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ADJOINTS, TORSION THEORY AND PURITY

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Abstract: This is a study of the sufficiency of pure injectives by means of torsion theory within the category of finitely presented contravariant functors.

Some 30 years ago R. B. Warfield established [13] that, over an arbitrary ring, any module could be embedded purely into an algebraically compact (i.e. pure injective) module. This result is known as 'sufficiency of pure injectives'.

In this paper we seek to further analyze the original result and generalize to M-purity where M is a collection of finitely presented modules. Warfield's result is then the 'global' case. The aim is to give a constructive proof based on 'local' cases (when M consists of a single module). As a consequence we obtain a new proof of Warfield's result and some added insight into the structure of M-pure injectives.

The natural context for the problem lies within various torsion theories in the category of finitely presented contravariant functors which vanish on projective modules (originally called 'coherent functors' by M. Auslander [4]). In particular torsion-torsion free (T.T.F.) theory plays a prominent role [7].

At a crucial stage of the development a well-known isomorphism due to M. Auslander [1] is required. Thus an interesting mathematical connection is revealed between R. B. Warfield and M. Auslander. This paper is dedicated to their memory.

After presenting these results at Miskolc, July 1996, William Crawley-Boevey informed me that results concerning M-purity appeared in the paper 'On Γ-pure injective modules', R. Kiełpiński, Bull. de L'Académie Polonaise des Sciences, Serie des sciences math. astr. et phys. 15 (1967), 127–131. His approach involves generalizing the methods of Y. Łoś and Y. M. Maranda from the abelian group setting to modules over an arbitrary ring.

1. Background

Throughout this paper, Λ will denote an arbitrary ring and Mod Λ the category of left Λ-modules. Let M be a set of finitely presented modules.

A short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is called M-pure (respectively, $A \longrightarrow B$ is a M-pure monomorphism and $B \longrightarrow C$ is a M-pure epimorphism) if the natural maps $\mathrm{Hom}_\Lambda(M, B) \longrightarrow \mathrm{Hom}_\Lambda(M, C)$ are onto for each M in M. An object N is M-pure injective if the natural maps $\mathrm{Hom}_\Lambda(B, N) \longrightarrow \mathrm{Hom}_\Lambda(A, N)$ are onto for every M-pure monomorphism $A \longrightarrow B$. Mod Λ is said to have sufficient M-pure injectives if for each A in Mod Λ there exists a M-pure injective A′ and a M-pure monomorphism $A \longrightarrow A'$. We will investigate when Mod Λ has sufficient M-pure injectives.

Let (T, F) be a torsion theory in an abelian category A. An object D is called divisible (with respect to (T, F)) if $\mathrm{Ext}^1_\Lambda(T, D)$ for each T in T. (T, F) is called localizing if for each A in A there exists a divisible and torsion free (i.e. in F) object D, along with a morphism $A \longrightarrow D$ whose kernel and cokernel are torsion (i.e. in T). Dually one has the notion of a colocalizing torsion theory. A triple (T, S, R) is a T.T.F. (torsion-torsion free) theory if the pairs (T, S) and (S, R) are torsion theories.

For the remainder of the paper A will denote the abelian category consisting of finitely presented contravariant functors from Mod Λ to abelian groups, which vanish on projective modules. To each such functor F in A there is associated a short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, such that F is the cokernel of $\mathrm{Hom}_\Lambda(-, B) \longrightarrow \mathrm{Hom}_\Lambda(-, C)$. In such a way A can be identified with the category of short exact sequences with the subcategory of split short exact sequences factored out [8]. The injectives of A have the form $\mathrm{Ext}^1_\Lambda(-,\mathrm{X})$, [3] [8]. The projectives have the form $\pi(-,\mathrm{X})$, where $\pi(\mathrm{Y},\mathrm{X}) = {}^{\mathrm{Hom}_\Lambda(\mathrm{Y},\mathrm{X})}/_{P(\mathrm{Y},\mathrm{X})}$ and $P(\mathrm{Y},\mathrm{X})$ consists of those morphisms Y to X which factor through a projective module.

Let S_M be the full subcategory of A consisting of those functors arising from M-pure short exact sequences, equivalently those functors vanishing on M. Define R_M and T_M by H is in R_M if and only if $\mathrm{Hom}_A(\mathrm{G},\mathrm{H}) = 0$ all G in S_M, and F is in T_M if and only if $\mathrm{Hom}_A(\mathrm{F},\mathrm{G}) = 0$ all G in S_M.

