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MEASURE AND CATEGORY VERSIONS OF ERDŐS–RADO THEOREM

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Abstract: Recently, Zara I. Abud and Francisco Miraglia proved a measure theoretic analogue of Erdős–Rado Theorem, in which the Lebesgue measure, instead of cardinality, is used. In this paper we will prove that the dual statement, with respect to measure and category duality, is also true. Our proof is direct and uses transfinite induction and the continuum hypothesis. Moreover we will show how the proofs of both these theorems can be deduced directly from the original Erdős–Rado Theorem shortly but by use of somewhat complicated tools.

1. Preliminary results

If A is a set, we denote by $|A|$ the cardinal number of A. We write m^* for the Lebesgue outer measure in [0, 1] and $A\backslash B$ for the set-theoretic difference of two sets A and B. The following definition gives a natural generalization of the concept of disjoint sets.

Definition. A family $\{S_i : i \in I\}$ of sets is a *quasi-disjoint family* (or a) Δ*–system)* if there is a fixed set J such that $S_i \cap S_j = J$ for all distinct $i, j \in I$. Especially, a family of pairwise disjoint sets is a quasi-disjoint family where $J = \emptyset$. Let now κ and α be two cardinals. Then α is called *strongly κ–inaccessible*, written $\kappa \ll \alpha$, if $\kappa < \alpha$ and $\beta^\lambda < \alpha$ whenever $\beta < \alpha$ and $\lambda < \kappa$.

In [3] (p.5) as well as in [2] we read the following theorem, known as Erdős–Rado Theorem.

Theorem A (Erdős–Rado). *Let κ, α be infinite cardinals such that $\kappa \ll \alpha$, with α regular. Let $\{S_\beta : \beta \in \alpha\}$ be a family of sets such that $|S_\beta| < \kappa$ for all $\beta \in \alpha$. Then there exist $A \subset \alpha$ with $|A| = \alpha$ and a set J such that $S_\beta \cap S_\gamma = J$ for all distinct $\beta, \gamma \in A$.*

An extension of Erdős-Rado Theorem to singular cardinals can be found in [2].

Since $\aleph_1$ is regular and strongly inaccessible with respect to any finite cardinal, the following theorem can be deduced as a special case of Erdős–Rado Theorem.

Theorem B. *Let $\{S_\beta : \beta \in A\}$ be a family of finite sets with $|A| = \aleph_1$. Then there is $B \subset A$ such that $|B| = \aleph_1$ and $\{S_\beta \colon \beta \in B\}$ is a quasi-disjoint family.*

Recently, Zara I. Abud and Francisco Miraglia [1] using the continuum hypothesis proved the following measure theoretic analogue of Theorem B.

Theorem C. *Let A be a subset of $[0,1]$ with $m^*(A) > 0$ and $\{S_t \colon t \in A\}$ a family of finite sets. Then there is $B \subset A$ with $m^*(B) > 0$ such that $\{S_t \colon t \in B\}$ is a quasi-disjoint family of sets.*

2. Results

The Th. C has a dual theorem, with respect to the duality between measure and category, which reads as follows.

Theorem D. *Let A be a subset of $[0,1]$ of the second category in $[0,1]$ and $\{S_t \colon t \in A\}$ a family of finite sets. Then there is $B \subset A$ of the second category in $[0,1]$, such that $\{S_t \colon t \in B\}$ is a quasi-disjoint family of sets.*

In order to prove the Th. D we can follow a method similar to that of Zara and Miraglia, assuming the continuum hypothesis and using transfinite induction. However some changes in the method are necessary.

We will need the following Lemma whose proof is easy and can be found in [4].

Lemma. *Let $\mathcal{Q}$ denote the collection of subsets of $[0,1]$ given by the formula*

$$\mathcal{Q} = \Big\{ \Big(\mathbb{R} \setminus \bigcup_{i=1}^{\infty} F_i\Big) \cap [0,1] : \text{where each } F_i \text{ is a closed nowhere dense subset of } \mathbb{R} \Big\}.$$

If B, a subset of the interval $[0,1]$, has the property that $Q \cap B \neq \emptyset$ for every $Q \in \mathcal{Q}$, then B is a set of the second Baire category in $\mathbb{R}$.

