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PARAMETRIC MONOTONE GENERALIZED EQUATIONS IN REFLEXIVE BANACH SPACES

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Abstract: We study the continuity of the solutions of parametric generalized equations, including Hammerstein integral equations, by using implicit function theorems for monotone mappings in reflexive Banach spaces.

1. Introduction

The most important results on the surjectivity of monotone operators and in the same time on the solvability of monotone generalized equations appeared in the 60's and 70's in the papers by F. E. Browder, G. J. Minty, J. P. Aubin, V. Barbu, R. T. Rockafellar and others.

The study of the sensitivity of the solutions of parametric generalized equations had a great development since 1980. We can mention the papers by S. M. Robinson, R. T. Rockafellar, B. Kummer, B. Mordukhovich, H. Frankowska, J. B. Penot, Zs. Páles and others. Because of the importance of variational inequalities, sensitivity analysis for monotone generalized equations has been studied in many papers (S. Dafermos [3], W. Alt and I. Kolumbán [1], N. G. Yen [7], G. Kassay and I. Kolumbán [5]).

In these papers have been used the properties of the metric projection on nonempty, closed, convex sets in finite dimensional Euclidean spaces or infinite dimensional Hilbert spaces. These properties cannot be used in general Banach spaces.

In this paper a main result is stated for reflexive, strictly-convex Banach spaces with strictly-convex dual. We follow some of the ideas of G. Kassay [4], used in that paper for completly continuous, single-valued mappings. As application we study a special class of nonlinear equations, including the Hammerstein integral equations.

2. The main result

Let X be a reflexive, strictly-convex Banach space with X^* strictly-convex and let W be a topological space. Then the following theorem is true.

Theorem 1. *Let $F : W \times X \rightsquigarrow X^*$ be a set-valued map, let $(w_0, x_0) \in W \times X$ and let W_0 and X_0 be neighbourhoods of w_0 and x_0. It is supposed that:*

(i) $0 \in F(w_0, x_0)$;

(ii) *F is consistent in w at (w_0, x_0), i.e. there exists a function $\beta : W_0 \to \mathbb{R}_+$ which is continuous at w_0 and $\beta(w_0) = 0$ such that, for all $w \in W_0$, there exists $y_w \in F(w, x_0)$ satisfying $\|y_w\| \leq \beta(w)$;*

(iii) *for all $w \in W_0$ the set-valued mappings $F(w, \cdot)$ are maximal-monotone;*

(iv) *for all $w \in W_0$ the set-valued mappings $F(w, \cdot)$ are uniformly-monotone on X_0, i.e. there exists an increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, with $\varphi(0) = 0$ and $\varphi(r) > 0$ when $r > 0$, such that*

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq \varphi(\|x_1 - x_2\|)\|x_1 - x_2\|,$$
$$\forall x_1, x_2 \in X_0 \cap D(F(w, \cdot)),\ \forall x_1^* \in F(w, x_1) \text{ and } \forall x_2^* \in F(w, x_2).$$

Then there exist neighbourhoods W_1 of w_0, X_1 of x_0 and a unique mapping $x : W_1 \to X_1$ which is continuous at w_0 and such that $0 \in F(w, x(w))$ for all $w \in W_1$ and $x(w_0) = x_0$.

Remark 1. In [1] an assumption similar to (ii) was used. We can use another formulation:

(ii)′ *For all $\varepsilon > 0$ there exist neighbourhoods W_ε of w_0 and X_ε of x_0 such that*

$$F(w,x) \subseteq F(w_0,x) + \varepsilon B_{X^*}$$
for all $w \in W_\varepsilon$ *and* $x \in X_\varepsilon$.

In the case when in Th. 1 we suppose (ii)′ instead of (ii) it is enough to have uniform-monotonicity for $F(w_0,\cdot)$ and strict-monotonicity for $F(w,\cdot)$ when $w \in W_0$. Assumption (ii) says that we have a selection for $F(\cdot,x_0)$ which is continuous at w_0. In the proof of Th. 1 we do not need monotonicity properties of $F(w_0,\cdot)$. We need only that $0 \in F(w_0,x_0)$. But it is very useful to see the difference between the positivity of $F(w_0,\cdot)$ and $F(w,\cdot)$.

The proof of Th. 1 is based on two lemmas. In the following we take $x_0 = 0$ and suppose that the assumptions of Th. 1 are satisfied.