(T_M, S_M) will be a colocalizing torsion theory in A, and T_M is generated by the functors $\pi(-,\mathrm{M})$, M in M, [7]. We seek to determine when (S_M, R_M) is a localizing torsion theory.

Regarding M as a full subcategory of Mod Λ one can form the projective stabilization of M, i.e. the factor category M/P. The objects of M/P are those of M, but $\mathrm{Hom}_{M/P}(\mathrm{M}',\mathrm{M}) = \pi(\mathrm{M}',\mathrm{M})$. Let $\mathbf{S}_M$ be the restriction functor $A \longrightarrow (M/P^{op}, \mathrm{Ab})$, so that $\mathbf{S}_M\mathrm{F}$ is the restriction of F to M/P. $\mathbf{S}_M$ is an exact functor with a fully faithful left adjoint $\mathbf{T}_M$. $\mathbf{T}_M$ is the unique right exact extension of the assignment which sends the representable functor $\mathrm{Hom}_{M/P}(-,\mathrm{M})$ to $\pi(-,\mathrm{M})$ in A, [7]. We seek to determine when $\mathbf{S}_M$ has a fully faithful right adjoint $\mathbf{R}_M$.

The following variation of the Yoneda Lemma holds for A: $\mathrm{Hom}_A(\pi(-,\mathrm{X}),\mathrm{F}) \cong \mathrm{F}(\mathrm{X})$. Now suppose $\mathbf{R} = \mathbf{R}_M$ is right adjoint to $\mathbf{S} = \mathbf{S}_M$ then $\mathbf{R}\mathrm{F}(\mathrm{X}) \cong \mathrm{Hom}_A(\pi(-,\mathrm{X}),\mathbf{R}\mathrm{F}) \cong \mathrm{Hom}_C(\mathbf{S}\pi(-,\mathrm{X}),\mathrm{F})$ where C is $(M/P^{op}, \mathrm{Ab})$. Hence, up to natural isomorphism, $\mathbf{R}\mathrm{F}(X)$ will be the group of natural transformations from $\pi(-,X)$, as a functor on M/P, to F. One could then use this as the definition of $\mathbf{R}\mathrm{F}$ provided such a definition yielded a finitely presented functor (since it will clearly vanish on projectives). If one can use this definition then for M in M/P, $\mathbf{SR}\mathrm{F}(\mathrm{M}) = \mathrm{Hom}_C(\mathbf{S}\pi(-,\mathrm{M}),\mathrm{F}) \cong \mathrm{F}(\mathrm{M})$ by the Yoneda Lemma in C, since then $\mathbf{S}\pi(-,\mathrm{M})$ is just the hom functor. Hence the counit $\mathbf{SR} \longrightarrow$ $\longrightarrow$ I will be a natural isomorphism and consequently [10] $\mathbf{R}$ will be fully faithful.

2. Results

The three investigations introduced in the background section are tied together by the following theorem.

Theorem A. *The following are equivalent:*

(1) *There are sufficient M-pure injectives in* $\mathrm{Mod}\,\Lambda$.
(2) *The pair* (S_M, R_M) *is a localizing torsion theory in* A.
(3) *The functor* $\mathbf{S}_M : A \longrightarrow (M/P^{op}, \mathrm{Ab})$ *has a right adjoint.*

Proof. Since M will be fixed, the subscript M will be omitted throughout the proof.

