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TREE MARTINGALES AND A.E. CONVERGENCE OF VILENKIN–FOURIER SERIES

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Abstract: Using martingale theory we give a new and simple proof of Carleson's theorem for Vilenkin–Fourier series. An atomic decomposition of spaces containing tree martingales is formulated. With the help of this a sufficient condition for the weak boundedness of an operator on the tree martingale spaces is given. Weak and strong type (p,p) inequalities are proved for the martingale transform and for the quadratic variations. Since the partial sums of a Vilenkin–Fourier series are a special martingale transform, we obtain that the supremum of the partial sums is of weak type (p,p) $(1 < p < \infty)$, whenever the Vilenkin system is bounded. This implies the a.e. convergence of the Vilenkin–Fourier series to the function. Finally, we prove also the L_p convergence for arbitrary Vilenkin systems.

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1. Introduction

In 1966 Carleson [4] proved his famous and very deep theorem: the trigonometric-Fourier series of a function $f \in L_2$ converges a.e. to the function f. Hunt [8] extended this theorem for all $f \in L_p$ $(1 < p < < \infty)$. This result for the Walsh system was proved by Billard [1] for $p = 2$ and by Sjölin [16] for $1 < p < \infty$ while for bounded Vilenkin systems by Gosselin [7] (see also Schipp [12], [13]; $p = 2$).

Using martingale theory we give a new and simple proof of this last result, i.e. for bounded Vilenkin systems. We introduce tree martingales and tree martingale difference sequences. Note that not every tree martingale difference sequence define a martingale. An atomic decomposition of Hardy spaces consisting of tree martingale difference sequences is formulated. With the help of this we shall prove that, for the boundedness of a sublinear operator from the tree martingale Hardy spaces to the weak L_p spaces, it is enough to check the operators on atoms. Some one-parameter martingale inequalities given in Section 2 are extended to tree martingales. A maximal inequality as well as Burkholder–Gundy inequality are verified. Moreover, it will be shown that the L_p norm of the conditional quadratic variation and of the maximal function of a martingale transform can be estimated by the L_p norm of the martingale in case the stochastic basis is regular.

The partial sums of the Vilenkin–Fourier series of an integrable function can be majorized by the maximal function of a suitable martingale transform. As a consequence we obtain that the supremum of the partial sums is of weak and strong type (p, p) $(1 < p < \infty)$, whenever the Vilenkin system is bounded. A usual density argument implies then Carleson's theorem for Vilenkin-Fourier series. Finally, we verify that, for an arbitrary Vilenkin system and for $f \in L_p$ $(1 < p < \infty)$, the Vilenkin–Fourier series of f converges to f in L_p norm. This result was shown by Schipp [11] and by Young [20].

2. One-parameter martingales

In this section we summarize shortly all results on one-parameter martingales, which will be used or extended later. Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\mathcal{F} = (\mathcal{F}_n, n \in \mathbb{N})$ be a sequence of non-decreasing σ-algebras. The σ-algebra generated by an arbitrary set system $\mathcal{H}$ will be denoted by $\sigma(\mathcal{H})$. Suppose that $\sigma(\cup_{n\in\mathbb{N}}\mathcal{F}_n) = \mathcal{A}$.

The expectation operator and the conditional expectation operators relative to $\mathcal{F}_n$ $(n \in \mathbb{N})$ are denoted by E and E_n, respectively. We briefly write L_p instead of the complex $L_p(\Omega, \mathcal{A}, P)$ space while the norm (or quasinorm) of this space is defined by $\|f\|_p := (E|f|^p)^{1/p}$.

An integrable sequence $f = (f_n, n \in \mathbb{N})$ is said to be a *martingale* if

(i) it is *adapted*, i.e. f_n is $\mathcal{F}_n$ measurable for all $n \in \mathbb{N}$
(ii) $E_n f_m = f_n$ for all $n \leq m$.

The *maximal function* and the *martingale differences* of a martingale $f = (f_n, n \in \mathbb{N})$ are denoted by

$$f^* := \sup_{m\in\mathbb{N}} |f_m|, \quad d_n f := f_{n+1} - f_n \quad (n \geq 1), \quad d_0 f := f_0.$$

It is easy to show that $(d_n f)$ is an integrable sequence, $d_n f$ is $\mathcal{F}_{n+1}$ measurable $(n \in \mathbb{N})$ and $E_n d_n f = 0$. Conversely, if a function sequence (d_n) has these three properties then $(f_n, n \in \mathbb{N})$ is a martingale where $f_n := \sum_{k=0}^{n-1} d_k$.

