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CONTINUOUS INCREASING WEAKLY BISYMMETRIC GROUPOIDS AND QUASI-GROUPS IN $\mathbb{R}$

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Abstract: In this paper a method is described for the construction of the solutions of the functional equation $F[F(x,y),F(x,y)] = F[F(x,x),F(y,y)]$, $F:\mathbb{R}^2\to\mathbb{R}$, which are continuous and strictly increasing in each variable.

1. Introduction

The celebrated stability theorem of D. H. Hyers ([6]) about the Cauchy functional equation

$$f(x+y)=f(x)+f(y)$$

is valid when x and y belong to a commutative semigroup and f assumes values in a Banach space.

Many generalizations have been proved in the last twenty years (see, for instance, [5]) and the first and most natural way to extend Hyers's result is to substitute the commutative semigroup with a set $\mathcal{X}$ endowed with a binary operation $F:\mathcal{X}\times\mathcal{X}\to\mathcal{X}$, i.e., to consider the functional equations of the form

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$$f\big[F(x,y)\big] = f(x) + f(y).$$

In order to prove stability theorems for this class of functional equations, it is necessary to require some property on F which takes the place of the commutativity. In various papers ([3], [4], [7]) it is assumed that F satisfies the functional equation

$$F\big[F(x,y), F(x,y)\big] = F\big[F(x,x), F(y,y)\big] \tag{1}$$

or a consequence of equation (1), i.e.,

$$F^{\nu}\big[F(x,y), F(x,y)\big] = F\big[F^{\nu}(x,x), F^{\nu}(y,y)\big] \tag{1'}$$

for some integer $\nu \geq 2$, where

$$F^{n}(x,x) = F\big[F^{n-1}(x,x), F^{n-1}(x,x)\big], \quad F^{1}(x,x) = F(x,x).$$

In this note we take an open interval $I \subset \mathbb{R}$ ($\mathbb{R}$ denotes the field of real numbers) and consider the functional equation (1) where $F : I^2 \to I$ is a continuous function strictly increasing in each variable and our goal is to present a method for the construction of all solutions of equation (1). As a consequence we can construct also the solutions of equation ($1'$).

The classical functional equation of bisymmetry (or mediality) is

$$F\big[F(x,y), F(u,v)\big] = F\big[F(x,u), F(y,v)\big],$$

so equation (1) may be considered as a form of the bisymmetry equation on restricted domain, i.e., $u = x$ and $v = y$.

First we study the case $I = \mathbb{R}$.

To solve equation (1) means to describe all continuous increasing weakly bisymmetric groupoids in $\mathbb{R}$. Among them it is possible to pick up those which are quasi-groups (for this terminology see [2]).

By using standard procedures (see [1] and [2]) we transform our problem into another functional equation, whose solutions allow us to get the solutions of (1).

Theorem 1. *The function F is a solution of* (1)*, continuous and strictly increasing in each variable if and only if there exist functions $G : \mathbb{R}^2 \to \mathbb{R}$, continuous, strictly increasing in each variable and reflexive and $\phi : \mathbb{R} \to \mathbb{R}$ continuous and strictly increasing such that $F(x,y) = \phi[G(x,y)]$ and the pair (ϕ, G) satisfies the functional equation*

$$\phi\big[G(x,y)\big] = G\big[\phi(x), \phi(y)\big]. \tag{2}$$

The functions ϕ and G are uniquely determined by F.

Proof. Let F be a solution of (1) and consider the function $\phi(z) = F(z,z)$; clearly ϕ is defined on $\mathbb{R}$, continuous and strictly increasing. We show that the range of ϕ coincides with the range of F. Fix $x < y$ and let $a = F(x,y)$; by (1) we have $F(a,a) = F[F(x,x), F(y,y)]$ and, since F is strictly increasing in each variable, we obtain $F(x,x) < a < F(y,y)$. Thus the continuity of F implies that $a = F(z,z) = \phi(z)$ for some z between x and y. Thus for every pair (x,y) in $\mathbb{R}^2$ there exists a unique $z = G(x,y)$ satisfying

$$F\big[G(x,y), G(x,y)\big] = F(x,y). \tag{3}$$

Obviously the function G is continuous and strictly increasing in each variable. Moreover, putting $x = y$ into (3), we get $F[G(x,x), G(x,x)] = F(x,x)$ and so we obtain $G(x,x) = x$ for every $x \in \mathbb{R}$. Thus we can write

$$F(x,y) = \phi\big[G(x,y)\big],$$

where ϕ is continuous and strictly increasing. By substituting the previous relation in both sides of equation (1) we have

$$F\big[F(x,y), F(x,y)\big] = \phi[G(F(x,y), F(x,y))] = = \phi\big[F(x,y)\big] = \phi\big[\phi(G(x,y))\big]$$

$$F\big[F(x,x), F(y,y)\big] = \phi\big[G(F(x,x), F(y,y))\big] = \phi\big[G(\phi(x), \phi(y))\big].$$

Thus if $F(x,y) = \phi[G(x,y)]$ is a solution of (1) then

$$\phi\big[G(x,y)\big] = G\big[\phi(x), \phi(y)\big].$$

The converse is obvious. $\Diamond$

From the representation given by the previous theorem we have that if F is a quasi–group then the function ϕ must be surjective.

From now on we study equation (2) under the following assumptions:

(A)
- G continuous, reflexive and strictly increasing in each variable
- $\phi : \mathbb{R} \to \mathbb{R}$ continuous and strictly increasing.

Instead of writing that G is *continuous, reflexive and strictly increasing in each variable* in its domain, we simply say that G is a *CRI–function.*

Remark 1. Observe that if the pair (ϕ, G) is a solution of (2) satisfying (A), then also the pair (ψ, K), where $\psi(x) = -\phi(-x)$ and $K(x,y) = -G(-x,-y)$ satisfies (2) and (A).

2. Properties of the solutions of equation (2)

In this section we deduce some conditions that the pair (ϕ, G) must satisfy if it is a solution of equation (2) satisfying the assumptions (A). Fix $u \in \mathbb{R}$ and define

$$\Lambda(u) := \big\{(x,y) \in \mathbb{R}^2 : G(x,y) = u\big\},$$

i.e., $\Lambda(u)$ is the level–set of G relative to the value u. In Section 3, during the construction of the solutions, we will have functions $G_0, G', \cdots$

defined on a subset S of $\mathbb{R}^2$ and we will still use the symbol $\Lambda(u)$ to denote the level–set of the function under consideration; obviously in this case $\Lambda(u) \subset S$.

Theorem 2. *Suppose that* (ϕ, G) *is a solution of equation* (2) *satisfying* (A).

For every $u \in \mathbb{R}$, $(u,u) \in \Lambda(u)$. *If* $(x,y),(z,w) \in \Lambda(u)$ *and* $z > x$, *then* $w < y$.

The set $\Lambda(u)$ *is unbounded in both directions, i.e., if* $\sup\{x : (x,y) \in \Lambda(u)\} < +\infty$ *then* $\inf\{y : (x,y) \in \Lambda(u)\} = -\infty$, *if* $\inf\{x : (x,y) \in \Lambda(u)\} > -\infty$ *then* $\sup\{y : (x,y) \in \Lambda(u)\} = +\infty$.

Define $E_u := \{x \in \mathbb{R} : (x,y) \in \Lambda(u) \text{ for some } y\}$; E_u *is an open interval and there exists a continuous strictly decreasing function* $f_u : E_u \to \mathbb{R}$ *such that*

$$\Lambda(u) = \{(x, f_u(x)) : x \in E_u\}.$$

Moreover, $(x,y) \in \Lambda(u)$ *if and only if* $(\phi(x), \phi(y)) \in \Lambda(\phi(u))$.

Proof. The first two properties follow from the reflexivity and the strict monotonicity of G.

By the continuity of G, the set $\Lambda(u)$ is closed in $\mathbb{R}^2$. Moreover, the strict monotonicity implies that every line parallel to a coordinate axis meets $\Lambda(u)$ at most in one point. Assume $(x,y) \in \Lambda(u)$ and take $z > x$ and $w < y$; then $G(x,w) < u < G(z,y)$ so by the continuity there exists $(t,s) \in \Lambda(u)$ with $x < t < z$ and $w < s < y$. This implies that the set $\Lambda(u)$ is connected. Suppose $\bar{x} = \sup\{x : (x,y) \in \Lambda(u)\} < +\infty$ and $\bar{y} = \inf\{y : (x,y) \in \Lambda(u)\} > -\infty$. Take $(x,y) \in \Lambda(u)$; then $G(x,\bar{y}) < u < G(\bar{x},y)$ and, by continuity, $G(\bar{x},\bar{y}) = u$. Now arguing as before we can find $(t,s) \in \Lambda(u)$ with $\bar{x} < t$ and $s < \bar{y}$; a contradiction. Similarly we prove the unboundedness in the other direction.