Lemma 1. *For all numbers* δ, ε *with* $0 < \delta \le \varepsilon$, $B(0,\varepsilon) \subseteq X_0$ *we have*

$$\inf\left\{\inf\left\{\frac{1}{\|x\|}\langle x,x^*\rangle; x^* \in F(w_0,x)\right\}; \delta \le \|x\| \le \varepsilon\right\} > 0.$$

Proof. Let $x \in X$ with $\delta \le \|x\| \le \varepsilon$, and let $x^* \in F(w_0,x)$. Then

$$\langle x,x^*\rangle \ge \varphi(\|x\|)\|x\| \ge \inf_{\delta \le r \le \varepsilon} \varphi(r)\delta = d > 0,$$

so

$$\inf\left\{\inf\left\{\frac{1}{\|x\|}\langle x,x^*\rangle : x^* \in F(w_0,x)\right\}; \delta \le \|x\| \le \varepsilon\right\} \ge d > 0. \quad \Diamond$$

Lemma 2. *Let* $T : X \rightsquigarrow X^*$ *be a maximal-monotone set-valued map. For all integers* $k \ge 1$ *let* $P_k : X^* \to X$ *be the single-valued mapping defined by* $P_k = (J + kT)^{-1}$, *where* $J : X \to X^*$ *is the normalized duality mapping defined by*

$$J(x) = \{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2, \|x^*\| = \|x\|\},$$

which in our case of a reflexive, strictly-convex Banach space with strictly-convex dual, is single-valued and bijective. If a sequence (x_k), *with* $x_{k+1} = P_k(Jx_k)$, *is bounded, then there exists* $x' \in X$ *such that* $0 \in T(x')$ *and* (x_k) *has a subsequence weakly converging to* x'.

Proof. We will use the mappings $Q_k : X \to X^*$ defined by

$$Q_k(x) = \frac{1}{k}\left(J(x) - J \circ P_k \circ J(x)\right).$$

Then $0 \in T(x)$ if and only if $0 \in Q_k(x)$. We have

$$\|Q_k(x_k)\| \le \frac{1}{k}\|J(X_k)\| + \frac{1}{k}\|J(P_k(J(x_k)))\| = \frac{1}{k}\|x_k\| + \frac{1}{k}\|x_{k+1}\|.$$

Taking into account that the sequence (x_k) is bounded, we have

$$Q_k(x_k) \to 0 \quad \text{when} \quad k \to \infty.$$

We have $Q_k(x) \in T(P_k(J(x)))$, because of

$$P_k \circ J(x) = (J + kT)^{-1}(J(x)) \Leftrightarrow$$
$$\Leftrightarrow J(x) \in J \circ P_k \circ J(x) + kT\left(P_k \circ J\left(x\right)\right) \Leftrightarrow$$
$$\Leftrightarrow \frac{1}{k}\left(J\left(x\right) - J \circ P_k \circ J\left(x\right)\right) \in T\left(P_k \circ J\left(x\right)\right).$$

The space X being reflexive, we can suppose that there exists a subsequence of (x_k) (denoted also by (x_k)), such that $x_k \rightharpoonup x' \in X$. The monotonicity of T implies that

$$\langle x - P_k \circ J(x_k), y - Q_k(x_k)\rangle \geq 0$$

for all $k \in N, x \in X, y \in T(x)$. Using the weak convergence of (x_k) and the strong convergence of $(Q_k(x_k))$, we conclude that

$$\langle x - x', y\rangle \geq 0$$

for all $x \in X, y \in T(x)$. The maximal monotonicity of T implies that $0 \in T(x')$. $\Diamond$

Proof of Th. 1. Let $0 < \delta \leq \varepsilon$ be such that $B(x_0, \varepsilon) \subseteq X_0$. We define the function $f : W \times X \to \mathbb{R}$ by

$$f(w, x) = \inf\left\{\frac{1}{\|x\|}\langle x, x^*\rangle; x^* \in F(w, x)\right\}.$$

From Lemma 1 we have

$$\inf_{\delta \leq \|x\| \leq \varepsilon} f(w_0, x) = d > 0.$$

Let $x \in X$ be such that $\delta \leq \|x\| \leq \varepsilon$, let $w \in W_0$ and $x^* \in F(w, x)$. Assumption (ii) implies the existence of $y_w \in F(w, 0)$ with $\|y_w\| \leq$ $\leq \beta(w)$. Then

$$\varphi\left(\|x\|\right)\|x\| \leq \langle x, x^* - y_w\rangle \leq \langle x, x^*\rangle + \varepsilon\beta(w)$$

and hence

$$\langle x, x^*\rangle \geq \varphi\left(\|x\|\right)\|x\| - \varepsilon\beta(w) \geq \varphi(\delta)\delta - \varepsilon\beta(w).$$

