Mathematica Pannonica
8/2 (1997), 285–292

ASYMPTOTIC FORMULAE CONCERNING ARITHMETICAL FUNCTIONS DEFINED BY CROSS-CONVOLUTIONS, VI. A–k-FREE INTEGERS

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Received: February 1997

MSC 1991: 11 A 25, 11 N 37

Keywords: Narkiewicz's regular convolution, k-free integers, unitarily k-free integers, asymptotic formula.

Abstract: We define the $A - k$-free integers as a common generalization of the k-free and unitarily k-free integers in terms of Narkiewicz's regular A-convolutions. We establish asymptotic formulae concerning arithmetical functions involving $A - k$-free integers if A is a cross-convolution, investigated in our previous papers.

1. Introduction

I. First we list some of the more important **symbols** and **definitions** we need in this article.

f, g: arithmetical functions,
$p \in \mathbb{P}$: a prime,
$n \in \mathbb{N}$: a natural number,
$\phi(n)$: Euler's arithmetical function,

$\omega(n)$: the number of the distinct prime divisors of n,

$\gamma(n)$: the core of n (the product of the prime divisors of n),

$A(n)$: a set of the positive divisors of n,

A-divisors of n: the elements of the set $A(n)$,

A-convolution: $(f *_A g)(n) = \sum_{d \in A(n)} f(d)g(n/d)$,

regular A-convolution (W. Narkiewicz [7]): the set of arithmetical functions is a commutative ring with unity with respect to ordinary addition and the A-convolution, the A-convolution of multiplicative functions is multiplicative, the function I, defined by $I(n) = 1$ for all $n \in \mathbb{N}$, has an inverse μ_A with respect to the A-convolution and $\mu_A(p^a) \in \{-1, 0\}$ for every prime power p^a,

μ_A: the generalized Möbius function in the definition above,

D-convolution: Dirichlet convolution with $D(n) = \{d \in \mathbb{N} : d|n\}$,

U-convolution: unitary convolution with $U(n) = \{d \in \mathbb{N} : d||n\} = \{d \in \mathbb{N} : d|n, (d, n/d) = 1\}$,

$t = t_A(p^a)$: the type of p^a with respect to a regular convolution A, i.e. that divisor of a, for which $A(p^{it}) = \{1, p^t, p^{2t}, \dots, p^{it}\}$ for all $i \in \{0, 1, \dots, a/t\}$.

cross-convolution: a regular convolution A such that for every $p \in \mathbb{P}$, $t_A(p^a) = 1$ for every $a \in \mathbb{N}$ or $t_A(p^a) = a$ for every $a \in \mathbb{N}$,

$P_A = P := \{p \in \mathbb{P} : t_A(p^a) = 1\}$ for a cross-convolution A,

$Q_A = Q := \{p \in \mathbb{P} : t_A(p^a) = a\}$ for a cross-convolution A,

$(P), (Q)$: the multiplicative semigroups generated by 1 and P or Q, for a cross-convolution A,

n_P, n_Q: an element of (P) and (Q), respectively, such that $n = n_P n_Q$,

$\sigma^*{}_r(n)$: the sum of r-th powers of the unitary divisors of n,

$\zeta_T(s) = \prod_{p \in T} (1 - 1/p^s)^{-1}$ for some $T \subseteq \mathbb{P}$,

$\zeta(s) := \zeta_{\mathbb{P}}(s)$ the Riemannian Zeta function.

II. New definitions and symbols

n is an $A - k$*-free integer* (for a regular convolution A and for $k \in \mathbb{N}$, $k \geq 2$): the greatest k-th power A-divisor of n is 1;

$\alpha_{A,k}$: the *characteristic function* of the set of $A - k$-free integers,

$$\sigma_{A_1,s}^{A,k}(n) = \sum_{d \in A_1(n)} \alpha_{A,k}(d) d^s$$

the sum of s-th powers of the $A - k$-free A_1-divisors of n (for a regular convolution A_1).

III. Assertions

Assertion 1. (W. Narkiewicz [7]) *An A-convolution is regular if and only if* $A(mn) = \{de : d \in A(m),\ e \in A(n)\}$ *for every coprime pairs* $m, n \in \mathbb{N}$ *and* $t_A(p^a)$ *exists for every prime power* p^a.

Some further assertions which can be verified easily:

Assertion 2. *The D and U-convolutions are cross-convolutions,* $P_D = \mathbb{P}, Q_D = \emptyset, P_U = \emptyset$ *and* $Q_U = \mathbb{P}$.