From these results we have that E_u is an open interval and $\Lambda(u) = \{(x, f_u(x)) : x \in E_u\}$ where f_u is a strictly decreasing function. Since the graph of f_u is the set $\Lambda(u)$, it is connected and the monotonicity of f_u implies its continuity. Let now $(x,y) \in \Lambda(u)$; by equation (2) we have

$$\phi[G(x,y)] = \phi(u) = G[\phi(x), \phi(y)],$$

and so $(\phi(x), \phi(y)) \in \Lambda(\phi(u))$ and vice-versa. $\Diamond$

From now we denote by $\mathrm{Gr}(f)$ the graph of the function f.

Given the function ϕ we define $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ as $\Phi(x,y) = (\phi(x), \phi(y))$ and we study the number of periodic points of the function ϕ.

Theorem 3. *Suppose that* (ϕ, G) *is a solution of equation* (2) *satisfying* (A).

(i) *Either* $\phi(x) \equiv x$ *or it has no more than one fixed point.*

(ii) *If* ϕ *has one fixed point, say* p, *then* $\phi(x) < x$ $[\phi(x) > x]$ *for* $x < p$ *and* $\phi(x) > x$ $[\phi(x) < x]$ *for* $x > p$.

Proof. (i): Suppose that p, q are fixed points of ϕ with $p < q$; from equation (2) we get

$$\phi[G(p,q)] = G(p,q) \in (p,q),$$

so $G(p,q)$ is a fixed point of ϕ in the interval (p,q). This, with the continuity of ϕ, implies that the set of the fixed points of ϕ is connected. Thus we suppose that the interval $[p,q]$, $p < q$, is the set of the fixed points of ϕ and consider the level set $\Lambda(q)$. If $(x,y) \in \Lambda(q)$, then also $\Phi(x,y) \in \Lambda(q)$ and if $y \in (p,q)$ (and so $x > q$) we obtain that both points (x,y) and $\Phi(x,y) = (\phi(x), y)$ belong to $\Lambda(q)$; a contradiction.

(ii): Assume $\phi(x) \geq x$ for all $x \in \mathbb{R}$ and consider the level set $\Lambda(p)$; for any point $(x,y) \in \Lambda(p)$ we must have $\Phi(x,y) \in \Lambda(p)$; from $\phi(x) > x$ and $\phi(y) > y$ we have a contradiction since the function f_p is strictly decreasing. ◊

3. Construction of the solutions

In this section we describe a method for the construction of the solutions of the functional equation (2). More precisely we construct the pairs (ϕ, G) with $\phi : \mathbb{R} \to \mathbb{R}$ continuous and strictly increasing and $G : \mathbb{R}^2 \to \mathbb{R}$ continuous, strictly increasing in each variable and reflexive. To do this we assume ϕ given and construct G so that (2) is satisfied. The procedure is different depending on the number of fixed points and on the range of ϕ. In the following $\mathbb{Z}$ and $\mathbb{N}$ denote the integers and the non-negative integers respectively.

I. The function ϕ has no fixed points and is surjective.

By Remark 1 we can suppose, without loss of generality, that $\phi(x) > x$. The surjectivity of ϕ implies that the function Φ is invertible on $\mathbb{R}^2$.

First assume that (ϕ, G) is a solution of (2).

We fix arbitrarily a value a and set $g_0 = f_a$. Now we define the sequence of functions

$$g_n(x) = \Phi^n g(x) := \phi^n \circ g_0 \circ \phi^{-n}(x), \quad x \in \phi^n(E_a), \quad n \in \mathbb{Z}.$$

Every function g_n is continuous and strictly decreasing and $\mathrm{Gr}(g_n) = \Lambda(\phi^n(a))$. About the function G, we observe that it is completely determined by the values assumed in the set

(4) $$\mathcal{F} = \mathcal{F}_0 \cap \mathcal{F}_1^c$$

where

$$\mathcal{F}_i = \{(x,y) \in \mathbb{R}^2 : y \geq g_i(x), x \in \phi^i(E_a)\} \cup \cup\{(x,y) \in \mathbb{R}^2 : x \geq \sup \phi^i(E_a)\}, \quad i = 0, 1.$$

Indeed, take a point $(x,y) \in \mathbb{R}^2 \setminus \mathcal{F}$ and let $u = G(x,y)$; there exists a unique $n \in \mathbb{Z}$ such that $a \leq \phi^n(u) < \phi(a)$ and so $\Phi^n(x,y) \in \mathcal{F}$. By the equation we obtain

$$G(x,y) = \phi^{-n}\left[G(\Phi^n(x,y))\right].$$

Following the properties stated in Th. 2 and the considerations above we can easily describe how to construct the solutions of (2), when we are given a function ϕ increasing, without fixed points, surjective and such that $\phi(x) > x$ for every $x \in \mathbb{R}$.

We choose $a \in \mathbb{R}$ and an open interval E_a with $a \in E_a$, then we take an arbitrary continuous strictly decreasing function g_0 defined on E_a such that its graph is unbounded in both directions and $g_0(a) = a$. Now we construct the function

$$g_1(x) = \Phi g_0(x), \quad x \in \phi(E_a).$$

Clearly the fixed point of g_1 is $\phi(a) > a$ so $\mathrm{Gr}(g_1)$ is in the upper-right region of the plane determined by $\mathrm{Gr}(g_0)$.

Theorem 4. *Let $\mathcal{F}$ be the set defined as in* (4) *and let $G_0 : \mathcal{F} \to \mathbb{R}$ be a CRI-function with the following properties:*

(i) $\mathrm{Gr}(g_0) = \Lambda(a)$;

(ii) $\lim_{(t,s)\to(x,g_1(x))} G_0(t,s) = \phi(a)$ *for every* $x \in \phi(E_a)$.

Then G_0 can be uniquely extended to a CRI-function $G : \mathbb{R}^2 \to \mathbb{R}$ such that (ϕ, G) is a solution of (2).

Proof. Condition (ii) assures that if we extend G_0 to the closure of $\mathcal{F}$ by assigning the value $\phi(a)$ on $Gr(g_1)$, such an extension is continuous. Now we extend G_0 to the whole $\mathbb{R}^2$. Define

$$\mathcal{F}^n = \{\Phi^n(x,y) : (x,y) \in \mathcal{F}\}, \quad n \in \mathbb{Z}, \quad \mathcal{F}^0 = \mathcal{F}.$$

Obviously the sets $\mathcal{F}^n$ are pairwise disjoint and $\mathbb{R}^2 = \bigcup_{n\in\mathbb{Z}} \mathcal{F}^n$. Thus for every $(x,y) \in \mathbb{R}^2$ there exists a unique $n \in \mathbb{Z}$ such that $\Phi^{-n}(x,y) \in \mathcal{F}$; we define

$$G(x,y) = \phi^n\left[G_0(\Phi^{-n}(x,y))\right].$$

We immediately recognize that the function G has all properties requested and the pair (ϕ, G) is a solution of equation (2). $\Diamond$

With the previous construction in general we have a groupoid and not a quasi–group. To get a quasi–group we have to choose the function g_0 so that its domain and its range are the whole $\mathbb{R}$. In this case for fixed x and y the functions $x \mapsto G(x,y)$ and $y \mapsto G(x,y)$ assume all real values and so do the functions $x \mapsto \phi[G(x,y)]$ and $y \mapsto \phi[G(x,y)]$.

II. The function ϕ has no fixed points and is not surjective.

By Remark 1, it is enough to study the case ϕ bounded below and so $\phi(x) > x$. Set $m = \inf_x \phi(x) = \lim_{x\to\infty} \phi(x)$ and define

$$\mathcal{D} = \mathbb{R}^2 \setminus (m, +\infty)^2.$$

Note that for any $(x,y) \in \mathbb{R}^2$ the point $\Phi(x,y)$ does not belong to $\mathcal{D}$.

From now on, given a function f we shall write $\inf f$ and $\sup f$ instead of $\inf_x f(x)$ and $\sup_x f(x)$.

Theorem 5. *Assume* (ϕ, G) *is a solution of* (2) *satisfying* (A), *with* $\phi(x) > x$ *and* $m = \inf \phi > -\infty$.

There exists $U_1 \leq +\infty$ *such that for every* $u < U_1$ *the interval* E_u *is bounded above and* $u < v < U_1$ *implies* $\sup E_u < \sup E_v$. *Moreover,* $\sup\{\sup E_u : u < U_1\} = +\infty$ *and* $\inf\{\sup E_u : u \in \mathbb{R}\} = -\infty$.

There exists $U_2 \leq +\infty$ *such that for every* $u < U_2$ *the function* f_u *is bounded above and* $u < v < U_2$ *implies* $\sup f_u < \sup f_v$. *Moreover,* $\sup\{\sup f_u : u < U_2\} = +\infty$ *and* $\inf\{\sup f_u : u \in \mathbb{R}\} = -\infty$.

