Mathematica Pannonica
8/2 (1997), 245–250

ON EMBEDDING OF INVOLUTION RINGS

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To Professor R. Wiegandt in his 65th birthday

Received: August 1996

MSC 1991: 16 A 28, 46 K 10

Keywords: Involution ring, $*$-biideal, chain conditions, $*$-extension, Dorroh extension, Jacobson and Baer radicals.

Abstract: Let A be an involution ring with dcc on $*$-biideals. Then A is embeddable into an involution ring with identity and with dcc on $*$-biideals if and only if A^+ has no direct summand $(Z(P^\infty)$. Furthermore, A has acc on $*$-biideals if and only if A^+ has no direct summand $Z(P^\infty)$. By defining involution in a natural way, other embeddings and extensions for rings are still valid for involution rings.

The structure of the additive groups for rings with chain conditions have been used to investigate many ring-theoretic properties and to prove some embedding theorems (cf. [3] and [4]). Here we prove the involutive version for some of these properties and embeddings using the structure of the additive groups of involution rings with descending chain condition (dcc) on $*$-biideals. Moreover, we show that the well known Dorroh extension and other extensions due to J. Szendrei are still valid for involution rings if we extend the involution in a natural way.

All rings considered are associative. By an *involution ring* A

(called also as a *ring with involution*) we mean a ring with an additional unary operation $*$ (called *involution*) such that for all $a, b \in A$,

(1) $$(a+b)^* = a^* + b^*, \quad (ab)^* = b^*a^*, \quad (a^*)^* = a.$$

If A is an R-algebra over a commutative ring R with identity, then A is called an *involution algebra* (or *algebra with involution*) if in addition to identities (1), we have

$$(ra)^* = ra^*, \quad \text{for all} \quad r \in R.$$

It has turned out that the notion $*$-biideal can be used successfully in describing the structure of involution rings as one $*$-sided ideals in rings without involution (cf. [1]). A *biideal* B of a ring A is a subring of A satisfying the inclusion $BAB \subseteq B$. An ideal (biideal) B of an involution ring A is called a *$*$-ideal* (*$*$-biideal*) of A if B is closed under involution, that is $B^{(*)} = \{a^* \in A | a \in B\} \subseteq B$. An involution ring K is said to be *$*$-extension* of the involution ring A if K contains A as a $*$-ideal, while K is called *direct $*$-extension* of A if $K = A \oplus B$ with B a $*$-ideal of K. The Jacobson and the Baer radicals of a ring A will be denoted by $J(A)$ and $\beta(A)$, respectively.

Now the Dorroh extension for involution rings is :

Proposition 1. *Every involution ring A can be embedded as a $*$-ideal into an involution ring with identity.*

Proof. If $\mathbb{Z}$ is the ring of integers, define on the Dorroh extension $A^1 = A \times \mathbb{Z}$ of A a unary operation $\Box$ as: $(a,n)^\Box = (a^*, n)$, for all $a \in A$ and $n \in \mathbb{Z}$. One can easily verify that $\Box$ is an involution on A^1 and A is a $\Box$-ideal of A^1. In fact $\Box$ is an extension of $*$ and therefore we may identify them to be $*$ only. $\Diamond$

The following generalization of Prop. 1 can be easily verified.

Proposition 2. *Let R be a commutative ring with identity and A an R-algebra with involution $*$. If on the Cartesian product $A^1_R = A \times R$, one defines a componentwise addition, a multiplication by*

$$(a,r)(b,s) = (ab + rb + sa, rs),$$

a scalar multiplication by

$$r(a,s) = (ra, s),$$

and an involution as

$$(a,r)^\Box = (a^*, r),$$

for all $a, b \in A$ and $r, s \in R$, then A^1_R becomes an involution R-algebra with identity such that A is embeddable into A^1_R as a $\Box$-ideal and $A^1_R / A \simeq R$, with identical involution.