Assertion 3. *For a regular convolution A the function* μ_A *is multiplicative and for all prime powers* p^a *we have*

$$\mu_A(p^a) = \begin{cases} -1, & \text{if } t_A(p^a) = a \\ 0, & \text{otherwise.} \end{cases}$$

$\mu_D = \mu$ *is the classical Möbius function and* $\mu_U(n) = (-1)^{\omega(n)}$ *is the Liouville function.*

Assertion 4. *If A is a cross-convolution, then* $A(n) = \{d \in \mathbb{N} : d|n, (d, n/d) \in (P)\}$.

Assertion 5. *Let A be a regular convolution. An integer*

$$n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$$

is $A - k$*-free if and only if for the type* $t_i = t_A(p_i^{a_i})$ *of* $p_i^{a_i}$ *one of the following conditions holds:*

(i) $k \not| t_i, k \not| 2t_i, \dots, k \not| a_i$ *for all* $i \in \{1, 2, \dots, r\}$,

(ii) $a_i/t_i \leq k/(k, t_i) - 1$ *for all* $i \in \{1, 2, \dots, r\}$,

(iii) $a_i + t_i \leq [k, t_i]$ *for all* $i \in \{1, 2, \dots, r\}$.

The integer n is $D - k$*-free if it is k-free, i.e.* $a_i < k$ *for every* $i \in \{1, 2, \dots, r\}$ *and it is* $U - k$*-free if it is unitarily k-free or k-skew, i.e.* $k \not| a_i$ *for every* $i \in \{1, 2, \dots, r\}$.

Assertion 6. *If A is a cross-convolution, then* $\alpha_{A,k}(n) = \alpha_{D,k}(n_P) \cdot \alpha_{U,k}(n_Q)$ *for every* $n \in \mathbb{N}$. *Hence if A is a cross-convolution, then n is* $A - k$*-free if and only if* n_P *is k-free and* n_Q *is unitarily k-free.*

For further results see also P. J. McCarthy [6], V. Sita Ramaiah [8] (regular convolutions), [13], [14], [15], [16], [17] (asymptotic properties of arithmetical functions defined by cross-convolutions) and E. Cohen [2], [3] (unitarily k-free integers).

2. Results

Our main result is the following theorem:

Theorem. *Let A be a cross-convolution and let* $k \in \mathbb{N}, k \geq 2, u \in \mathbb{N}$ *and* $s \geq 0$. *For an arbitrary* $\varepsilon \in (0, \frac{1}{k})$ *we have*

$$\sum_{\substack{n\le x\\(n,u)=1}} \alpha_{A,k}(n)n^s = \frac{C_{A,k}}{s+1} f_{A,k}(u)x^{s+1} + O(x^{s+1/k}\sigma^*{}_{-\varepsilon}(u)),$$

where

$$C_{A,k} = \frac{1}{\zeta_P(k)} \prod_{p\in Q}\left(1-\frac{p-1}{p(p^k-1)}\right)$$

and

$$f_{A,k}(u) = \prod_{p|u}\left(1-\frac{1}{p}\right)\prod_{p|u_P}\left(1-\frac{1}{p^k}\right)^{-1}\prod_{p|u_Q}\left(1-\frac{p-1}{p(p^k-1)}\right)^{-1}.$$

Remark 1. The following are some properties of $f_{A,k}$ and $C_{A,k}$: $f_{A,k}(n) = f_{A,k}(\gamma(n)) \le 1$ for all $n\in\mathbb{N}$ and

$$C_{A,k} = \frac{\zeta_Q(k)}{\zeta_P(k)} \prod_{p\in Q}\left(1-\frac{2}{p^k}+\frac{1}{p^{k+1}}\right).$$

The Theorem has important corollaries.

Corollary 1. *The asymptotic density of the $A-k$-free integers coprime to u is $C_{A,k}f_{A,k}(u)$. The asymptotic density of the $A-k$-free integers is $C_{A,k}$.*

Remark 2. If A and A_1 are two cross-convolutions and $A(n)\subseteq A_1(n)$ for every $n\in\mathbb{N}$, then $\alpha_{A_1,k}(n)\le\alpha_{A,k}(n)$ and $C_{A_1,k}\le C_{A,k}$ for every $n,k\in\mathbb{N}, k\ge 2$. In particular,

$$C_{D,k} = \frac{1}{\zeta(k)} \le C_{A,k} \le C_{U,k} = \prod_{p\in\mathbb{P}}\left(1-\frac{p-1}{p(p^k-1)}\right)$$

for every cross-convolution A.