Proof. Take a point $(x,y) \in \mathbb{R}^2$ and its corresponding point $\Phi(x,y)$. We hold x fixed and let y go to $-\infty$; the point $\Phi(x,y)$ goes to $(\phi(x), m)$. If $G(x,y) \to -\infty$ for $y \to -\infty$, then the functional equation and the continuity of G imply $G[\phi(x), m] = m$. Thus the level curve $\Lambda(m)$ is not strictly decreasing; a contradiction.

The previous argument shows that for u small enough E_u is bounded above; let U_1 be the supremum of these values. Take $u < < v < U_1$, obviously $\sup E_u \leq \sup E_v$; assume $\sup E_u = \sup E_v = s$, then $\lim_{x\to s} f_u(x) = \lim_{x\to s} f_v(x) = -\infty$ and so $\lim_{x\to\phi(s)} f_{\phi(u)}(x) = = \lim_{x\to\phi(s)} f_{\phi(v)}(x) = m$, i.e., $(\phi(s), m) \in \Lambda(\phi(u)) \cap \Lambda(\phi(v))$; a contradiction.

If $U_1 = +\infty$, obviously $\sup\{\sup E_u : u < U_1\} = +\infty$. Let now $U_1 < +\infty$ and suppose $\sup\{\sup E_u : u < U_1\} = \sigma < +\infty$. Then $\sup E_{U_1} = \sigma$: if not, take $(x,y) \in \mathbb{R}^2$ with $x > \sigma$ and $y < f_{U_1}(x)$; from $U_1 < G(x,y) < U_1$ we have a contradiction.

Take now $x_0 > \sigma$ and let $y \to -\infty$; clearly $G(x_0, y) \to U_1$, since the line $x = x_0$ crosses, as $y \to -\infty$, all sets $\Lambda(v)$ for v in a right neighbourhood of U_1. We have $\Phi(x_0, y) \to (\phi(x_0), m)$ and, by the equation and the continuity of G, we obtain

$$G\big(\Phi(x_0, y)\big) \to \phi(U_1) = G\big(\phi(x_0), m\big),$$

i.e., $(\phi(x_0), m) \in \Lambda(\phi(U_1))$ for every $x_0 > \sigma$; a contradiction.

To prove the last statement, assume $\inf\{\sup E_u : u \in \mathbb{R}\} = L > -\infty$, take a point (τ, m) with $\tau < \phi(L)$ and the corresponding level set $\Lambda(t)$. If we consider the level set $\Lambda(\phi^{-1}(t))$ we immediately get $\sup E_{\phi^{-1}(t)} < L$; a contradiction.

Similarly we prove the second part of the theorem. $\Diamond$

The function G is completely determined by its values in $\mathcal{D}$. Indeed, the sets $\mathcal{D}_n = \Phi^n \mathcal{D}$, $n \geq 0$, are pairwise disjoint and $\bigcup_{n=0}^{+\infty} \mathcal{D}_n = \mathbb{R}^2$. By the equation if $(x, y) \in \mathcal{D}_n$ we have

$$G(x, y) = \phi^n\big[G(\Phi^{-n}(x, y))\big],$$

and $\Phi^{-n}(x, y) \in \mathcal{D}$.

Now we show how to construct the solutions when we are given the function ϕ which is bounded below.

Theorem 6. *Let $\mathcal{D}$ be the set defined above and let $G_0 : \mathcal{D} \to \mathbb{R}$ be a CRI–function with the following properties:*

(i) *the level sets of G_0 in $\mathcal{D}$ satisfies the conditions of* Th. 5*;*

(ii) *for every $u \in \mathbb{R}$ such that* $\sup E_u < +\infty$, $G_0(\phi(\sup E_u), m) = \phi(u)$*; for every $v \in \mathbb{R}$ such that* $\sup f_v < +\infty$, $G_0(m, \phi(\sup f_u)) = \phi(u)$*;*

Then G_0 can be uniquely extended to a CRI–function $G : \mathbb{R}^2 \to \mathbb{R}$ such that (ϕ, G) is a solution of (2).

Proof. We extend G_0 to the whole $\mathbb{R}^2$ as follows. Define

$$\mathcal{D}^n = \big\{\Phi^n(x, y) : (x, y) \in \mathcal{D}\big\}, \quad n \in \mathbb{N}, \quad \mathcal{D}^0 = \mathcal{D}.$$

Obviously the sets $\mathcal{D}^n$ are pairwise disjoint and $\mathbb{R}^2 = \bigcup_{n\in\mathbb{N}} \mathcal{D}^n$. Thus for every $(x, y) \in \mathbb{R}^2$ there exists a unique $n \in \mathbb{N}$ such that $\Phi^{-n}(x, y) \in \mathcal{D}$; we define

$$G(x, y) = \phi^n\big[G_0(\Phi^{-n}(x, y))\big].$$

Condition (ii) guarantees that the function G is continuous on $\mathbb{R}^2$ and obviously by our construction the pair (ϕ, G) is a solution of equation (2). $\Diamond$

Remark 2. If we look to the statement of Th. 6, the following problem arises: *How to construct a CRI–function satisfying conditions* (i) *and* (ii)*?* Clearly the problem concerns condition (ii), since we have to give the values to G_0 in points which depend on G_0 itself.

Now we show a *constructive procedure*.

Take a continuous strictly decreasing function h_0 defined on $(-\infty, A)$, $m < A \leq +\infty$, with $h_0(m) = m$ and define a CRI–function G_0 in

$$\mathcal{R}_0 = \{(x,y) : y \leq h_0(x), x \in (-\infty, A)\}$$

such that $G_0(x, h_0(x)) = m$ and satisfying the conditions of Th. 5.

Assume $A = +\infty$. In this case all sets E_u bounded above correspond to values of u less than m. Thus we can compute for each of these E_u the value $\phi(\sup E_u)$. Now we define G_0 on the line $(m, +\infty) \times \times \{m\}$ so that $G_0(\phi(\sup E_u), m) = \phi(u)$. The next step simply consists in defining G_0 below the line $y = m$ in order to get a CRI–function continuous also in the points of the line $(m, +\infty) \times \{m\}$.

Suppose now $A < +\infty$. Thus for every $u \leq m$ we have $\sup E_u < < +\infty$. For these values of u we compute $\phi(\sup E_u)$ and define G_0 on $(m, \phi(A)) \times \{m\}$ so that $G_0(\phi(\sup E_u), m) = \phi(u)$. Now we take a continuous strictly decreasing function h_1 defined on $[\phi(A), B)$, $B \leq \leq +\infty$, such that $h_1(\phi(A)) = m$; furthermore, we define $G_0(x, h_1(x)) = = \phi(m)$ and extend G_0 to the set

$$\mathcal{R}_1 = \mathcal{R} \cap \{(x,y) : y \leq h_1(x), x < B\}$$

so that G_0 is a CRI–function satisfying the conditions of Th. 5. We proceed iteratively to get G_0 on the whole set below the line $y = m$.

The analogous procedure permits to construct G_0 on the left of the line $x = m$ and satisfying all conditions of Th. 6.

III. The function ϕ has one fixed point and is surjective.

Let p be the fixed point of ϕ and assume $\phi(x) > x$ for $x > p$ and $\phi(x) < x$ for $x < p$. Let f_p the function corresponding to the level curve $\Lambda(p)$, by Th. 1 we have $\Phi(x, f_p(x)) \in \Lambda(p)$ and this implies $f_p(\phi(x)) = \phi(f_p(x))$, i.e., f_p and ϕ are a pair of commuting functions.

Theorem 7. *For every* $u \in \mathbb{R}$ *it is* $E_u = \mathbb{R}$ *and* $f_u(\mathbb{R}) = \mathbb{R}$.

Proof. First we prove that $E_p = \mathbb{R}$ and $f_p(\mathbb{R}) = \mathbb{R}$. Suppose that $\sup E_p = \sigma < +\infty$, then as $x \to \sigma$ we have $f_p(x) \to -\infty$ and $(\phi(x), \phi(f_p(x))) \to (\phi(\sigma), -\infty)$; this implies $\phi(\sigma) = \sigma$; a contradiction since $\sigma > p$. Similarly we get $\inf E_p = -\infty$ and $f_p(\mathbb{R}) = \mathbb{R}$.

Fix now $u > p$ and consider the function f_u. Since the graph of f_u is above that of f_p, from the previous part of the proof we obtain $\sup E_u = \sup f_u = +\infty$.