(1) $\Longrightarrow$ (2) Let $\underline{\mathrm{F}}$ be an element of A and $0 \longrightarrow \mathrm{A} \longrightarrow \mathrm{B} \longrightarrow \longrightarrow C \longrightarrow 0$ a short exact sequence yielding $\underline{\mathrm{F}}$. $\underline{\mathrm{F}}$ embeds in the injective $\mathrm{Ext}^1_\Lambda(-, \mathrm{A})$. Let $\mathrm{A} \longrightarrow \mathrm{A}'$ be a M-pure monomorphism with A' M-pure injective. Consider the compositional map $\mathrm{F} \longrightarrow \mathrm{Ext}^1_\Lambda(-, \mathrm{A}) \longrightarrow \longrightarrow \mathrm{Ext}^1_\Lambda(-, \mathrm{A}')$. The kernel of this map is the functor associated with the short exact sequence $0 \longrightarrow \mathrm{A} \longrightarrow \mathrm{B} \oplus \mathrm{A}' \longrightarrow \mathrm{N} \longrightarrow 0$ and is clearly in S since $A \longrightarrow \mathrm{A}'$ is M-pure. The image of this map is in R since it is a subobject of $\mathrm{Ext}^1_\Lambda(-, A')$ (which is clearly in R and R is closed under subobjects). Hence for each $\underline{\mathrm{F}}$ in A there exists a short exact sequence in A of the form $0 \longrightarrow \underline{\mathrm{K}} \longrightarrow \underline{\mathrm{F}} \longrightarrow \underline{\mathrm{L}} \longrightarrow 0$ with $\underline{\mathrm{K}}$ in S and $\underline{\mathrm{L}}$ in R. It follows that (S, R) is a torsion theory in A. Now suppose $\underline{\mathrm{F}}$ is in R, then $\underline{\mathrm{F}}$ would be zero and hence $\underline{\mathrm{F}}$ embeds in $\mathrm{Ext}^1_\Lambda(-, \mathrm{A}')$ which is injective and lies in R. It follow from general torsion theory [6] [8] [11] [12] that (S, R) is localizing.

(2) $\Longrightarrow$ (3) This implication utilizes T.T.F. theory, see [7] for specific details.

Assuming (S, R) is a localizing torsion theory then (T, S, R) will be a T.T.F. theory in A and localization restricted to the full subcategory $T \cap R$ will be the right adjoint to $\mathbf{S}' : A \longrightarrow T \cap R$ where $\mathbf{S}' = \mathrm{r}'\mathrm{t}$, with r' the torsion free functor with respect to (S, R) and t the torsion functor with respect to (T, S). Furthermore $T \cap R$ is equivalent to $(M/P^{op}, \mathrm{Ab})$ and $\mathbf{S}$ will be the composition $A \longrightarrow T \cap R \cong (M/P^{op}, \mathrm{Ab})$. Hence $\mathbf{S}$ has a right adjoint.

(3) $\Longrightarrow$ (1) For each A the unit of the adjoint pair $(\mathbf{S}, \mathbf{R})$ yields an exact sequence $0 \longrightarrow \underline{\mathrm{K}} \longrightarrow \mathrm{Ext}^1_\Lambda(-, \mathrm{A}) \longrightarrow \mathbf{RS}(\mathrm{Ext}^1_\Lambda(-, \mathrm{A}))$. Since the counit is an isomorphism (see background section) it follows from the triangular identities for adjoints that $\mathbf{S}$ converts each unit map to an isomorphism, and since $\mathbf{S}$ is also exact it follows that $\mathbf{S}\underline{\mathrm{K}} = 0$. Thus $\underline{\mathrm{K}}$ is in S, since S can be regarded as those functors in A which vanish on M.

Now the functor category $(M/P^{op}, \mathrm{Ab})$ has sufficient injectives, hence $\mathbf{S}(\mathrm{Ext}^1_\Lambda(-,\mathrm{A}))$ embeds in some injective $\underline{\mathrm{E}}$. Since $\mathbf{R}$ is a right adjoint it will be left exact, and since $\mathbf{S}$ is exact $\mathbf{R}$ will preserve injectives. Thus one obtains an embedding of $\mathbf{RS}(\mathrm{Ext}^1_\Lambda(-A))$ into the injective $\mathbf{R}\underline{\mathrm{E}}$ of A. But this injective must then have the form $\mathrm{Ext}^1_\Lambda(-,\mathrm{A}')$ for some A'. Now any object in the image of $\mathbf{R}$ will be in R, since for $\underline{\mathrm{F}}$ in S, $\mathbf{S}\underline{\mathrm{F}} = 0$ and hence $\mathrm{Hom}_{\mathrm{A}}(\underline{\mathrm{F}}, \mathbf{R}\underline{G}) \cong \mathrm{Hom}_{\mathrm{C}}(\mathbf{S}\underline{\mathrm{F}}, \mathrm{G}) = 0$. Thus $\mathrm{Ext}^1_\Lambda(-,\mathrm{A}')$ is in R and consequently A' is M-pure injective. Finally $\underline{\mathrm{K}}$ can be computed as the kernel of $\mathrm{Ext}^1_\Lambda(-,\mathrm{A}) \longrightarrow \mathrm{Ext}^1_\Lambda(-,\mathrm{A}')$ and hence arises from a short exact sequence of the form $0 \longrightarrow \mathrm{A} \longrightarrow \mathrm{A}' \oplus \mathrm{I} \longrightarrow \mathrm{N} \longrightarrow 0$, where $\mathrm{A} \longrightarrow \mathrm{I}$ is a monomorphism with I an injective module. Since $\underline{\mathrm{K}}$ is in S this shows A can be embedded M-purely into the M-pure injective $\mathrm{A}' \oplus \mathrm{I}$. ◊