We can find a neighbourhood $W' \subseteq W_0$ of w_0 such that $\varepsilon\beta(w) \leq \frac{d}{2}$ for all $w \in W'$. Then

$$\inf_{\delta \leq \|x\| \leq \varepsilon} f(w, x) \geq d - \frac{d}{2} = \frac{d}{2} > 0$$

for all $w \in W'$. We can suppose that $D(F(w, \cdot)) = B(0, \varepsilon)$, otherwise we replace $F(w, \cdot)$ with $F(w, \cdot) + N_{B(0,\varepsilon)}(\cdot)$, where $N_{B(0,\varepsilon)}$ is the normal cone operator to $B(0, \varepsilon)$. Let $w \in W'$ be arbitrarily chosen. We denote

$$P_k(x) = \left(J + kF(w, \cdot)\right)^{-1}\left(J(x)\right).$$

Put $x_1 = 0$ and $x_{k+1} = P_k(x_k)$ for all $k \geq 1$. We will prove that $\|x_k\| \leq \delta$, for all $k \geq 1$.

We suppose that this is not true. Let k_0 be the first integer such that $\|x_{k_0}\| \leq \delta$ and $\|x_{k_0+1}\| > \delta$. Then

$$x_{k_0+1} = (J + k_0 F(w,\cdot))^{-1} (J(x_{k_0})),$$

which implies that

$$J(x_{k_0}) \in J(x_{k_0+1}) + k_0 F(w, x_{k_0+1}).$$

Hence $x_{k_0+1} \in D(F(w,\cdot)$ and $\|x_{k_0+1}\| \leq \varepsilon$. Then, there exists $u_{k_0+1} \in$ $\in F(w, x_{k_0+1})$, such that

$$J(x_{k_0}) = J(x_{k_0+1}) + k_0 u_{k_0+1}.$$

Then we have

$$\|J(x_{k_0})\|\|x_{k_0+1}\| \geq \langle x_{k_0+1}, J(x_{k_0})\rangle = \langle x_{k_0+1}, J(x_{k_0+1}) + k_0 u_{k_0+1}\rangle =$$

$$= \langle x_{k_0+1}, J(x_{k_0+1})\rangle + k_0\langle x_{k_0+1}, u_{k_0+1}\rangle \geq \|x_{k_0+1}\|^2 + k_0\frac{d}{2} > \|x_{k_0+1}\|^2.$$

Consequently

$$\|x_{k_0}\| = \|J(x_{k_0})\| > \|x_{k_0+1}\| > \delta,$$

which is a contradiction.

So we have proved that the sequence (x_k) is bounded. In view of Lemma 2 we can find $x \in X$ such that $0 \in F(w,x)$, and we can also find a subsequence of (x_k) weakly converging to x. The ball $B(0,\delta)$ being weakly compact, we have $x \in B(0,\delta)$.

In consequence, if we fix $\varepsilon_1 > \varepsilon > 0$ with $B(0,\varepsilon_1) \in X_0$, then, for all $0 < \delta \leq \varepsilon$, we can find a neighbourhood W_δ of w_0 such that for all $w \in W_\delta$ there exists a unique $x_\delta(w) \in X$ with $\|x_\delta(w)\| \leq \delta$ and $0 \in F(w, x_\delta(w))$.

Next we define $x : W_\varepsilon \to B(0,\varepsilon)$ by $x(w) = x_\varepsilon(w)$ and we take $W_1 = W_\varepsilon, X_1 = B(0,\varepsilon)$.

The mapping x is continuous at 0. Indeed, let $0 < \delta_1 < \varepsilon$. Then there exists a neighbourhood W_{δ_1} of w_0 and a unique mapping x_{δ_1} with the above mentioned properties. In accordance with the unicity we have $x_{\delta_1}(w) = x(w)$, and from this we get $\|x(w)\| \leq \delta_1$, for all $w \in W_\varepsilon \cap W_{\delta_1}$. $\Diamond$

Corollary 1. *Let (W,d) be a metric space. In addition to the hypotesis of* Th. 1 *suppose that the following two conditions are satisfied:*

(v) *$T(w_0,\cdot)$ is strongly-monotone on X_0 with a constant $a > 0$, i.e. for all $x_1, x_2 \in X_0$ and for all $x_1^* \in T(w_0,x_1), x_2^* \in T(w_0,x_2)$ we have*

$$\langle x_1^* - x_2^*, x_1 - x_2\rangle \geq a\|x_1 - x_2\|^2.$$

(vi) *There exist a neighbourhood U of 0_{X^*} and $\lambda \geq 0$ such that*

$$T(w,x) \cap U \subseteq T(w_0,x) + \lambda d(w,w_0) B_{X^*}$$

for all $x \in X_0, w \in W_0$, i.e. the mappings $T(\cdot,x)$ are pseudo-Lipschitz continuous at w_0.