If $E_u \subset [p, +\infty)$, then for every $n \in \mathbb{Z}$ we have $\phi^n(E_u) \subset [p, +\infty)$. This implies that the set $\{(x,y) \in \mathbb{R}^2 : G(x,y) > p\}$ is contained in the set $\{(x,y) \in \mathbb{R}^2 : x > p\}$. This is impossible, since for every point (x,y) with $x \in E_p \cap (-\infty, p)$ and $y > f_p(x)$ we have $G(x,y) > p$. The same argument proves that $f_u(E_u)$ is not contained in $[p, +\infty)$.

Assume now that $\inf E_u = \alpha > -\infty$. Since $\alpha < p$, we have $\phi(\alpha) < < \alpha$ and $\phi(u) > u$; this implies that $\Lambda(u) \cap \Lambda(\phi(u)) \neq \emptyset$: a contradiction. Thus $\inf E_u = -\infty$. Similarly we prove that $\inf f_u(E_u) = -\infty$.

Obviously the same result holds for $u < p$. ◊

As a consequence of Th. 7 in this case the groupoid F is in fact a quasi–group.

Now we fix two arbitrary values $a < p$ and $b > p$, we set $h_0 = f_a$ and $g_0 = f_b$ and define the sequences of functions

$$g_n = \Phi^{-n} g_0 \quad , \quad h_n = \Phi^{-n} h_0, \quad n \in \mathbb{Z}.$$

Clearly we have the following.

Theorem 8. *Let $\{g_n\}$ and $\{h_n\}$ be defined as above. For every $n \in \mathbb{Z}$ and $x \in \mathbb{R}$ we have*

$$g_n(x) > g_{n+1}(x) > f_p(x) > h_{n+1}(x) > h_n(x);$$

for every $x \in \mathbb{R}$ we have

$$\inf_n g_n(x) = \lim_{n \to +\infty} g_n(x) = f_p(x) = \lim_{n \to +\infty} h_n(x) = \sup_n h_n(x).$$

Moreover, the function G is completely determined by the values assumed in the set

$$\mathcal{E} = \{(x,y) \in \mathbb{R}^2 : g_1(x) < y \le g_0(x)\} \cup \cup \{(x,y) \in \mathbb{R}^2 : h_0(x) \le y < h_1(x)\}.$$

Following Ths. 7 and 8, now we construct the solutions of (2). Assume we are given a function ϕ with the fixed point p and such that $\phi(x) < x$ for $x < p$ and $\phi(x) > x$ for $x > p$. We choose two arbitrary numbers a_0 and b_0 with $a_0 < p < b_0$ and define

(5) $$a_n = \phi^n(a_0), \quad b_n = \phi^n(b_0), \quad n \in \mathbb{Z}.$$

Obviously the two sequences $\{a_n\}$ and $\{b_n\}$ have the following properties:

(6) $$a_{n+1} < a_n; \lim_{n \to +\infty} a_n = -\infty; \lim_{n \to -\infty} a_n = p,$$
$$b_{n+1} > b_n; \lim_{n \to +\infty} b_n = +\infty; \lim_{n \to -\infty} b_n = p.$$

Let g_0 and h_0 be two functions defined on (a_1, b_1) continuous, strictly decreasing, with

(7) $$h_0(a_0) = a_0, g_0(b_0) = b_0; h_0(x) < g_0(x), x \in (a_1, b_1).$$

Now we define four sequences of functions $\{h_n^l\}, \{h_n^r\}, \{g_n^l\}, \{g_n^r\}$, $n \ge 0$, with the following properties:

(8)

$$\begin{cases} h_n^l, g_n^l : (a_1, a_0] \to \mathbb{R} \quad \text{are continuous and strictly decreasing,} \\ h_0^l \text{and} g_0^l \quad \text{are the restrictions to } (a_1, a_0] \text{ of } h_0 \text{ and } g_0 \text{ respectively,} \\ h_n^l(x) < h_{n+1}^l(x) < g_{n+1}^l(x) < g_n^l(x), \\ h_{n+1}^l(a_0) = \lim\limits_{x \to a_1^+} \phi^{-1}[h_n^l(x)], \\ g_{n+1}^l(a_0) = \lim\limits_{x \to a_1^+} \phi^{-1}[g_n^l(x)], \\ \lim\limits_{n \to +\infty} [g_n^l(x) - h_n^l(x)] = 0 \quad \text{for every } x \in (a_1, a_0], \\ \text{the function } H^l(x) = \lim\limits_{n \to +\infty} g_n^l(x) = \lim\limits_{n \to +\infty} h_n^l(x) \text{ is strictly} \\ \text{decreasing and } H^l(a_0) > p; \end{cases}$$

(9)

$$\begin{cases} h_n^r, g_n^r : [b_0, b_1) \to \mathbb{R} \quad \text{are continuous and strictly decreasing,} \\ h_0^r \text{ and } g_0^r \quad \text{are the restrictions to } [b_0, b_1) \text{ of } h_0 \text{ and } g_0 \text{ respectively,} \\ h_n^r(x) < h_{n+1}^r(x) < g_{n+1}^r(x) < g_n^r(x), \\ h_{n+1}^r(b_0) = \lim\limits_{x \to b_1^-} \phi^{-1}[h_n^r(x)], \\ g_{n+1}^r(b_0) = \lim\limits_{x \to b_1^-} \phi^{-1}[g_n^r(x)], \\ \lim\limits_{n \to +\infty} [g_n^r(x) - h_n^r(x)] = 0 \quad \text{for every } x \in [b_0, b_1), \\ \text{the function } H^r(x) = \lim\limits_{n \to +\infty} g_n^r(x) = \lim\limits_{n \to +\infty} h_n^r(x) \text{ is strictly} \\ \text{decreasing and } H^r(b_0) < p. \end{cases}$$

In the next step we extend the functions h_0 and g_0 to the whole $\mathbb{R}$.

Lemma 1. *Let* $g_0, h_0 : (a_1, b_1) \to \mathbb{R}$ *be as in* (7). *Define* $g_0 : \mathbb{R} \setminus \setminus (a_1, b_1) \to \mathbb{R}$ *as*

$$\begin{cases} g_0(x) = \Phi^n g_n^r(x), x \in [b_n, b_{n+1}), \\ g_0(x) = \Phi^n g_n^l(x), x \in (a_{n+1}, a_n], \end{cases}$$

and $h_0 : \mathbb{R} \setminus (a_1, b_1) \to \mathbb{R}$ *as*

$$\begin{cases} h_0(x) = \Phi^n h_n^r(x), x \in [b_n, b_{n+1}), \\ h_0(x) = \Phi^n h_n^l(x), x \in (a_{n+1}, a_n], \end{cases}$$

where $n \geq 1$ *and* $\{h_n^l\}, \{h_n^r\}, \{g_n^l\}, \{g_n^r\}$ *are as in* (8) *and* (9).

Then g_0 *and* h_0 *are continuous and strictly decreasing on* $\mathbb{R}$.

Moreover,

$$\lim_{x\to+\infty} g_0(x) = \lim_{x\to+\infty} h_0(x) = -\infty$$
$$\lim_{x\to-\infty} g_0(x) = \lim_{x\to-\infty} h_0(x) = +\infty.$$

Proof. We prove the lemma for the function g_0. It is continuous. Indeed

$$g_0(b_n) = \Phi^n g_n^r(b_n) = \phi^n \circ g_n^r(b_0)$$

and

$$\lim_{x\to b_n^-} g_0(x) = \lim_{x\to b_n^-} \Phi^{n-1} g_{n-1}^r(x) = \lim_{x\to b_1^-} \phi^{n-1} \circ g_{n-1}^r(x) = \phi^n[g_n^r(b_0)],$$

thus g_0 is continuous on $[b_0, +\infty)$; in the same way we prove that it is continuous on $(-\infty, b_0]$. From this and the properties of the functions g_n^r and g_n^l we immediately have that g_0 is strictly decreasing. Since $H^r(b_0) < p$, for n large enough we have $g_n^r(b_0) < p$ and this implies that $\lim_{n\to+\infty} g_0(b_n) = -\infty$ and so $\lim_{x\to+\infty} g_0(x) = -\infty$.

In a completely analogous way we prove the other limit formulae. ◊

Now we define

$$g_n(x) = \Phi^{-n} g_0(x) \quad , \quad h_n(x) = \Phi^{-n} h_0(x) \quad , \quad x \in \mathbb{R}, n > 0.$$

Lemma 2. *For every $n > 0$ the restrictions of g_n and h_n to $[b_0, b_1)$ are g_n^r and h_n^r respectively. For every $n > 0$ the restrictions of g_n and h_n to $(a_1, a_0]$ are g_n^l and h_n^l respectively.*

Proof. Take $x \in [b_0, b_1)$, then $\phi^n(x) \in [b_n, b_{n+1})$ and, by the construction of g_0, $g_0 \circ \phi^n(x) = \phi^n \circ g_n^r(x)$ and so $g_n^r(x) = \Phi^{-n} g_0(x) = g_n(x)$.