Corollary B. *There are sufficient M-pure injectives for any collection M of finitely presented modules.*

Proof. Let $\mathrm{mod}\,\Lambda$ denote the category of all finitely presented Λ-modules. The restriction functor $\mathbf{S}_M$ factors as successive restrictions $A \longrightarrow (\mathrm{mod}\,\Lambda/P^{op}, \mathrm{Ab}) \longrightarrow (M/P^{op}, \mathrm{Ab})$. Since by Warfield's result [13] there are sufficient pure injectives the first restriction has a right adjoint by the theorem. The second restriction has a right adjoint by a general result on functor categories [7, Ex. 3.5] [2, Sec. 3]. Hence $\mathbf{S}_M$ has a right adjoint and the result follows from the theorem. ◊

We will proceed now to establish the sufficiency of M-pure injectives without recourse to Warfield's 'global' result. When M equals $\mathrm{mod}\,\Lambda$, Warfield's result is reobtained. When M consists of a single finite presented module M we will use the terminology M-pure and M-pure injective. The existence of sufficient M-pure injectives is then a 'local' result. The first step is to establish that 'local' results can be patched together.

Proposition C. *Given a collection M of finitely presented modules, if there are sufficient* M*-pure injectives for each* M *in M then there are sufficient M-pure injectives.*

Proof. Given any A, let $\mathrm{A} \longrightarrow \mathrm{A_M}$ be a M-pure monomorphism with $\mathrm{A_M}$ a M-pure injective. Consider $A \longrightarrow \prod_{\mathrm{M} \in M} \mathrm{A_M}$. Now S_M is contained in S_{M} so any M-pure injective is also M-pure injective. Then the product $\prod_{\mathrm{M} \in M} \mathrm{A_M}$ of M-pure injectives is again M-pure injective.

Now $\mathrm{A} \longrightarrow \mathrm{A_M}$ factors via the projection map as $\mathrm{A} \longrightarrow \prod_{\mathrm{X} \in M} \mathrm{A_X} \longrightarrow \mathrm{A_M}$, and hence $\mathrm{A} \longrightarrow \prod_{\mathrm{X} \in M} \mathrm{A_X}$ is M-pure for each

M. Thus the functor in A associated to the short exact sequence $0 \longrightarrow \longrightarrow \mathrm{A} \longrightarrow \prod_{\mathrm{M} \in M} \mathrm{A_M} \longrightarrow \mathrm{N} \longrightarrow 0$ will be in S_M for each M. Now $S_M = \bigcap_{\mathrm{M} \in M} S_{\mathrm{M}}$, so the functor actually lies in S_M and hence $\mathrm{A} \longrightarrow \longrightarrow \prod_{\mathrm{M} \in M} \mathrm{A_M}$ is M-pure. ◊

