S_5(x_0) = 6.01081\ldots$$

we see that

$$S_5(x_1) \ge S(\varepsilon)$$

for $\varepsilon \le x_1 \le x_0$ and $0 < \varepsilon \le \varepsilon_0$, with equality only if $x_1 = \varepsilon$. Hence

$$(25) \qquad S(P) \ge S(\varepsilon) \qquad (\varepsilon \le x_1 \le x_0)$$

with equality only if $x_1 = \varepsilon$ and $x_2 = x_3 = x_4 = x_5$.

It remains to be shown that

$$(26) \qquad S(P) > S(\varepsilon) \qquad (0 \le x_1 < \varepsilon),$$

which will be proved in three steps.

(a) First we consider the case

$$(27) \qquad x_2 = x_0$$

with $0 < \varepsilon \le \varepsilon_0 = \pi/12$. From

$$(28) \quad S(P) = \sum_{k=1}^{5} f(x_k) \ge f(x_1) + f(x_0) + 3f\left(\frac{\pi - x_1 - x_0}{3}\right) \equiv T(x_1)$$

by (20) we obtain

$$\begin{aligned} T'(x_1) =& f'(x_1) - f'(y_1) \\ =& (\cos^2 x_1 - \cos^2 y_1)\left(2 - \frac{1}{\cos^2 x_1 \cos^2 y_1}\right), \end{aligned}$$

where

$$y_1 = \frac{\pi - x_1 - x_0}{3}.$$

Because of $0 \le x_1 < y_1$, we have $\cos^2 x_1 - \cos^2 y_1 > 0$. If $x_0 + x_1 < \pi/4$, we have $y_1 > \pi/4$, so that $\cos x_1 \cos y_1 \le \cos y_1 < 1/\sqrt{2}$ and, by (29),

(30) $$T'(x_1) < 0.$$

If $x_0 + x_1 \geq \pi/4$, the inequalities $x_1 \geq \frac{\pi}{4} - x_0 > 12°$ and $y_1 > > \left(\pi - \frac{\pi}{12} - x_0\right)/3 > 44°$ imply that

$$\cos x_1 \cos y_1 < \cos 12° \cos 44° = 0.70362\ldots < \frac{1}{\sqrt{2}} = 0.70710\ldots.$$

Hence (30) is true also in this case. The required result follows from (28) and

$$T(x_1) > T(\varepsilon) = f(\varepsilon) + f(x_0) + 3f\left(\frac{\pi - \varepsilon - x_0}{3}\right)$$
$$> f(\varepsilon) + 4f\left(\frac{x_0}{4} + \frac{\pi - \varepsilon - x_0}{4}\right) = S(\varepsilon).$$

(b) Therefore, we can now assume, in the second step, that

(31) $$x_0 < x_2\,.$$

It may be that the polygon P is obtained from the original polygon by a process of reduction. For the moment we denote this original polygon by P'. According to the assumption of Thm. 1, we have

(32) $$\rho(P') \geq 2\varepsilon.$$

At the beginning of the proof we pointed out that reduction possibly decreases $\rho(P')$. However, we will show that this does not happen if (31) and $x_1 < \varepsilon$ are satisfied. Since reduction does not change any parameter $\geq x_0$, we can write the parameters of P' in the following way:

$$0 < y_1 \leq y_2 \leq \ldots \leq y_r < x_2 \leq x_3 \leq x_4 \leq x_5\,,$$

where $y_r < x_0$ and $r \geq 2$. Because $\sum_{i=1}^{r} y_i$ is invariant under reduction, we have

$$\sum_{i=1}^{r} y_i = x_1\,.$$

Since $x_1 < \pi/8$, we see that $y_i < \pi/8$ $(i = 1, \ldots, r)$, so that $\min\left(y_i\,, |\frac{\pi}{4} - y_i|\right) = y_i$. Hence we get

$$\rho(P') = \sum_{i=1}^{r} y_i + \sum_{k=2}^{5} \left|\frac{\pi}{4} - x_k\right| = \rho(P),$$

as required. This shows that

(33) $$\rho(P) \geq 2\varepsilon\,.$$

(c) Let us recall that

(34) $$0 \leq x_1 < \varepsilon < x_0 < x_2 \leq x_3 \leq x_4 \leq x_5 < \frac{\pi}{2}\,.$$