The other cases are analogous. ◊

Lemma 3. *The sequence $\{g_n\}$ is decreasing; the sequence $\{h_n\}$ is increasing. For every n and every $x \in \mathbb{R}$ we have $g_n(x) > h_n(x)$.*

Proof. First we consider the sequence $\{g_n\}$; if $x \in [b_0, b_1)$ then $g_{n+1}(x) = g_{n+1}^r(x) < g_n^r(x) = g_n(x)$ by the definition of the sequence $\{g_n^r\}$. Take $x \geq b_1$, so $x \in [b_N, b_{N+1})$ for some N and $\phi^{n+1}(x) \in$ $\in [b_{N+n+1}, b_{N+n+2})$, $\phi^n(x) \in [b_{N+n}, b_{N+n+1})$. Then

$$\begin{aligned} g_{n+1}(x) &= \phi^{-n-1} \circ \phi^{N+n+1} \circ g_{N+n+1}^r \circ \phi^{-N-n-1} \circ \phi^{n+1}(x) \\ &= \phi^N \circ g_{N+n+1}^r \circ \phi^{-N}(x); \gamma_n(x) \\ &= \phi^{-n} \circ \phi^{N+n} \circ g_{N+n}^r \circ \phi^{-N-n} \circ \phi^n(x) \\ &= \phi^N \circ g_{N+n}^r \circ \phi^{-N}(x); \end{aligned}$$

and the inequality $g_{n+1}(x) < g_n(x)$ follows from $g_{n+1}^r(x) < g_n^r(x)$. In a completely similar way we obtain the the inequality for $x \leq a_0$.

It remains to be considered the interval (a_0, b_0); take $x \in [b_{-1}, b_0)$ and so $\phi(x) \in [b_0, b_1)$. Then for $n \geq 1$ we have

$$g_n(x) = \phi^{-n} \circ g_0 \circ \phi^n(x) = \phi^{-n} \circ g_0 \circ \phi^n \circ \phi^{-1} \circ \phi(x)$$
$$= \phi^{-n} \circ g_0 \circ \phi^{n-1} \circ \phi(x) = \phi^{-1} \circ \phi^{1-n} \circ g_0 \circ \phi^{n-1} \circ \phi(x)$$
$$= \phi^{-1} \circ g_{n-1} \circ \phi(x) = \phi^{-1} \circ g^r_{n-1} \circ \phi(x)$$

and $g_{n+1} = \phi^{-1} \circ g^r_n \circ \phi(x)$, thus the inequality follows. For $n = 0$ we have

$$g_1(b_{-1}) = \phi^{-1} \circ g_0 \circ \phi(b_{-1}) = \phi^{-1} \circ g_0(b_0) = \phi^{-1}(b_0) < b_0,$$

so $g_1(x) < g_0(x)$ for every $x \in [b_{-1}, b_0)$.

Proceeding iteratively we get the inequality on the whole interval $[p, b_0)$.

Similarly we obtain the inequality on (a_0, p) and for the sequence $\{h_n\}$.

Arguing as before we prove the inequality $h_n(x) < g_n(x)$ starting from $h_0(x) < g_0(x)$ on (a_0, b_0). ◊

Lemma 4. *Suppose* $\{s_n\}$ *and* $\{u_n\}$ *are sequences of continuous decreasing functions defined on an open interval* I. *Moreover, assume that*

$$s_n(x) \le s_{n+1}(x) \le u_{n+1}(x) \le u_n(x), n \ge 1, x \in I;$$
$$\lim_{n\to+\infty} s_n(x) = \lim_{n\to+\infty} u_n(x) = Z(x), x \in I.$$

Then Z *is continuous in* I.

Proof. Obviously Z is non increasing. Suppose that Z is not continuous in r; this means that

$$\lim_{x\to r^-} Z(x) = A > B = \lim_{x\to r^+} Z(x).$$

Assume $B < Z(r) \le A$. For any ε with $0 < \varepsilon < (Z(r) - B)/4$ there exists ν such that

$$s_\nu(r) > Z(r) - \varepsilon > Z(r) - \frac{Z(r) - B}{4}$$

and, by the continuity of s_ν, there exists $\delta > 0$ such that for $x \in$ $\in (r - \delta, r + \delta) \cap I$ we have

$$s_\nu(x) > Z(r) - \frac{Z(r) - B}{2} = \frac{Z(r) + B}{2}.$$

Hence for every $n \ge \nu$ and $x \in (r-\delta, r+\delta) \cap I$ we have $s_n(x) > \frac{Z(r)+B}{2}$ and so $Z(x) > \frac{Z(r)+B}{2}$. This implies

$$\lim_{x\to r^+} Z(x) \ge \frac{Z(r) + B}{2};$$

a contradiction.

If $B = Z(r) < A$, the analogous proof is obtained by working on the sequence $\{u_n\}$. ◊

Lemma 5. *For every* $x \in \mathbb{R}$,

$$\lim_{n\to+\infty} g_n(x) = \lim_{n\to+\infty} h_n(x) = H(x).$$

The function H *is continuous and strictly decreasing and*

$$H_{|[b_0,b_1)} = H^r, H_{|(a_1,a_0]} = H^l.$$

Moreover, H *commutes with* ϕ: $H \circ \phi = \phi \circ H$.

Proof. Take $x \in (p, +\infty)$ and let $N \in \mathbb{Z}$ such that $\phi^N(x) \in [b_0, b_1)$. For n such that $n + N > 0$ we have

$$\begin{aligned} g_n(x) &= \phi^N \circ g_{n+N} \circ \phi^{-N}(x) &= \phi^N \circ g^r_{n+N} \circ \phi^{-N}(x), \\ h_n(x) &= \phi^N \circ h_{n+N} \circ \phi^{-N}(x) &= \phi^N \circ h^r_{n+N} \circ \phi^{-N}(x), \end{aligned} \tag{10}$$

thus from the continuity of ϕ and the property

$$\lim_{n\to+\infty} [g^r_n(x) - h^r_n(x)] = 0$$

for every $x \in [b_0, b_1)$, we get

$$\lim_{n\to+\infty} [g_n(x) - h_n(x)] = 0$$

for every $x \in (p, +\infty)$. So we define the function H as the common limit of the two sequences. In the same way we prove the existence of the limit function H in $x \in (-\infty, p]$. Obviously in $[b_0, b_1)$ and in $(a_1, a_0]$ we obtain H^r and H^l respectively. By Lemma 4, the function H is continuous. By (10) we have $H(x) = \phi^N \circ H^r \circ \phi^{-N}(x)$ in $[b_{-N}, b_{-N+1})$, since H^r is strictly decreasing, so it is H in $(p, +\infty)$. The same is true for $x \in (-\infty, p]$.

The commutativity of H and ϕ follows immediately. ◊

Theorem 9. *Let* $\mathcal{E}$ *be the set defined as in* Th. 8 *and let* $\mathcal{T} = \{(x, H(x)) : : x \in \mathbb{R}\}$. *Let* $G_0 : \mathcal{E} \cup \mathcal{T} \to \mathbb{R}$ *be a CRI–function with the following properties:*

(i) $Gr(g_0) = \Lambda(b_0)$;
(ii) $Gr(h_0) = \Lambda(a_0)$;
(iii) $\mathcal{T} = \Lambda(p)$;

$$\begin{aligned} &\lim_{(t,s)\to(x,g_1(x))} G_0(t, s) = b_{-1}, x \in \mathbb{R}, \\ &\lim_{(t,s)\to(x,h_1(x))} G_0(t, s) = a_{-1}, x \in \mathbb{R}. \end{aligned} \tag{iv}$$

Then the function G_0 *can be uniquely extended to a CRI–function* G *on* $\mathbb{R}^2$ *such that* (ϕ, G) *is a solution of equation* (2).

Proof. Since

$$\mathbb{R}^2 = \mathcal{T} \cup \Big(\bigcup_{n\in\mathbb{Z}} \mathcal{E}^n \Big)$$

and the sets $\mathcal{T}$ and $\mathcal{E}^n$ are pairwise disjoint, we extend G_0 to the whole

$\mathbb{R}^2$ by using the equation. The properties of the functions G_0 assures that G is a CRI–function. ◊

In the case $\phi(x) < x$ for $x > p$, $\phi(x) > x$ for $x < p$ we proceed in an analogous way.

IV. The function ϕ has one fixed point and is not surjective.

By Remark 1, we study only the case ϕ bounded below and so $\phi(x) > x$ for $x < p$ and $\phi(x) < x$ for $x > p$. Set $m = \inf \phi = \lim_{x \to -\infty} \phi(x)$. We have the following.

Theorem 10. *Le* (ϕ, G) *be a solution of* (2) *satisfying* (A) *with* ϕ *as above.*

For every $u \in \mathbb{R}$ *the interval* E_u *is bounded above and* $u < v$ *implies* $\sup E_u < \sup E_v$. *Moreover,* $\inf\{\sup E_u : u \in \mathbb{R}\} = -\infty$ *and* $\sup\{\sup E_u : u < p\} = \sup E_p = \inf\{\sup E_u : u > p\}$.