We say that a martingale $f = (f_n, n \in \mathbb{N})$ is *predictable in* L_p $(0 < p \leq \infty)$ if there exists a sequence $0 < \lambda_0 \leq \lambda_1 \leq \ldots$ of functions such that λ_n is $\mathcal{F}_{n-1}$ measurable and

$$|f_n| \leq \lambda_n, \qquad \lambda_\infty := \sup_{n\in\mathbb{N}} \lambda_n \in L_p.$$

We introduce the *martingale Hardy spaces* for $0 < p \leq \infty$; denote by H_p (resp. $\mathcal{P}_p$) the space of martingales (resp. predictable martingales) for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty$$

(resp.

$$\|f\|_{\mathcal{P}_p} := \inf \|\lambda_\infty\|_p$$

where the infimum is taken over all predictable sequences $(\lambda_n, n \in \mathbb{N})$ having the above property).

It is known (see Weisz [18]) that H_p is equivalent to $\mathcal{P}_p$ $(0 < p < < \infty)$, if the stochastic basis is *regular,* i.e. if there exists a number $R > 0$ such that for all $f \in L_1$

$$|E_n f| \leq R E_{n-1}|E_n f| \quad (n \in \mathbb{N}).$$

As we can see later the atomic decomposition is a useful characterization of Hardy spaces. Let us introduce first the concept of an atom: a measurable function a is a *p-atom* if there exists a non-increasing sequence $(A_n, n \in \mathbb{N})$ of adapted sets such that

(i) $$(E_n a)1_{A_{n-1}} = 0 \quad (n \in \mathbb{N}),$$

(ii) $$\|a^*\|_\infty \le P(\cup_{n=0}^\infty A_n^c)^{-1/p}$$

where $A_{-1} := \Omega$ and A^c denotes the complement of the set A.

Note that the original definition of p-atoms is formulated by using stopping times, however, it is easy to see that the two definitions are equivalent.

Observe that (i) is equivalent to

(i') $$d_n a 1_{A_n} = 0 \quad (n \in \mathbb{N}).$$

The basic result of atomic decomposition states that, if the martingale $f = (f_n; n \in \mathbb{N})$ is in $\mathcal{P}_p$ $(0 < p < \infty)$ then there exist a sequence $(a^k, k \in \mathbb{Z})$ of p-atoms and a sequence $(\mu_k, k \in \mathbb{Z})$ of real numbers such that for all $n \in \mathbb{N}$

$$\sum_{k=-\infty}^{\infty} \mu_k E_n a^k = f_n \quad \text{and} \quad \Big(\sum_{k=-\infty}^{\infty} |\mu_k|^p\Big)^{1/p} \le C_p \|f\|_{\mathcal{P}_p}.$$

The following martingale inequalities are basic theorems in the martingale theory and are due to Doob (see Neveu [10]) and to Burkholder, Davis and Gundy [3], [5]:

(1) $$\|f\|_p \le \|f^*\|_p \le \frac{p}{p-1}\|f\|_p \qquad (1 < p < \infty),$$

(2) $$c_p\|f^*\|_p \le \Big\|\Big(\sum_{n=0}^{\infty} |d_n f|^2\Big)^{1/2}\Big\|_p \le C_p\|f^*\|_p \qquad (1 \le p < \infty).$$

It was verified by Burkholder [2] (see also Weisz [19]) that

(3)
$$\|f^*\|_p \le C_p\Big\|\Big(\sum_{n=0}^{\infty} E_n|d_n f|^2\Big)^{1/2}\Big\|_p + C_p\|\sup_{n\in\mathbb{N}} |d_n f|\|_p \qquad (2 \le p < \infty).$$

Note that the converse of this inequality is also true (see Weisz [19]).

We will also use the next convexity and concavity theorem: if $(f_n, n \in \mathbb{N})$ is a sequence of non-negative measurable functions then we have

(4) $$E\Big[\Big(\sum_{n=0}^{\infty} E_n f_n\Big)^p\Big] \le C_p E\Big[\Big(\sum_{n=0}^{\infty} f_n\Big)^p\Big] \qquad (1 \le p < \infty)$$

and

$$E\Big[\Big(\sum_{n=0}^{\infty} f_n\Big)^p\Big] \le C_p E\Big[\Big(\sum_{n=0}^{\infty} E_n f_n\Big)^p\Big] \qquad (0 < p \le 1). \tag{5}$$

Note that the positive constants C_p depend only on p and may denote different constants in different contexts.