Suppose that $x_5 \leq \pi/4$. Then, by (33), we obtain

$$\rho(P) = x_1 + \pi - \sum_{k=2}^{5} x_k \geq 2\varepsilon$$

or

$$2x_1 \geq 2\varepsilon,$$

which contradicts (34). Hence

(35) $$x_5 > \frac{\pi}{4}.$$

Supposing that $x_2 \geq \pi/4$, we see, from (34) and (35), that $\sum_{k=2}^{5} x_k >$ $> \pi$, which is impossible. This shows that

(36) $$x_2 < \frac{\pi}{4}.$$

Let us denote the number of the parameters $x_k \in (x_0\,, \pi/4]$ by m and their sum by y. From (34), (35) and (36) we deduce that $1 \leq m \leq 3$ and that

(37) $$x_0 < \frac{y}{m} < \frac{\pi}{4} < \frac{\pi - x_1 - y}{4 - m}.$$

Inequality (33) yields

$$\rho(P) = x_1 + \left(m\frac{\pi}{4} - y\right) + (\pi - y - x_1) - (4 - m)\frac{\pi}{4} \geq 2\varepsilon$$

or

(38) $$y \leq m\frac{\pi}{4} - \varepsilon.$$

Making use of Jensen's inequality we get

(39)

$$S(P) = \sum_{k=1}^{5} f(x_k) \geq f(x_1) + mf\left(\frac{y}{m}\right) + (4 - m)f\left(\frac{\pi - x_1 - y}{4 - m}\right)$$
$$\equiv T(x_1, y).$$

From (37) and (38) we conclude that

$$\frac{\partial T}{\partial y} = f'\left(\frac{y}{m}\right) - f'\left(\frac{\pi - x_1 - y}{4 - m}\right) < 0$$

and

(40)
$$T(x_1, y) \geq T\left(x_1, m\frac{\pi}{4} - \varepsilon\right)$$
$$= f(x_1) + mf\left(\frac{\pi}{4} - \frac{\varepsilon}{m}\right) + (4 - m)f\left(\frac{\pi}{4} + \frac{\varepsilon - x_1}{4 - m}\right).$$

The derivative of the right side is

$$f'(x_1) - f'(y_1) = (\cos^2 x_1 - \cos^2 y_1)\left(2 - \frac{1}{\cos^2 x_1 \cos^2 y_1}\right),$$

where

$$y_1 = \frac{\pi}{4} + \frac{\varepsilon - x_1}{4 - m}.$$

From

$$0 \le x_1 < \varepsilon < \frac{\pi}{4} < y_1 \le \frac{\pi}{4} + \varepsilon < \frac{\pi}{2}$$

we see that $f'(x_1) - f'(y_1) < 0$ and in consequence

$$T\left(x_1, m\frac{\pi}{4} - \varepsilon\right) > T\left(\varepsilon, m\frac{\pi}{4} - \varepsilon\right)$$

(41) $$= f(\varepsilon) + mf\left(\frac{\pi}{4} - \frac{\varepsilon}{m}\right) + (4-m)f\left(\frac{\pi}{4}\right).$$

The desired inequality (26) follows from (39) and (40) by applying Jensen's inequality to the right side of (41). This concludes the case $n = 5$.

n = 6. The supposition (24) restricts the variable x_1 to

$$0 \le x_1 \le \pi - 5x_0,$$

where $\pi - 5x_0 < x_0$. Making use of Jensen's inequality, we get

$$S(P) = \sum_{k=1}^{6} f(x_k) \ge f(x_1) + 5f\left(\frac{\pi - x_1}{5}\right) \equiv S_6(x_1).$$

In [2], p. 80 it was shown that

$$S_6(x_1) \ge S_6(0) = 5f\left(\frac{\pi}{5}\right) = 6.01035\ldots > S(\varepsilon_0) \ge S(\varepsilon)$$

for any $\varepsilon \in (0, \varepsilon_0]$.

This completes the proof of Thm. 1. $\Diamond$

The following statement is equivalent to Thm. 1.

Theorem 2. *Let $S(\varepsilon_0) = \delta_0 = 6.00874\ldots$, and $6 \le \delta \le \delta_0$. If P is a convex polygon with $S(P) \le \delta$, then*

$$\rho(P) \le 2S^{-1}(\delta),$$

with equality if and only if P is a pentagon with the parameters $x_1 = \varepsilon = S^{-1}(\delta)$, $x_2 = x_3 = x_4 = x_5 = \frac{\pi - \varepsilon}{4}$. (Here S^{-1} denotes the inverse function of S.)