Every function f_u, $u \in \mathbb{R}$, *is bounded above and* $u < v$ *implies* $\sup f_u < \sup f_v$.

Moreover, $\sup\{\sup f_u : u < p\} = \sup f_p = \inf\{\sup f_u : u > p\}$, $\inf\{\sup f_u : u \in \mathbb{R}\} = -\infty$ *and, for every* $x \in E_p$, $\inf\{f_u(x) : u > p\} = f_p(x) = \sup\{f_u(x) : u < p\}$.

Proof. Consider the function f_p and a point $(x, f_p(x))$; if $\sup E_p = +\infty$, the point $\Phi(x, f_p(x)) = (\phi(x), f_p(\phi(x))$ is above the line $y = m$ and its first coordinate goes to $+\infty$ as $x \to +\infty$; this implies that $p > w = \inf f_p \geq m$. Thus $\phi(f_p(x)) = f_p(\phi(x)) \to \phi(w) = w$ as $x \to +\infty$, i.e., w is a fixed point of ϕ different from p; a contradiction. So $\sup E_p < +\infty$ and $\inf f_p = -\infty$. A similar argument proves that $\sup f_p < +\infty$. Thus the two properties hold for every $u < p$.

Let now $u > p$ and suppose $\sup E_u = +\infty$. If $f_u(x) \to -\infty$ as $x \to +\infty$, then $\phi(x) \to +\infty$ and $\phi(f_u(x)) \to m$; a contradiction since this implies $\Lambda(u) \cap \Lambda(\phi(u)) \neq \emptyset$. If $f_u(x) \to k$ as $x \to +\infty$, then $\phi(f_u(x)) \to \phi(k)$ and to avoid $\Lambda(u) \cap \Lambda(\phi(u)) \neq \emptyset$ we must have $\phi(k) \leq k$; the only possibility is $k \geq p$. Clearly, for every $n \in \mathbb{N}$ the functions $f_{\phi^n(u)}$ have the same property and $\phi^n(u) \to p$ as $n \to +\infty$. Thus $p = \inf\{u > p : \sup E_u = +\infty\}$. Since $\sup E_p < +\infty$ we get a contradiction since no level set can have points in the region over the curve $\Lambda(p)$ and under the line $y = p$. As in Th. 5 we prove that $\inf\{\sup E_u : u \in \mathbb{R}\} = -\infty$.

The other parts of the theorem follow immediately. ◊

Consider the set $\mathcal{D} = \mathbb{R}^2 \setminus (m, +\infty)^2$. For every point (x, y) belonging to the set $(m, +\infty)^2 \setminus [p, +\infty)^2$ there exists a unique $n \in \mathbb{N}$

such that $\Phi^{-n}(x,y) \in \mathcal{D}$. Thus the function G is completely determined on $\mathbb{R}^2 \setminus [p,+\infty)^2$ by the values assumed in $\mathcal{D}$. More precisely, if $s(p) = \sup E_p$, we immediately see that the values of G in the set $\{(x,y) : m \le x \le \phi(s(p)), m \le y \le f_p(x)\}$ are completely determined by those in $\mathcal{D} \cap \{(x,y) : x \in E_p, y \le f_p(x)\}$.

For what concern the set $[p,+\infty)^2$, we fix a value $b > p$ and set $g_0 = f_b$ and define $g_1 = f_{\phi(b)}$. Clearly the values of G on $[p,+\infty)^2$ are completely determined by those in the set

$$\mathcal{P} := \big\{(x,y) : x \ge p, \quad \max(g_1(x),p) \le y < g_0(x)\big\}.$$

If we consider the sequence $\{f_{\phi^n(m)}\}$, $n \in \mathbb{N}$, then we have $\bigcup_0^{+\infty} E_{\phi^n(m)} = E_p$. Thus every $x \in E_p$ belongs to $E_{\phi^n(m)}$ for n large enough and we have

$$\lim_{n\to+\infty} f_{\phi^n(m)}(x) = f_p(x).$$

If we start from f_u with $u > p$ and proceed as before, we obtain a sequence of functions pointwise converging to f_p from above.

Guided by Th. 10 and the previous discussion about the function G, we proceed by describing a *constructive procedure for the solutions.*

We choose four real sequences $\{s_n\}$, $\{u_n\}$, $\{r_n\}$ and $\{v_n\}$, $n \in \mathbb{N}$, satisfying the following conditions:

— for every n, $m < s_n < s_{n+1}$ and $s_n > \phi(s_{n-1})$; $\lim_{n\to+\infty} s_n = \gamma > p$;

— for every n, $m < u_n < u_{n+1}$ and $u_n > \phi(u_{n-1})$; $\lim_{n\to+\infty} u_n = \sigma > p$;

— for every n, $r_n > r_{n+1}$ and $r_n > \phi(r_{n-1})$; $\lim_{n\to+\infty} r_n = \gamma$;

— for every n, $v_n > v_{n+1}$ and $v_n > \phi(v_{n-1})$; $\lim_{n\to+\infty} v_n = \sigma$.

Now we fix $b_0 < \min\{r_0, v_0\}$ and take two strictly decreasing continuous functions h_0 and g_0 defined on $(-\infty, s_0)$ and $(-\infty, r_0)$ respectively and such that

— $h_0(m) = m$, $\lim_{x\to-\infty} h_0(x) = u_0$, $\lim_{x\to s_0^-} h_0(x) = -\infty$;

— $g_0(b_0) = b_0$, $\lim_{x\to-\infty} g_0(x) = v_0$, $\lim_{x\to r_0^-} g_0(x) = -\infty$.

Finally we take four sequences of strictly decreasing continuous functions $\{h_n^l\}$, $\{h_n^r\}$, $\{g_n^l\}$ and $\{g_n^r\}$, $n \ge 1$, with the following properties:

— for every $n \ge 1$, h_n^l and g_n^l are defined on $(-\infty, m]$ and

$$(11) \qquad \begin{cases} \lim_{x\to-\infty} h_n^l(x) = u_n, \ h_n^l(m) = \phi(u_{n-1}), \\ \lim_{x\to-\infty} g_n^l(x) = v_n, \ g_n^l(m) = \phi(v_{n-1}); \end{cases}$$

— for every $n \geq 1$, h_n^r is defined on $[\phi(s_{n-1}), s_n)$ and

(12) $$\lim_{x \to s_n^-} h_n^r(x) = -\infty, \; h_n^r(\phi(s_{n-1})) = m;$$

— for every $n \geq 1$, g_n^r is defined on $[\phi(r_{n-1}), r_n)$ and

(13) $$\lim_{x \to r_n^-} g_n^r(x) = -\infty, \; g_n^r(\phi(r_{n-1})) = m;$$

— for every $n \geq 1$,

$h_0(x) < h_n^l(x) < h_{n+1}^l(x), \quad g_0(x) > g_n^l(x) > g_{n+1}^l(x), \quad x \in (-\infty, m];$

$h_n^r(x) < h_{n+1}^r(x), \quad x \in [\phi(s_n), s_n);$

$g_n^r(x) > g_{n+1}^r(x), \quad x \in [\phi(r_n), r_n);$

$h_0(x) < h_1^r(x), \quad x \in [\phi(s_0), s_0); \&g_0(x) > g_1^r(x), \quad x \in [\phi(r_0), r_0);$

— for every $x \in (-\infty, m]$

$$\lim_{n \to +\infty} h_n^l(x) = \lim_{n \to +\infty} g_n^l(x) =: H^l(x)$$

and the function H^l is strictly decreasing;

— for every $y \in (-\infty, m]$

$$\lim_{n \to +\infty} (h_n^r)^{-1}(y) = \lim_{n \to +\infty} (g_n^r)^{-1}(y) \;\; =: (H^r)^{-1}(y)$$

and the function H^r is strictly decreasing.

If we define the function h_1 on the interval $(-\infty, s_1)$ as

$$h_1(x) = \begin{cases} h_1^l(x), x \in (-\infty, m] \\ \Phi h_0(x), x \in \big(m, \phi(s_0)\big) \\ h_1^r(x), x \in \big[\phi(s_0), s_1\big) \end{cases}$$

then conditions (11) and (12) guarantee that h_1 is continuous and strictly decreasing. Iteratively, assume h_{n-1} has been defined on (−$-\infty, s_{n-1}$) and construct h_n on $(-\infty, s_n)$ as

$$h_n(x) = \begin{cases} h_n^l(x), x \in (-\infty, m] \\ \Phi h_{n-1}(x), x \in \big(m, \phi(s_{n-1})\big) \\ h_n^r(x), x \in \big[\phi(s_{n-1}), s_n\big) \end{cases}$$

Analogously we define the sequence $\{g_n\}$, $n \geq 1$, starting from g_0 and the sequences $\{g_n^l\}$ and $\{g_n^r\}$. Moreover, we define $g_{-1} = \Phi^{-1} g_0$; note that g_{-1} is defined in $(-\infty, r_{-1})$, where

$$r_{-1} = \phi^{-1}\big(g_0^{-1}(m)\big)$$

and

$$\lim_{x \to -\infty} g_{-1}(x) = v_{-1} = \phi^{-1}\big(g_0(m)\big).$$

Iteratively for every $n < 0$ we define $g_n = \Phi^{-1} g_{n+1}$. In this way we obtain a bilateral sequence $\{g_n\}$, $n \in \mathbb{Z}$.