The Dorroh extension A^1 of A, however, is not the only extension of A with identity, nor it is, in general, a minimal extension with identity. Nevertheless, A^1 may have zero divisors even if A has none. For a ring A without zero divisors, J. Szendrei [6] proved that there is always a minimal extension of A with identity. In fact he proved : *every ring A without zero divisors is embeddable as an ideal (up to isomorphism) into one and only one ring $\overline{A^1}$ with identity and without zero divisors such that $\overline{A^1}$ is a minimal extension of A possessing identity.*

We can adapt Szendrei's result for involution rings in the following way. Let us consider the set

$$S = \{a \in A | 0 \neq a^2 = ta, \quad t \in \mathbb{Z}, \quad \text{and} \quad a^* = a\}.$$

If S is empty, we put $a = 0 = t$. If S is not empty, then following a proof similar to that in [6] we see that there exists an $e \in S$ such that the corresponding coefficient t is the least positive integer among the coefficients occuring in the set S, both e and t are uniquely determined. Szendrei [6] also proved that either the additive group A^+ is torsionfree or $pA = 0$ with a prime p.

Proposition 3. *Every involution ring A without zero divisors is embeddable as a $*$-ideal (up to isomorphism) into one and only one involution ring $\overline{A^1}$ with identity and without zero divisors such that $\overline{A^1}$ is a minimal extension of A possessing identity. Moreover, if $(p,t) = 1$, then $A = \overline{A^1}$; otherwise $\overline{A^1}$ is the factor ring of the Dorroh extension A^1 modulo the ideal generated by the element $(e,-t) \in A^1$, (here $(e,-t) = (0,0)$ is possible) with involution $\square$ defined by*

$$((a,r) + I)^{\square} = (a^*,r) + I$$

and the correspondence

$$a \to (a,0) + I$$

embeds A into $\overline{A^1}$.

In [7] J. Szendrei characterized all rings which admit only direct extensions. Noting that the identity of an involution ring is invariant under involution, following a similar proof to that of Th. 1 in [7] we obtain:

Proposition 4. *Every $*$-extension of an involution ring A is a direct $*$-extension if and only if A contains an identity element.*

To prove one of our main results, that is an embedding theorem for involution rings with dcc on $*$-biideals, we need two auxiliary results. If Q is the field of rational, $Z(pk)$ denotes the cyclic group of order p^k and $Z(P^\infty)$ denotes the quasi-cyclic group, then following a similar proof to that of Th. 58.2-(I) in [4] one gets:

Lemma 1. *The additive group A^+ of an involution ring A having dcc on $*$-biideals can be decomposed in the form*

$$A^+ \simeq \bigoplus_{\alpha} Q^+ \oplus \bigoplus_{\text{finite}} Z(p_i^{\infty}) \oplus \bigoplus_{\beta} Z(p_j^{k_j}),$$

where $p_j^{k_j}/m$ and α, β are arbitrary cardinals, p_i and p_j are primes and m is a fixed positive integer.

Lemma 2. *The additive group A^+ of an involution ring A with identity and having dcc on $*$-biideals has a decomposition*

$$A^+ \simeq \bigoplus_{\alpha} Q^+ \oplus \bigoplus_{\beta} A(p_j^{k_j}),$$

where $p_j^{k_j}/m$ and α, β are arbitrary cardinals, each p_j is a prime and m is a fixed positive integer.

Proof. From Lemma 1, A^+ has a direct decomposition $A^+ = T \oplus F$, where

$$T = \bigoplus_{\text{finite}} Z(p_i^{\infty}) \oplus \bigoplus_{\beta} Z(p_j^{k_j})$$

is the maximal torsion subgroup of A^+ and $F = \bigoplus_{\alpha} Q^+$ is a torsionfree divisible subgroup of A^+. By Prop. 57.8 in [4], every subgroup of type $Z(P^{\infty})$ annihilates the ring A. Since A has identity the annihilator of A is the zero ideal, whence A^+ has the desired decomposition. ◊

Now, we are ready to prove the embedding theorem, that is :

Theorem 1. *An involution ring A with dcc on $*$-biideals can be embedded as $*$-ideal into an involution ring with identity and having dcc on $*$-biideals if and only if A^+ has no direct summand $Z(P^{\infty})$.*

Proof. If A is embeddable as a $*$-ideal into an involution ring with identity and having dcc on $*$-biideals, then it follows from Lemma 2 that A^+ has no direct summand $Z(P^{\infty})$Z.