In considering the local case of a singleton M there will be an equivalence of categories between $(M/P^{op}, \mathrm{Ab})$ and $\mathrm{Mod}\,\Gamma$ the category of right Γ-modules where $\Gamma = \pi(\mathrm{M},\mathrm{M})$ (also denoted frequently as $\underline{\mathrm{End}\,\mathrm{M}}$). This equivalence is given by $\mathrm{F} \mapsto \mathrm{F(M)}$ where F(M) is a right Γ-module via $x \cdot g = \mathrm{F}(g)(x)$ for x in F(M), g in Γ. As indicated in the background section the required right adjoint **R** (again suppressing the subscript M) if it exists must satisfy in a natural way $\mathbf{R}\mathrm{F(X)} \cong \mathrm{Hom_C}(\mathbf{S}\pi(-,\mathrm{X}),\mathrm{F})$ where again $C = (M/P^{op}, \mathrm{Ab})$. Passing to $\mathrm{Mod}\,\Gamma$ it follows that $\mathbf{R}\mathrm{N(X)} \cong \mathrm{Hom}_\Gamma(\pi(\mathrm{M},\mathrm{X}),\mathrm{N})$ for N a right Γ-module. Consequently the existence of a right adjoint amounts to establishing that the functor $\mathrm{Hom}_\Gamma(\pi(\mathrm{M},-),\mathrm{N})$ is in A for any Γ-module N. It will suffice to verify this for injective Γ-modules since if $0 \longrightarrow \longrightarrow \mathrm{N} \longrightarrow \mathrm{I_1} \longrightarrow \mathrm{I_2} \cdots$ is an injective coresolution of N then **R**N can be computed in A as the kernel of $\mathbf{R}\mathrm{I_1} \longrightarrow \mathbf{R}\mathrm{I_2}$. Fortunately this final step is established by means of a frequently used isomorphism due to M. Auslander [1, Prop. 3.3]. Namely for I injective right Γ-module and M a left finitely presented Λ-module there exists a natural isomorphism $\mathrm{Hom}_\Gamma(\pi(\mathrm{M},-),\mathrm{I}) \cong \mathrm{Ext}^1_\Lambda(-,\mathrm{Hom}_\Gamma(\mathrm{TrM},\mathrm{I}))$ where TrM is the transpose of M (regarded as an object of M/P where it is uniquely determined by M). If $\mathrm{P_2} \longrightarrow \mathrm{P_1} \longrightarrow \mathrm{M} \longrightarrow O$ is exact with $\mathrm{P_1}$ and $\mathrm{P_2}$ finitely generated projectives then TrM can be computed as the cokernel of $\mathrm{Hom}_\Lambda(\mathrm{P_1},\Lambda) \longrightarrow \mathrm{Hom}_\Lambda(\mathrm{P_2},\Lambda)$.

Remark. The isomorphism results from combining the isomorphism $\pi(\mathrm{M},-) \cong \mathrm{Tor}^1_\Lambda(\mathrm{TrM},-)$ established by M. Auslander [1, Prop. 3.2] for any ring Λ, with the isomorphism $\mathrm{Hom}_\Gamma(\mathrm{Tor}^1_\Lambda(\mathrm{N},-),\mathrm{I}) \cong \cong \mathrm{Ext}^1_\Lambda(-,\mathrm{Hom}_\Gamma(\mathrm{N},\mathrm{I}))$ from [5, VI, Prop. 5.1], which is an extension of the adjoint relationship between $\mathrm{N}_\Lambda \otimes -$ and $\mathrm{Hom}_\Gamma(\mathrm{N},-)$.

Summarizing one has for I an injective Γ-module then defining $\mathbf{R}\mathrm{I} = \mathrm{Ext}^1_\Lambda(-,\mathrm{Hom}_\Gamma(\mathrm{TrM},\mathrm{I}))$ and extending **R** to a left exact functor $\mathrm{Mod}\,\Gamma \longrightarrow A$ then **R** is the right adjoint to **S**. This establishes the 'local' result of sufficient M-pure injectives from which the general result (any M) is obtained by means of Prop. C (and hence also providing a proof of Warfield's 'global' result). However much more can be obtained from this approach.

From [7, Th. 2.8] the adjoint pair $(\mathbf{S}_M, \mathbf{R}_M)$ will induce an equivalence between the injectives in A which lie in R_M and the injectives of $(M/P^{op}, \text{Ab})$. Furthermore R_M is cogenerated by these injectives. In the 'local' case this implies that R_M can be cogenerated by injectives of the form $\text{Ext}^1_\Lambda(-, \text{Hom}_\Gamma(\text{TrM}, \text{I}))$ where I varies over the injective hulls of the simple Γ-modules. Now if N is M-pure injective then $\text{Ext}^1_\Lambda(-, \text{N})$ will be in R_M and injective in A. Hence this injective will be a direct summand of some product of the above-mentioned cogenerators, and thus N will be a direct summand of a product of modules of the form $\text{Hom}_\Gamma(\text{TrM}, \text{I})$, and possibly an injective Λ-module containing N. Combining these results with (the proof of) Prop. C establish the following theorem.

Theorem D. *For any collection of finitely presented modules M there are sufficient M-pure injectives. Furthermore the set of modules of the form* $\text{Hom}_\Gamma(\text{TrM}, \text{I})$ *where* $\Gamma = \pi(\text{M}, \text{M})$, I *varies over the injective hulls of simple right* G*-modules and* M *varies over M, along with the injective hulls of simple left Λ-modules will be a split cogenerating set for M-pure injectives (i.e. any M-pure injective will be a direct summand of some product of such modules).* $\diamondsuit$

We conclude this paper with some remarks concerning simple objects in A. From [7, Sec. 4] there is a bijection between the simple objects of A not in S_M and the simple objects of $R_M \cap T_M$ (and thus of the equivalent category $(M/P^{op}, \text{Ab})$).