Finally, if we define H on $(-\infty, \gamma)$ as

$$H(x) = \begin{cases} H^l(x), x \in (-\infty, m] \\ \Phi^n H^l(x), x \in \big(\phi^{n-1}(m), \phi^n(m)\big], n \geq 1, \\ p, x = p \\ \Phi^n H^r(x), x \in \big[\phi^{n+1}(\gamma), \phi^n(\gamma)\big), n \geq 1, \\ H^r(x), x \in \big[\phi(\gamma), \gamma\big) \end{cases}$$

then H is continuous, strictly decreasing and

$$\lim_{n \to +\infty} h_n(x) = H(x) = \lim_{n \to +\infty} g_n(x) \quad x \in (-\infty, \gamma).$$

Moreover, it is immediately checked that the function H commutes with ϕ.

From the previous considerations we know that the function G is completely determined by its values in $\mathcal{D} \cup \mathcal{P}$.

Theorem 11. *Consider the sequences $\{h_n\}$, $n \geq 0$, and $\{g_n\}$, $n \in \mathbb{Z}$, and the function H defined as above. Let $G_0 : \mathcal{D} \to \mathbb{R}$ be a CRI–function with the following properties:*

(i) *for each $n \geq 0$, $Gr(h_n) = \Lambda(\phi^n(m))$;*

(ii) *for each $n \in \mathbb{Z}$, $Gr(g_n) = \Lambda(\phi^n(b_0))$;*

(iii) *$Gr(H) = \Lambda(p)$;*

(iv) *for every $t, w \in \mathbb{R}$ with $t < w$, it is*

$$\sup\big\{x : G_0(x,y) = t\big\} < \sup\big\{x : G_0(x,y) = w\big\}$$
$$\sup\big\{y : G_0(x,y) = t\big\} < \sup\big\{y : G_0(x,y) = w\big\};$$

moreover,

$$\inf\big\{\sup\{x : G_0(x,y) = u\}\big\} = -\infty$$
$$\inf\big\{\sup\{y : G_0(x,y) = u\}\big\} = -\infty;$$

(v) *for every $t \in \mathbb{R}$*

$$G_0\big(\phi\big(\sup\{x : G_0(x,y) = t\}\big), m\big) = \phi(t)$$
$$G_0\big(m, \phi\big(\sup\{y : G_0(x,y) = t\}\big)\big) = \phi(t);$$

Then G_0 can be uniquely extended to a CRI-function $G' : \mathbb{R}^2 \setminus [p, +\infty)^2 \to$ $\to \mathbb{R}$ such that the pair (ϕ, G') is a solution of (2) in $\mathbb{R}^2 \setminus [p, +\infty)^2$.

Here we can repeat Remark 2 almost word by word.

Theorem 12. *Let* $G_1 : \bar{\mathcal{P}} \to \mathbb{R}$ *be a CRI–function such that* $\mathrm{Gr}(g_0) = \Lambda(b_0)$ *and* $\mathrm{Gr}(g_1) = \Lambda(\phi(b_0))$ *and such that the function*

$$\begin{cases} G_1(x,y), (x,y) \in \mathcal{P} \\ G'(x,y), (x,y) \in \mathbb{R}^2 \setminus [p,+\infty)^2 \end{cases}$$

is continuous (*G' is the function defined in* Th. 11). *Then* G_1 *can be uniquely extended to a CRI–function* $G'' : [p,+\infty)^2 \to \mathbb{R}$ *such that the function*

$$G(x,y) = \begin{cases} G'(x,y), (x,y) \in \mathbb{R}^2 \setminus [p,+\infty)^2 \\ G''(x,y), (x,y) \in [p,+\infty)^2 \end{cases}$$

is a solution of equation (2).

To finish we consider the case ϕ bilaterally bounded, so $\phi(x) > x$ for $x < p$, $\phi(x) < x$ for $x > p$ and $m = \inf \phi$, $M = \sup \phi$. In this case the following theorem holds.

Theorem 13. *There exist* $\alpha, \beta \in \mathbb{R}$ *such that*

(i) $\sup E_\alpha = +\infty$ *and* $\lim_{x\to+\infty} f_\alpha(x) = -\infty$;

(ii) *for every* $u < \alpha$ *the interval* E_u *is bounded above and* $u < v < \alpha$ *implies* $\sup E_u < \sup E_v$. *Moreover,* $\sup\{\sup E_u : u < \alpha\} = +\infty$ *and* $\inf\{\sup E_u : u < \alpha\} = -\infty$;

(iii) *for every* $u > \alpha$, $\sup E_u = +\infty$, f_u *is bounded below and* $u > v > \alpha$ *implies* $\inf f_u > \inf f_v$. *Moreover,* $\inf\{\inf f_u : u > \alpha\} = -\infty$ *and* $\sup\{\inf f_u : u > \alpha\} = +\infty$;

(iv) $\inf E_\beta = -\infty$ *and* $\lim_{x\to-\infty} f_\beta(x) = +\infty$;

(v) *for every* $u > \beta$ *the interval* E_u *is bounded below and* $u > v > \beta$ *implies* $\inf E_u > \inf E_v$. *Moreover,* $\sup\{\inf E_u : u > \beta\} = +\infty$ *and* $\inf\{\inf E_u : u > \beta\} = -\infty$;

(vi) *for every* $u < \beta$, $\inf E_u = -\infty$, f_u *is bounded above and* $u < v < \beta$ *inplies* $\sup f_u < \sup f_v$. *Moreover,* $\sup\{\sup f_u : u < \beta\} = +\infty$ *and* $\inf\{\sup f_u : u < \beta\} = -\infty$.

Proof. (i): Define α as the (unique) number such that the point (m, M) belongs to the level set $\Lambda(\phi(\alpha))$. If we consider the point $(x, f_\alpha(x))$ and let x increase, the corresponding point $\Phi(x, f_\alpha(x)) = (\phi(x), f_{\phi(\alpha)}(\phi(x)))$ can reach (m, M) if and only if x goes to $+\infty$ and $f_\alpha(x)$ goes to $-\infty$.

(ii): Take $u < \alpha$ and suppose $\sup E_u = +\infty$ (so $\lim_{x\to+\infty} f_u(x) = -\infty$). Arguing as before we obtain that the point (m, M) belongs to $\Lambda(\phi(u))$ as well; a contradiction. The other parts of ii) follow as in Theorems 5 and 10.

The other parts of the theorem follow in an analogous way. ◊

Note that if $p > \alpha$, then $\lim_{x\to+\infty} f_p(x) = a_p$ and $f_p(M) = \phi(a_p)$. If $p < \alpha$, then $\sup E_p = b_p$ and $f_p(\phi(b_p)) = m$. The analogous remark holds with respect to β. It has to be noted that either α or β or both can coincide with p.

The function G is completely determined by its values in $\mathcal{R} = \mathbb{R}^2 \setminus (m, M)^2$. Indeed, the sets

$$\Phi^n(\mathcal{R}) = \left(\phi^{n-1}(m), \phi^{n-1}(M)\right)^2 \setminus \left(\phi^n(m), \phi^n(M)\right)^2, \quad n \in \mathbb{N},$$

are disjoint and their union is the whole $\mathbb{R}^2$; if $(x, y) \in \Phi^n(\mathcal{R})$, then there exists a unique pair $(u, v) \in \mathcal{R}$ such that $(x, y) = (\phi^n(u), \phi^n(v))$ and $G(x, y) = \phi^n[G(u, v)]$.

Now, guided by Th. 13, we construct the solutions. We describe in detail the construction only in the case $m < \alpha < p < \beta < M$; the other cases are analogous.

We take two strictly decreasing continuous functions k_α and k_β defined on $(L_\alpha, +\infty)$ and $(-\infty, L_\beta)$ respectively, where $\alpha < L_\alpha < M$, $m < L_\beta < \beta$ and such that

— $\lim_{x\to L_\alpha^+} k_\alpha(x) = m, \ \lim_{x\to+\infty} k_\alpha(x) = -\infty,$

— $\lim_{x\to L_\beta^-} k_\beta(x) = M, \ \lim_{x\to-\infty} k_\beta(x) = +\infty.$

Then we define $N_1, N_2 \in \mathbb{N}$ as the (unique) numbers such that $\phi^{-N_1}(\alpha) < m$ and $\phi^{-N_2}(\beta) > M$.