3. Tree martingales and Hardy spaces

Let $\mathbf{T}$ be a countable, upward directed index set with respect to the partial ordering $\le$ satisfying the following two conditions: for every $t \in \mathbf{T}$

$$\mathbf{T}^t := \{u \in \mathbf{T} : u \le t\}$$

is finite and the set

$$\mathbf{T}_t := \{u \in \mathbf{T} : t \le u\}$$

is linearly ordered. Thus $\mathbf{T}$ is a tree and every non-empty subset of $\mathbf{T}$ has at least one minimum element. Denote by $\mathbf{T}_0$ the set of the minimum elements of $\mathbf{T}$.

Let $(\Omega, \mathcal{A}, P)$ be a probability space and let us fix a non-decreasing sequence $\mathcal{F} = (\mathcal{F}_t, t \in \mathbf{T})$ of sub-σ-algebras of $\mathcal{A}$ with respect to the partial ordering. Assume that $(\mathcal{F}_t, t \in \mathbf{T})$ can be ordered linearly and $\mathcal{A} = \sigma(\cup_{t \in \mathbf{T}} \mathcal{F}_t)$. Denote again by E_t the conditional expectation operator with respect to $\mathcal{F}_t$.

In the tree case it is more useful to work with projections instead of the conditional expectation operators. We consider the *projections*

$$P_t f := \phi_t E_t(f \overline{\phi}_t) \qquad (f \in L_1, t \in \mathbf{T}) \tag{6}$$

where the functions ϕ_t are given and $|\phi_t| = 1$ for every $t \in \mathbf{T}$. P_t is a projection, indeed, since $\|P_t\| \le 1$ and $P_t \circ P_t = P_t$ $(t \in \mathbf{T})$. It is obvious that the conditional expectation operators are projections of the form (6) with $\phi = 1$.

The sequence $(\mathcal{F}_t, P_t; t \in \mathbf{T})$ is called a *tree basis* if for the projections defined in (6) we have

(i) $P_t f = \phi_u E_t(f \overline{\phi}_u)$ for every $u \le t$ and $f \in L_1$,
(ii) $P_t P_u = 0$ for each incomparable u and t from $\mathbf{T}$.

Note that the equality $P_t P_u = P_u P_t = P_u$ for $u \le t$ follows from (i). We say that a sequence $(f_t, t \in \mathbf{T})$ of integrable functions is a *tree martingale* if $u \le t$ implies $P_u f_t = f_u$ $(u, t \in \mathbf{T})$. If $f \in L_1$ then the sequence $(P_t f, t \in \mathbf{T})$ is obviously a tree martingale and it is denoted also by f.

The succeeding element of $t \in \mathbf{T}$, namely, the minimum element of $\mathbf{T}_t \setminus \{t\}$ is denoted by t^+. For simplicity we suppose that if $t = u^+ = r^+$ then $\mathcal{F}_u = \mathcal{F}_r$. This common σ-algebra will be denoted by $\mathcal{F}_{t-}$. For $t \in \mathbf{T}_0$ we define an element $t^- < t$ and set $\mathcal{F}_{t^-} = \mathcal{F}_t$. We say in this case that $(t^-)^+ = t$. Let $\mathbf{T}_0^- := \{t^- : t \in \mathbf{T}_0\}$ and $\overline{\mathbf{T}} := \mathbf{T} \cup \mathbf{T}_0^-$.

$df := (d_t f, t \in \overline{\mathbf{T}})$ is called a *martingale difference sequence* if

(i) $d_t f$ is integrable for every $t \in \overline{\mathbf{T}}$,
(ii) $P_t(d_t f) = 0$ for every $t \in \mathbf{T}$,
(ii) $P_{t^+}(d_t f) = d_t f$ for every $t \in \overline{\mathbf{T}}$.

In case $f = (f_t, t \in \mathbf{T})$ is a martingale then $df := (d_t f, t \in \overline{\mathbf{T}})$ is clearly a martingale difference sequence, where

$$d_t f := f_{t^+} \quad (t \in \mathbf{T}_0^-), \qquad d_t f := f_{t^+} - f_t \quad (t \in \mathbf{T}).$$

The converse is not true because from a martingale difference sequence we cannot define f_t uniquely. However, if df is a martingale difference sequence and

$$f_u := \sum_{v \le t < u} d_t f = \sum_{s \le t < u} d_t f$$

for every $u \in \mathbf{T}$ and $v, s \in \mathbf{T}_0^-$ with $v, s < u$, then $(f_u, u \in \mathbf{T})$ is a martingale.