Then, the unique mapping $x : W_1 \to X_1$ *from* Th. 1 *is Lipschitz-continuous at* w_0.

Proof. Let $w \in W_1$. Then there exists $x(w) \in X_1$ such that $0 \in T(w, x(w))$. Using the assumption (vi) we can choose $u \in T(w_0, x(w))$ such that $\|u\| \leq \lambda d(w, w_0)$. Then

$$a\|x(w) - x_0\|^2 \leq \langle u - 0, x(w) - x_0 \rangle \leq$$
$$\leq \|u\| \cdot \|x(w) - x_0\| \leq \lambda d(w, w_0)\|x(w) - x_0\|$$

and hence

$$\|x(w) - x_0\| \leq \frac{\lambda}{a} d(w, w_0). \quad \diamond$$

3. Application

Now we will study the continuity of the solution of the following equation

(1) $$w = u + G \circ H(u),$$

which can be written in equivalent form

(2) $$0 \in H(u) - G^{-1}(w - u),$$

where X is a reflexive, strictly-convex Banach space with strictly-convex dual, and $H : X \to X^*, G : X^* \to X$. This kind of equations includes the Hammerstein integral equations of the form

(3) $$u(x) + \int_a^b K(x, y)F(y, u(y))dy = w(x),$$

where $(a, b) \subset R$ is bounded, $X = L^p(a, b)$, $p > 1$, $1/p + 1/q = 1$,

$$Hu(x) = F(x, u(x)),\ Gv(x) = \int_a^b K(x, y)v(y)dy.$$

The following two results from [6], show that there is possible to include the equations of the form (3) in the class of the equations of the form 1).

Proposition 1. *If* $F : (a, b) \times \mathbb{R} \to \mathbb{R}$ *satisfies the following properties:*

(1) $F(\cdot, r)$ *is measurable for all* $r \in \mathbb{R}$,
(2) $F(x, \cdot)$ *is continuous for almost all* $x \in (a, b)$,
(3) *there exist* $g \in L^q(a, b)$ *and* $c > 0$ *such that*
$$|F(x, r)| \leq g(x) + c|r|^{p-1},$$
(4) $(F(x, r_1) - F(x, r_2))(r_1 - r_2) \geq 0$ *for all* $x \in (a, b)$, $r_1, r_2 \in \mathbb{R}$,

then H is a well-defined monotone, continuous operator from $L^p(a,b)$ to $L^q(a,b)$.

Proposition 2. *Suppose that $p-q>1$ and*

$$\left(\int_a^b |K(x,y)|^q dx\right)^{\frac{1}{q}} \leq c \quad \textit{for almost all} \quad y\in(a,b),$$

$$\left(\int_a^b |K(x,y)|^{p-q} dy\right)^{\frac{1}{p-q}} \leq d \quad \textit{for almost all} \quad x\in(a,b).$$

Then G is a linear, continuous (not necessarily compact) operator from $L^q(a,b)$ to $L^p(a,b)$.

Theorem 2. *Let $H : X \to X^*$ be a uniformly-monotone and continuous mapping with $D(T) = X$, let $G : X^* \to X$ be a strictly-monotone and linear mapping with $D(T) = X^*$. Then there exists a unique continuous mapping $u : X \to X$ such that, for all $w \in X$, $u(w)$ is the unique solution of* (1).

Proof. The existence of a solution has been discussed in many papers or books ([2], [6]). In our case, for all $w \in X$ there exists a unique solution $u(w) \in X$. Since G is linear and monotone, it is continuous and because of $D(T) = X^*$, it is maximal-monotone. So, the mappings $T(w,\cdot) = -G^{-1}(w-\cdot)$ are single-valued, monotone, linear and continuous. The sums $H(\cdot)+T(w,\cdot)$ are uniformly-monotone and maximal-monotone for all $w \in X$. The continuity of G^{-1} implies assumption (ii). If we choose a $w_0 \in X$ arbitrarily, we can find a neighbourhood W_0 of w_0 and a unique mapping u_{w_0} defined on W_0, which is continuous at w_0 and such that, for all $w \in W_0$, $u_{w_0}(w)$ is the unique solution of (1). So, $u_{w_0}(w) = u(w)$ for all $w \in W_0$, and u is continuous at w_0. $\Diamond$

Remark 2. In the case when G is a nonlinear set-valued mapping and if we suppose that G is uniformly-monotone, then we can obtain the same conclusions as in Th. 2.

Remark 3. When H and G are strongly-monotone, H is Lipschitz-continuous on X, we can use Cor. 1 to prove the Lipschitz-continuity of the solution mapping.

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