Remark. It is easy to see that $\mathcal{Q}$ is contained in the σ–algebra of Borel sets of $\mathbb{R}$ and contains the sets of the form $[0, 1]\setminus\{x\}$ for all $x \in [0, 1]$. Therefore $\mathcal{Q}$ has the power of the continuum and by the continuum hypothesis it has a well ordering of the form $\mathcal{Q} = \{Q_\alpha\colon \alpha < \Omega\}$ where Ω is the first uncountable ordinal.

The next proposition is the first step in the proof of Th. D.

Proposition 1. *Let A be a subset of $[0,1]$ of the second category in $[0,1]$ and $\{S_t\colon t \in A\}$ be a family of singletons. Then there is $B \subset A$ of the second category in $[0,1]$ and $\{S_t\colon t \in B\}$ is a quasi-disjoint family of sets. This means that either all singletons $S_t, t \in B$ are equal, or they are pairwise disjoint.*

Proof. Let $\mathcal{A} = \bigcup_{t\in A} S_t$ and for each $x \in \mathcal{A}$ put $A_x = \{t \in A\colon S_t = \{x\}\}$. If for some $x \in \mathcal{A}$ the set A_x is of the second category in $[0, 1]$, take $B = A_x$. Then all sets $S_t, t \in B$ are equal to $\{x\}$ and the family $\{S_t\}_{t\in B}$ is quasi-disjoint. If this is not the case, the set A_x is of the first category in $[0, 1]$ for every $x \in \mathcal{A}$.

Consider the collection $\mathcal{Q}$ mentioned in the Lemma and its well ordering $\mathcal{Q} = \{Q_\alpha\colon \alpha < \Omega\}$ according to the Remark. Since Q_1 is a residual set (the complement of a set of the first category) we have $A \cap Q_1 \neq \emptyset$. Therefore we can pick $t_1 \in A \cap Q_1$ and let $S_{t_1} = \{x_1\}$. Obviously $t_1 \in A_{x_1}$.

Suppose that for each $\beta < \alpha$ a point $t_\beta \in (A \setminus \bigcup_{\gamma<\beta} A_{x_\gamma}) \cap Q_\beta$ has been chosen, for which $S_{t_\beta} = \{x_\beta\}$ and $t_\beta \in A_{x_\beta}$. Then the set $\bigcup_{\beta<\alpha} A_{x_\beta}$ is of the first category in $[0,1]$ and consequently we can choose a point

$$t_\alpha \in (A \setminus \bigcup_{\beta<\alpha} A_{x_\beta}) \cap Q_\alpha \tag{1}$$

for which $S_{t_\alpha} = \{x_\alpha\}$ and $t_\alpha \in A_{x_\alpha}$.

Put now, $B = \{t_\alpha\colon \alpha < \Omega\} \subset A$. Then, by Lemma, the set

B is of the second category in [0, 1] since its intersection with each member of the collection $\mathcal{Q}$ is nonempty. Moreover if t_β, t_α are arbitrary points of B with $\beta < \alpha$, then (1) implies that $t_a \notin A_{x_\beta}$. This means that $S_{t_a} \neq \{x_\beta\} = S_{t_\beta}$, or $S_{t_\alpha} \cap S_{t_\beta} = \emptyset$. Therefore the family $\{S_t\}_{t\in B}$ is pairwise disjoint and the proof has been completed. ◊

Now we will generalize the Prop. 1.

Proposition 2. *Let $A \subset [0,1]$ be a set of the second category in $[0,1]$ and $\{S_t\colon t \in A\}$ a family of finite sets, each of them having $k \geq 0$ elements. Then there exists a set $B \subset A$ of the second category in $[0,1]$ such that the family $\{S_t\colon t \in B\}$ is quasi-disjoint.*

Proof. For $k = 0$ the proposition is obvious and for $k = 1$ is exactly the Prop. 1. Suppose now that the statement is true for all $k \leq n$. We will show that it is true for $k = n + 1$.

Let $\mathcal{A} = \bigcup_{t\in A} S_t$ and for every $M \subset \mathcal{A}$ put $A_M = \{t \in A\colon M \subset \subset S_t\}$. If there is a $M_0 \subset \mathcal{A}$ with $1 \leq |M_0| \leq n+1$ such that A_{M_0} is of the second category in [0, 1], then the family $\{S_t \setminus M_0\}_{t\in A_{M_0}}$ consists of finite sets with $n + 1 - |M_0| \leq n$ elements. Then, according to our induction assumption, there is a set $B \subset A_{M_0}$ of the second category in [0, 1] such that the family $\{S_t \setminus M_0\}_{t\in B}$ is quasi-disjoint. Hence it follows that the family $\{S_t\}_{t\in B}$ is also quasi-disjoint.