Let δ and P be as before. If $S(P) \le \delta$, the combination of Thm. 2 and the lemma proves the existence of a square Q such that

(42) $$\rho^H(P, Q) \le 4S^{-1}(\delta).$$

We proceed to derive from (42) a simpler upper bound to $\rho^H(P, Q)$. The following properties of the function f defined by (18) can easily be verified.

$$f(0) = 0;\ f\left(\frac{\pi}{4}\right) = \frac{3}{2}; f'(0) = f'\left(\frac{\pi}{4}\right) = 2;\ f''(0) = 0;\ f''\left(\frac{\pi}{4}\right) = 2;$$

$$f'''(x) < 0 \quad \text{for} \quad 0 \le x \le \frac{\pi}{12}, \quad f'''(x) > 0 \quad \text{for} \quad x_0 \le x < \frac{\pi}{2}.$$

As an immediate consequence we note that

(43) $S(0) = 6;\ S'(0) = 0;\ S''(0) = \frac{1}{2};\ S'''(\varepsilon) < 0$ for $0 \le \varepsilon \le \varepsilon_0$.

With a constant C we consider the function

$$F(\varepsilon) = S(\varepsilon) - (6 + C\varepsilon^2)$$

and obtain from (43)

(44) $F(0) = F'(0) = 0;\ F''(0) = \frac{1}{2} - 2C;\ F'''(\varepsilon) < 0$ for $0 \le \varepsilon \le \varepsilon_0$.

By taking $C = \frac{1}{4}$ we see, on the one hand, that

(45) $$S(\varepsilon) \le 6 + \frac{1}{4}\varepsilon^2 \qquad (0 \le \varepsilon \le \varepsilon_0).$$

From (44) it follows that F' is strictly concave in $[0, \varepsilon_0]$. Now we choose $C = 1/8$ and calculate $F'(\varepsilon_0) = S'(\varepsilon_0) - \varepsilon_0/4 = -0.02724\ldots <$ < 0. Therefore, F' passes from positive to negative values, and F attains its minimum in $\varepsilon = 0$ or $\varepsilon = \varepsilon_0$. Since $F(0) = 0$ and $F(\varepsilon_0) =$ $= 0.000177\ldots > 0$, [4] we have, on the other hand,

(46) $$S(\varepsilon) \ge 6 + \frac{1}{8}\varepsilon^2 \qquad (0 \le \varepsilon \le \varepsilon_0).$$

Now, (45) and (46) imply the desired inequalities

(47) $$2\sqrt{\delta - 6} \le S^{-1}(\delta) \le 2\sqrt{2(\delta - 6)}.$$

Combining (42) and (47), we obtain the following corollary.

Corollary. *Let* $6 \le \delta \le \delta_0 = 6.00874\ldots$ *and let* P *be a convex polygon with* $S(P) \le \delta$. *There exists a square* Q *such that*

(48) $$\rho^H(P, Q) \le 8\sqrt{2(\delta - 6)}.$$

Set $S^{-1}(\delta) = \varepsilon$, and let $\overline{P}$ be the pentagon with the parameters $x_1 = \varepsilon$, $x_2 = x_3 = x_4 = x_5 = (\pi - \varepsilon)/4$, so that $S(\overline{P}) = \delta$. From Theorem 2, (10) and (47) we conclude that

(49) $$\rho^H(\overline{P}, Q) \ge 4\pi^{-1}\left(\cos\frac{\pi}{8} - \cos\frac{\pi}{4}\right)\sqrt{\delta - 6}$$

for every square Q. The numerical constants in (48) and (49) are $11.313\ldots$ and $0.276\ldots$, respectively.

References

[1] ACZÉL, J. and FUCHS, L.: A minimum-problem on areas of inscribed and circumscribed polygons of a circle, *Compositio Math.* **8** (1950), 61–67.

[2] FLORIAN, A.: On the area sum of a convex polygon and its polar reciprocal, *Math. Pannonica* **6**/1 (1995), 77–84.

[3] FLORIAN, A.: On the area sum of a convex set and its polar reciprocal, *Math. Pannonica* **7**/2 (1996), 171–176.

[4] GROEMER, H.: Stability of Geometric Inequalities, in: *Handbook of Convex Geometry* (eds. P. M. Gruber and J. M. Wills), North-Holland, Amsterdam 1993, 125–150.

[5] KUIPERS, L. and MEULENBELD, B.: Two minimum-problems. I, II, III., *Nederl. Akad. Wetensch. Proc. Ser. A* **54** = *Indagationes Math.* **13** (1951), 135–142, 143–151, 237–242.

[6] TROST, E.: Beweis einer Minimaleigenschaft des Quadrates, *Elem. Math.* **6** (1951), 26–28.