Corollary 2. *Let A and A_1 be cross-convolutions with $P_{A_1}=P_1$, $Q_{A_1}=Q_1$ and let $k\in\mathbb{N}, k\ge 2, u\in\mathbb{N}$ and $s\ge 0$. For an arbitrary $\varepsilon\in(0,1/k)$ we have*

$$\sum_{\substack{n\le x\\(n,u)\in(P_1)}} \alpha_{A,k}(n)n^s = \frac{C_{A,k}}{s+1} f_{A,k}(u_{Q_1})x^{s+1} + O(x^{s+1/k}g(u)),$$

with

$$g(u) = \begin{cases} 1, & \text{if } Q_1 \text{ is finite}\\ \sigma^*{}_{-\varepsilon}(u), & \text{otherwise.}\end{cases}$$

Corollary 3. *Let A and A_1 be cross-convolutions and let $k\in\mathbb{N}, k\ge 2$ and $s>0$. Then*

$$\sum_{n\le x}\sigma^{A,k}_{A_1,s}(n)=C_{A,k}E_{A,A_1,k,s}\frac{x^{s+1}}{s+1}+O(R_{k,s}(x)),$$

where

$$E_{A,A_1,k,s}=\zeta(s+1)\prod_{p\in Q\cap Q_1}\left(1-\frac{1}{p^{s+1}}+\frac{(p-1)(p^k-1)}{p^{s+1}(p^{k+1}-2p+1)}\right)\times$$
$$\times\prod_{p\in Q_1\setminus Q}\left(1-\frac{1}{p^{s+1}}+\frac{p^k(p-1)}{p^{s+2}(p^k-1)}\right)$$

and $R_{k,s}(x)=x^{s+1/k}$ $(s+1/k>1)$, $x\log x$ $(s+1/k=1)$, x $(s+1/k<1)$.

Remark 3. The Theorem is a generalization of some known results as e.g. L. Gegenbauer [5] ($A=D, u=1, s=0$), D. Suryanarayana [9], Th. 1 ($A=D, s=0$), E. Cohen [3], Th. 4.2, [4] ($A=U, u=1, s=0$), [2], Th. 6.1, Cor. 6.1.1 ($A=U, u=1, s=0, k=2$).

For $A=D$ and $A=U$ the error terms can be improved, see [10], [11].

Special cases of Cor. 3 were investigated in the literature, see for example E. Cohen [2], Cor. 5.1.1 and J. Chidambaraswamy [1], Cor. B/(ii), case $A=D, A_1=U, k=2, s\ge 1$.

3. Proof of the Theorem

The proof of Theorem is based on the following 4 lemmas.

Lemma 1. *If A is a cross-convolution and $k\in\mathbb{N}, k\ge 2$, then*

$$\alpha_{A,k}(n)=\sum_{d^k\in A(n)}\mu_A(d)=\sum_{\substack{d^ke=n\\(d,e)\in(P)}}\mu_A(d)$$

holds for every $n\in\mathbb{N}$.

Proof. By the multiplicativity it is enough to prove that

$$\alpha_{A,k}(p^a)=\begin{cases}\sum_{d^k|p^a}\mu(d), & \text{if } p\in P\\ \sum_{d^k||p^a}\mu_U(d), & \text{if } p\in Q\end{cases}$$

for every prime power p^a, see Assertions 6 and 4. $\diamondsuit$

Lemma 2 ([12], Lemma 2.1). *If $s\ge 0$ and $u\in\mathbb{N}$, then for every $\varepsilon\in[0,1)$ we have*

$$\sum_{\substack{n\le x\\(n,u)=1}}n^s=\frac{\phi(u)}{u(s+1)}x^{s+1}+O(x^{s+\varepsilon}\sigma^*{}_{-\varepsilon}(u)).$$

Lemma 3. *Let A be a cross-convolution, $k \in \mathbb{N}, k \geq 2$ and $u \in \mathbb{N}$. Then we have*

$$\sum_{\substack{n=1\\(n,u)=1}}^{\infty} \frac{\mu_A(n)\phi(n_Q)}{n^k n_Q} = \frac{1}{\zeta_P(k)} \prod_{p|u_p} \left(1-\frac{1}{p^k}\right)^{-1} \prod_{\substack{p\in Q\\ p\not| u}} \left(1-\frac{p-1}{p(p^k-1)}\right).$$