Let now that for every $M \subset \mathcal{A}$ with $1 \leq |M| \leq n + 1$ the set $A_M = \{t \in A\colon M \subset S_t\}$ is of the first category in [0, 1] and consider the collection $\mathcal{Q}$ of the Lemma together with its well ordering according to the Remark. Since Q_1 is a residual set we have $A \cap Q_1 \neq \emptyset$. Therefore we can pick $t_1 \in A \cap Q_1$ and let $S_{t_1} = \{x_1^1, x_2^1, \ldots, x_{n+1}^1\}$ and $A_1 = \{t \in A\colon S_{t_1} \cap S_t \neq \emptyset\} = \bigcup_{j=1}^{n+1} A_{\{x_j^1\}}$. Obviously, $t_1 \in A_1$. Suppose that for each $\beta < \alpha$ a point $t_\beta \in (A \setminus \setminus \bigcup_{\gamma<\beta} A_\gamma) \cap Q_\beta$ has been chosen, for which $S_{t_\beta} = \{x_1^\beta, x_2^\beta, \ldots, x_{n+1}^\beta\}$ and $A_\beta = \{t \in A\colon S_{t_\beta} \cap S_t \neq \emptyset\} = \bigcup_{j=1}^{n+1} A_{\{x_j^\beta\}}$. Of course, $t_\beta \in \in A_\beta$.

It is easy to see that the set $\bigcup_{\beta<\alpha} A_\beta$ is of the first category in [0, 1] and consequently we can choose a point

$$t_\alpha \in (A \setminus \bigcup_{\beta<\alpha} A_\beta) \cap Q_\alpha \tag{2}$$

for which $S_{t_\alpha} = \{x_1^\alpha, x_2^\alpha, \ldots, x_{n+1}^\alpha\}$, and $A_\alpha = \{t \in A\colon S_{t_\alpha} \cap S_t \neq \emptyset\} = = \bigcup_{j=1}^{n+1} A_{\{x_j^\alpha\}}$ and $t_\alpha \in A_\alpha$.

Thus, a transfinite sequence $B = \{x_\alpha\colon \alpha < \Omega\} \subset A$ has been constructed by transfinite induction and by the Lemma, B is of the second category in [0, 1].

Now, if we take $t_\alpha, t_\beta \in B$ with $\beta < \alpha$ then (2) implies that $t_\alpha \notin$ $\notin A_\beta$ which means that $S_{t_\alpha} \cap S_{t_\beta} = \emptyset$. Therefore the family $\{S_t\}_{t \in B}$ is pairwise disjoint and the conclusion follows. ◊

Proof of Th. D. To finish the proof of Th. D it is sufficient to observe that $A = \bigcup_{k=0}^{\infty} A_k$, where $A_k = \{t \in A\colon |S_t| = k\}$. Since A is of the second category in [0, 1], A_k is also of the second category in [0, 1] for at least one k. Then the conclusion follows if we apply the Prop. 2 to A_k. ◊

3. Another approach

In this paragraph we will describe a manner to obtain Th. C and D directly from Th. B which is a special case of the original Erdős–Rado Theorem.

Proof of Th. C. Since $m^*(A) > 0$, it follows that A contains a Sierpiński set, that is, an uncountable subset A_1 every uncountable subset of which has positive outer measure (see [5], p. 78). By the continuum hypothesis we have $|A_1| = \aleph_1$.

Consider the subfamily $\{S_t\colon t \in A_1\}$. Then Th. B implies that there is a set $B \subset A_1$ with $|B| = \aleph_1$ and consequently with $m^*(B) > 0$, such that the family $\{S_t : t \in B\}$ is quasi-disjoint. ◊

Proof of Th. D. Since A is of the second category in [0, 1], it contains a Lusin set, that is, an uncountable subset A_1 every uncountable subset of which is of the second category in [0, 1] (see [5], p. 78). By the continuum hypothesis we have $|A_1| = \aleph_1$.

Consider the subfamily $\{S_t\colon t \in A_1\}$. Then, by Th. B, there is $B \subset A_1$ with $|B| = \aleph_1$ and consequently of the second category in [0, 1], such that the subfamily $\{S_t\colon t \in B\}$ is quasi-disjoint. ◊

Thms. C and D constitute one more pair of dual theorems with respect to the duality between measure and category (see [5], p. 74–85).

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