For the converse implication, assume that A^+ has no direct summand $Z(P^{\infty})$ and let T denote the maximal torsion $*$-ideal of A. Since A has dcc on $*$-biideals, it follows from Prop. 5 in [5] that $A = T \oplus F$, where F is the uniquely determined maximal torsionfree $*$-ideal of A. Furthermore, by Lemma 1, A^+ has the form

$$A^+ \simeq \bigoplus_{\alpha} Q^+ \oplus \bigoplus_{p \in P} A_p^+$$

with P a finite set of primes. Hence $T^+ \simeq \bigoplus_{p \in P} A_p^+$ and necessarily $F^+ \simeq \bigoplus_{\alpha} Q^+$. Since F is a Q-algebra, it follows from Prop. 2 that F can be embedded as a $*$-ideal into an involution ring F_Q^1 with identity and $F^1_Q/F \simeq Q$. Since F^1_Q is an extension of an involution ring having dcc on $*$-biideals by an involution ring with dcc on $*$-biideals, by Prop. 5 in [1], F^1_Q has also dcc on $*$-biideals. Nevertheless, for every $p \in P$, A_p is a$\mathbb{Z}_{p^k}$-algebra, where $p^k A_p^+ = 0$ and $\mathbb{Z}_{p^k}$ is the ring of integers modulo p^k. Again A_p can be embedded as a $*$-ideal into an involution ring $(A)^1_{\mathbb{Z}_{p^k}}$ with identity and having dcc on $*$-biideals. Thus clearly A can be embedded as a $*$-ideal into the involution ring

$$F^1_Q \oplus \bigoplus_{p\in P} (A)^1_{\mathbb{Z}_{p^k}}$$

which has identity and dcc on $*$-biideals, by Cor. 1 in [1]. $\Diamond$

It is well known that an artinian ring A is noetherian if and only if A^+ has no subgroup of type $Z(P^\infty)$. The involutive version of this property is :

Theorem 2. *An involution ring A with dcc on $*$-biideals has ascending chain condition (acc) on $*$-biideals if and only if A^+ has no subgroup of type $Z(P^\infty)$.*

Proof. If A has dcc on $*$-biideals, then by Th. 3 in [2], A is left and right artinian and the Jacobson radical $J(A)$ has dcc on subgroups and $J(A) = \beta(A)$. Since A has also acc on $*$-biideals, in view of [2] Th. 6, $J(A)^+$ is finitely generated, so $J(A)^+$ does not contain $Z(P^\infty)$ subgroup. Since $A/J(A)$ is semisimple artinian, by [4] Th. 58.2-(III) its additive group has no direct summand $Z(P^\infty)$, and hence also A^+ has no such subgroup.

Conversely, suppose that A has dcc on $*$-biideals and A^+ has no direct summand $Z(P^\infty)$. By [2] Th. 3, $J(A)$ is a nilpotent artinian ring. Since $J(A)^+$ has no $Z(P^\infty)$ subgroup, it follows from Th. 58.3 in [4] that $J(A)^+$ is finite and consequently $J(A)$ has acc on $*$-biideals. Nevertheless, $A/J(A)$ as a semisimple artinian ring is also semiprime and hence by Th. 1 in [1] it has dcc on $*$-biideals. Consequently, Th. 2 of [1] implies that $A/J(A)$ has also acc on $*$-biideals. Finally, applying Prop. 5 in [1], A has acc on $*$-biideals. $\Diamond$

The connection of Ths. 1 and 2 yields immediately

Corollary. *Let A be an involution ring with dcc on $*$-biideals. The following conditions are equivalent:*

(i) *A can be embedded as a $*$-ideal into an involution ring with identity having dcc on $*$-biideals;*

(ii) *A^+ has no subgroup of type $Z(P^\infty)$;*

(iii) *$J(A)$ is finite;*

(iv) *A has acc on $*$-biideals.*

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