Proof. The series is well defined as $\sum_{n=1}^{\infty} 1/n^2$ is a majorant series. Furthermore

$$\sum_{n=1}^{\infty} \frac{\mu_A(n)\phi(n_Q)\delta((n,u))}{n^k n_Q}$$

with

$$\delta(n) = \begin{cases} 1, & \text{if } n=1 \\ 0, & \text{if } n>1 \end{cases}$$

has an Euler-product representation. ◊

Lemma 4 ([4]). *If $k \in \mathbb{N}, k \geq 2$, then for every $\varepsilon \in (0, 1/k)$ we have*

$$\sum_{n\leq \sqrt[k]{x}} \frac{\sigma^*_{-\varepsilon}(n)}{n^{k\varepsilon}} = O(x^{1/k-\varepsilon}).$$

Proof of the Theorem. Observe that for a cross-convolution A we have $(m,n) \in (P)$ if and only if $(m, \gamma(n_Q)) = 1$. Therefore, by Lemmas 1, 2, 3 we obtain

$$\sum_{\substack{n\leq x\\(n,u)=1}} \alpha_{A,k}(n) n^s = \sum_{\substack{d^k e=n\leq x\\(d,e)\in(P)\\(d^k e,u)=1}} \mu_A(d) d^{ks} e^s = \sum_{\substack{d\leq \sqrt[k]{x}\\(d,u)=1}} \mu_A(d) d^{ks} \sum_{\substack{e\leq x/d^k\\(e,u\gamma(d_Q))=1}} e^s =$$

$$= \sum_{\substack{d\leq \sqrt[k]{x}\\(d,u)=1}} \mu_A(d) d^{ks} \left(\frac{\phi(u\gamma(d_Q))}{(s+1)u\gamma(d_Q)} \left(\frac{x}{d^k}\right)^{s+1} + O\left(\left(\frac{x}{d^k}\right)^{s+\varepsilon} \sigma^*_{-\varepsilon}(u\gamma(d_Q))\right)\right) =$$

$$= \frac{x^{s+1}\phi(u)}{(s+1)u} \sum_{\substack{d\leq \sqrt[k]{x}\\(d,u)=1}} \frac{\mu_A(d)\phi(d_Q)}{d^k d_Q} + O\left(\sum_{\substack{d\leq \sqrt[k]{x}\\(d,u)=1}} \left(\frac{x}{d^k}\right)^{s+\varepsilon} d^{ks} \sigma^*_{-\varepsilon}(u\gamma(d_Q)) \right) =$$

$$= \frac{\phi(u)}{(s+1)u\zeta_P(k)} \prod_{p|u_p} \left(1-\frac{1}{p^k}\right)^{-1} \prod_{\substack{p\in Q\\ p\not| u}} \left(1-\frac{p-1}{p(p^k-1)}\right) x^{s+1} +$$

$$+ O\left(x^{s+1} \sum_{d>\sqrt[k]{x}} \frac{1}{d^k} \right) + O\left(\sigma^*_{-\varepsilon}(u) x^{s+\varepsilon} \sum_{d\leq \sqrt[k]{x}} \frac{\sigma^*_{-\varepsilon}(d)}{d^{k\varepsilon}} \right).$$

We apply Lemma 4 to derive the error term. ◊

Proof of Cor. 1. We apply the Theorem for the special cases $s = 0$ and $s = 0, u = 1$, respectively. ◊

Proof of Cor. 2. As $(n, u) \in (P_1)$ if and only if $(n, \gamma(u_{Q_1})) = 1$ and $f_{A,k}(u) = f_{A,k}(\gamma(u))$, this is an other special case of the Theorem. ◊

Proof of Cor. 3. As

$$\sum_{n\le x} \sigma_{A_1,s}^{A,k}(n) = \sum_{d\le x} \sum_{\substack{e\le x/d \\ (e,d)\in(P_1)}} \alpha_{A,k}(e)e^s,$$

it can be derived from Cor. 2. The series

$$\sum_{n=1}^{\infty} \frac{f_{A;k}(n_{Q_1})}{n^{s+1}}$$

is well defined as $\sum_{n=1}^{\infty} 1/n^{s+1}$ is a majorant series and we apply the Euler product representation to derive the final form of $E_{A,A_1,k,s}$. ◊

Acknowledgement. The author is grateful to the referee for useful suggestions on the presentation of this paper.

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