Now we take two strictly decreasing continuous functions h_0 and g_0 defined on $(-\infty, A_0)$ and $(B_0, +\infty)$ respectively, where $A_0 > \phi^{-N_1}(\alpha)$ and $B_0 < \phi^{-N_2}(\beta)$ and such that

— $h_0(\phi^{-N_1}(\alpha)) = \phi^{-N_1}(\alpha), \ \lim_{x\to-\infty} h_0(x) = v_0, \ \lim_{x\to A_0^-} h_0(x) = -\infty.$

— $g_0(\phi^{-N_2}(\beta)) = \phi^{-N_2}(\beta), \ \lim_{x\to+\infty} g_0(x) = r_0, \ \lim_{x\to B_0^+} g_0(x) = +\infty.$

We choose four real sequences $\{s_n\}$, $\{r_n\}$, $\{u_n\}$ and $\{v_n\}$, $n \in \mathbb{N}$, satisfying the following conditions:

— $s_0 < m$ and, for every n, $s_n < s_{n+1}$, $s_n < \phi(s_{n-1})$; $\lim_{n\to+\infty} s_n = \gamma < p$.

— for every n, $r_n > r_{n+1}$ and $r_n < \phi(r_{n-1})$; $\lim_{n\to+\infty} r_n = \gamma$.

— $u_0 > M$ and, for every n, $u_n > u_{n+1}$, $u_n > \phi(u_{n-1})$;
$$\lim_{n\to+\infty} u_n = \sigma > p.$$

— for every n, $v_n < v_{n+1}$ and $v_n > \phi(v_{n-1})$; $\lim_{n\to+\infty} v_n = \sigma$.

Finally we take two sequences of functions $\{g_n^+\}$, $n \geq 1$, and $\{g_n^-\}$, $n \in \mathbb{N}$, defined on $(M, +\infty)$ and two other sequences $\{h_n^+\}$, $n \in \mathbb{N}$, and $\{h_n^-\}$, $n \geq 1$, defined on $(-\infty, m)$, with the following properties:

— for every $x \in (B_0, +\infty) \cap (M, +\infty)$, $g_0(x) > g_1^+(x)$;

— for every $n \geq 1$, $g_n^+(x) > g_{n+1}^+(x)$, $\lim_{x \to M^+} g_n^+(x) = \phi(r_{n-1})$ and $\lim_{x \to +\infty} g_n^+(x) = r_n$;

— for every $n \geq 1$, $g_{n-1}^-(x) < g_n^-(x)$, $\lim_{x \to M^+} g_n^-(x) = \phi(s_{n-1})$, $\lim_{x \to +\infty} g_n^-(x) = s_n$, $\lim_{x \to M^+} g_0^-(x) = m$ and $\lim_{x \to +\infty} g_0^-(x) = s_0$;

— for every $x \in (M, +\infty)$
$$\inf_n g_n^+(x) = \lim_{n \to +\infty} g_n^+(x) = H^r(x) \ = \lim_{n \to +\infty} g_n^-(x) = \sup_n g_n^-(x)$$
and the function H^r is strictly decreasing.

— for every $x \in (-\infty, A_0) \cap (-\infty, m)$, $h_0(x) < h_1^-(x)$;

— for every $n \geq 1$, $h_n^-(x) < h_{n+1}^-(x)$, $\lim_{x \to m^-} h_n^-(x) = \phi(v_{n-1})$, $\lim_{x \to -\infty} h_n^-(x) = v_n$;

— for every $n \geq 1$, $h_{n-1}^+(x) > h_n^+(x)$, $\lim_{x \to m^-} h_n^+(x) = \phi(u_{n-1})$, $\lim_{x \to -\infty} h_n^+(x) = u_n$, $\lim_{x \to m^-} h_0^+(x) = M$ and $\lim_{x \to -\infty} h_0^+(x) = u_0$;

— for every $x \in (-\infty, m)$
$$\inf_n h_n^+(x) = \lim_{n \to +\infty} h_n^+(x) = H^l(x) \ = \lim_{n \to +\infty} h_n^-(x) = \sup_n h_n^-(x)$$
and the function H^l is strictly decreasing.

Now we can state the theorem about the construction of the solutions.

Theorem 14. *Let $G_0 : \mathcal{R} \to \mathbb{R}$ be a CRI-function with the following properties:*

(i) *for every $n \geq 0$, $Gr(h_n^-) = \Lambda(\phi^n(m))$;*
(ii) *for every $n \geq 0$, $Gr(h_n^+) = \Lambda(\phi^{n+1}(\beta))$;*
(iii) *$Gr(k_\beta) = \Lambda(\beta)$;*
(iv) *for every $n \geq 0$, $Gr(g_n^+) = \Lambda(\phi^n(M))$;*
(v) *for every $n \geq 0$, $Gr(g_n^-) = \Lambda(\phi^{n+1}(\alpha))$;*
(vi) *$Gr(k_\alpha) = \Lambda(\alpha)$;*
(vii) *$Gr(H^l) \cup Gr(H^r) = \Lambda(p)$;*
(viii) *if $t_1 < t_2 < \alpha$, then*
$$\sup\{x : G_0(x, y) = t_1\} < \sup\{x : G_0(x, y) = t_2\} < +\infty;$$
moreover, $\inf\{\sup\{x : G_0(x, y) = t\} t < \alpha\} \ = -\infty;$

(ix) *if* $t_1 > t_2 > M$, *then*

$$r_0 < \inf\{y : G_0(x,y) = t_2\} < \inf\{y : G_0(x,y) = t_1\};$$

moreover, $\sup\{\inf\{y : G_0(x,y) = t\}t > M\} = +\infty$;

(x) *if* $t_1 > t_2 > \beta$, *then*

$$-\infty < \inf\{x : G_0(x,y) = t_2\} < \inf\{x : G_0(x,y) = t_1\};$$

moreover, $\sup\{\inf\{x : G_0(x,y) = t\}t > \beta\} = +\infty$;

(xi) *if* $t_1 < t_2 < m$, *then*

$$\sup\{y : G_0(x,y) = t_1\} < \sup\{y : G_0(x,y) = t_2\} < v_0;$$

moreover, $\inf\{\sup\{y : G_0(x,y) = t\}t < m\} = -\infty$;

(xii) *for every* $t < \alpha$,

$$G_0\big(\phi\big(\sup\{x : G_0(x,y) =\}\big), m\big) = \phi(t)$$

and, for every $t > \alpha$,

$$G_0\big(M, \phi\big(\inf\{y : G_0(x,y) = t\}\big)\big) = \phi(t);$$

(xiii) *for every* $t > \beta$,

$$G_0\big(\phi\big(\inf\{x : G_0(x,y) = t\}\big), M\big) = \phi(t)$$

and, for every $t < \beta$,

$$G_0\big(m, \phi\big(\sup\{y : G_0(x,y) = t\}\big)\big) = \phi(t).$$

Then G_0 *can be uniquely extended to a CRI–function* $G : \mathbb{R}^2 \to \mathbb{R}$ *such that the pair* (ϕ, G) *is a solution of equation* (2).

Also here we can repeat Remark 2 with the obvious modifications.

4. Final remarks

We now consider equation (1) where $F : I^2 \to I$ and I is a proper open real interval. Let $h : I \to \mathbb{R}$ be an increasing homeomorphism and suppose $F : I^2 \to I$ is a solution of (1). Then the function

$$T(x,y) = h\big[F(h^{-1}(x)), h^{-1}(y))\big]$$

is a solution of equation (1) and $T : \mathbb{R}^2 \to \mathbb{R}$. Conversely, if $T : \mathbb{R}^2 \to \mathbb{R}$ is a solution of (1) then

$$F(x,y) = h^{-1}\big[T(h(x)), h(y))\big]$$

is a solution of (1) in I. Thus the solutions of (1) on an open interval I can be obtained from those on the whole $\mathbb{R}$. Obviously we can extend this remark to the case of an ordered topological space E such that there exists an increasing homeomorphism $h : E \to \mathbb{R}$.

We now turn to equation (1′) with a fixed $\nu \geq 2$. We can reformulate Th. 1 for this equation, and we obtain that any solution F of (1′) has the form

$$F(x,y) = \phi\big[G(x,y)\big]$$

where the pair (ϕ, G) satisfies the functional equation

$$\phi^{\nu}\big[G(x,y)\big] = G\big[\phi^{\nu}(x), \phi^{\nu}(y)\big].$$

Thus setting $\phi^{\nu} = \psi$ we have again equation (2), and so the previous construction produces also the solutions of equation (1′).

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