Suppose then that F is a simple not in S_M, so that it must be in $T_M \cap R_M$. But T_M is generated by the set of projectives $\pi(-, \text{M})$ as M varies over M. Hence F will be the epimorph of $\pi(-, \text{M})$ some M. For such an M, F will be in T_{M} and hence not in S_{M}, and thus in R_{M} since it is simple. Consequently F will be in $R_{\text{M}} \cap T_{\text{M}}$ for some M and to determine the nature of simples not in S_{M} one can work locally.

Now R_M is cogenerated by functors of the form $\text{Ext}^1_\Lambda(-, \text{Hom}_\Gamma(\text{TrM}, \text{I}))$, so a simple object F will embed in one such functor. Since F is also an epimorph of $\pi(-, \text{M})$ it follows that F arises from a short exact sequence of the form $0 \longrightarrow \text{Hom}_\Gamma(\text{TrM}, \text{I}) \longrightarrow \text{E} \longrightarrow \text{M} \longrightarrow 0$. Such short exact sequences are familiar to those working in the representation theory of finite dimensional algebras. Indeed a short exact sequence $0 \longrightarrow \text{A} \longrightarrow \text{B} \longrightarrow \text{C} \longrightarrow 0$ represents a simple object of A if any $\text{X} \longrightarrow \text{C}$ which is not a split epimorphism factors over B for any module X (or equivalently any A $\longrightarrow$ Y which is not a split monomorphism factors through B for any module Y) [9]. Such sequences resemble 'almost split short exact sequences' (also known as 'AR sequences' in honor of the originators M. Auslander and I. Reiten), except that there is no assumption that A and C are indecomposable (in fact with local endomorphism rings) and no restriction for A, B, C, X or Y to be finitely presented.

Gathering the above results leads to the following theorem.

Theorem E. *A simple object of $\mathcal{A}$, which does not come from a $\mathcal{M}$-pure short exact sequence, will arise from a short exact sequence of the form* $0 \longrightarrow \mathrm{Hom}_\Gamma(\mathrm{TrM}, \mathrm{I}) \longrightarrow \mathrm{E} \longrightarrow \mathrm{M} \longrightarrow 0$, *for some* M *in* $\mathcal{M}$, $\Gamma = \pi(\mathrm{M}, \mathrm{M})$ *and* I *the injective hull of some simple right* Γ*-module.* $\lozenge$

Remark. M itself need not have a local endomorphism ring, however the indecomposable injective I will have a local endomorphism ring.

Furthermore

$$\mathrm{End}_{\mathcal{A}}(\mathbf{RI}) = \mathrm{Hom}_{\mathcal{A}}(\mathbf{RI}, \mathbf{RI}) \cong \mathrm{Hom}_\Gamma(\mathbf{SRI}, \mathrm{I}) \cong \mathrm{Hom}_\Gamma(\mathrm{I}, \mathrm{I}) = \mathrm{End}_{\mathrm{G}}\, \mathrm{I}$$

Now it is easily verified that

$$\mathrm{Hom}_{\mathcal{A}}(\mathrm{Ext}^1_\Lambda(-, \mathrm{A}), \mathrm{Ext}^1_\Lambda(-, B)) \cong \overline{\mathrm{Hom}_\Lambda}(\mathrm{A}, \mathrm{B})$$

where $\overline{\mathrm{Hom}_\Lambda}(\mathrm{A}, \mathrm{B}) = \mathrm{Hom}_\Lambda(\mathrm{A}, \mathrm{B})/I(\mathrm{A}, \mathrm{B})$ and $I(\mathrm{A}, \mathrm{B})$ consists of those morphisms which factor through an injective. Hence $\overline{\mathrm{End}}_\Lambda(\mathrm{Hom}_\Gamma(\mathrm{TrM}, \mathrm{I}))$ will be a local ring.

In the ideal case M itself will have a local endomorphism ring, in which case I is uniquely determined by M and there is only one simple arising from a short exact sequence with end term M. For rings Λ in which every finitely presented module is a direct sum of modules with local endomorphism rings, then the M's used in Prop. C, Th. D and Th. E can be taken to be those of $\mathcal{M}$ with local endomorphism rings.

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