Suppose that there is a distinguished minimal element $v_0 \in \mathbf{T}_0$ for which $d_{v_0} f = 0$ for all tree martingale difference sequences. For a martingale difference sequence $(d_t f, t \in \overline{\mathbf{T}})$ we define the *maximal function* by

$$f_t^* := \sup_{u \in \mathbf{T}_0^-} \sup_{u \le r < t} \left| \sum_{s=u}^{r} d_s f \right|, \qquad f^* := \sup_{t \in \mathbf{T}} f_t^*.$$

Since for an integrable function f

$$|P_t f| \le E_t |f|$$

and $(\mathcal{F}_t, t \in \mathbf{T})$ can be linearly ordered, (1) holds also in the tree case.

We are going to introduce the quasi-norm $\|\cdot\|_{\mathbf{H}_p^q}$. Let $g = (g_t, t \in \mathbf{T})$ be a sequence of $\mathcal{A}$ measurable functions defined on Ω. For $0 < p, q < \infty$ set

$$\|g\|_{\mathbf{H}_p^q} := \sup_{y>0} y \left(\int_\Omega \left[\sum_{t \in \mathbf{T}} 1_{\{g_t^* > y,\ g_u^* \le y\ (\forall u < t)\}} \right]^{p/q} dP \right)^{1/p} \tag{7}$$

while for $0 < p < \infty$, $q = \infty$

$$\|g\|_{\mathbf{H}_p^\infty} := \sup_{y>0} y P(g^* > y)^{1/p} \tag{8}$$

where $u < t$ means that $u \le t$ but $u \ne t$ and

$$g_t^* := \sup_{u\leq t} |g_u|, \qquad g^* := \sup_{t\in \mathbf{T}} g_t^*.$$

Denote by $\mathbf{H}_p^q$ the set of such sequences g which satisfy $\|g\|_{\mathbf{H}_p^q} < \infty$. Note that the right hand side of (8) is the weak L_p norm of g^*. Observe that, for each fixed sequence g, the function $q \mapsto \|g\|_{\mathbf{H}_p^q}$ decreases and $p \mapsto \|g\|_{\mathbf{H}_p^q}$ increases.

For a martingale difference sequence df we define

$$\|df\|_{\mathbf{H}_p^q} := \|(f_t^*, t\in \mathbf{T})\|_{\mathbf{H}_p^q} \qquad (0<p<\infty, 0<q\leq\infty).$$

Of course, if $f = (f_t, t \in \mathbf{T}) \in L_1$ is a tree martingale, then $\|df\|_{\mathbf{H}_p^q} = \|f\|_{\mathbf{H}_p^q}$. It is proved in Schipp [13] that $\|\cdot\|_{\mathbf{H}_p^q}$ is really a quasi-norm and that the map $g \mapsto \|g\|_{\mathbf{H}_p^q}$ is non-decreasing in the following sense: if $|g_t| \leq |h_t|$ for all $t\in \mathbf{T}$ then $\|g\|_{\mathbf{H}_p^q} \leq \|h\|_{\mathbf{H}_p^q}$ $(0<p<\infty, 0<q\leq\infty)$.

Notice that if $\mathbf{T}$ is linearly ordered then the sets on the right hand side in (7) are pairwise disjoint and

$$\|g\|_{\mathbf{H}_p^q} = \sup_{y>0} yP(g^*>y)^{1/p}$$

for any $0<p<\infty$ and $0<q\leq\infty$. Thus, in the linear case, by using the $\mathbf{H}_p^q$ quasi-norms, the maximal inequality (1) can be reformulated as follows:

$$\left\|(f_t, t\in\mathbf{T})\right\|_{\mathbf{H}_p^q} \leq \|f\|_p \qquad (1<p<\infty, 0<q\leq\infty).$$

This form of the maximal inequality can be transfered to the tree case. Let Δ denote the closure of the triangle in $\mathbf{R}^2$ with vertices $(0,0)$, $(1/2,1/2)$ and $(1,0)$ except the points $(x, 1-x)$, $1/2 < x \leq 1$. Let $\delta : \mathbf{T} \longrightarrow \mathbf{T}$ be a map such that $\delta(t) \leq t$ and $\mathcal{F}_{\delta(u)} \subset \mathcal{F}_{\delta(t)}$ for each $u \leq t$. Set

$$f_t^\delta := E_{\delta(t)}|f_t| \qquad (t\in\mathbf{T}).$$

Theorem 1. *Let $f = (f_t, t\in\mathbf{T}) \in L_p$ be a tree martingale and suppose that $1<p,q<\infty$ satisfy $(1/p, 1/q) \in \Delta$. Then*

$$\|(f_t^\delta, t\in\mathbf{T})\|_{\mathbf{H}_p^q} \leq C_{p,q}\|f\|_p \qquad (f\in L^p).$$

We omit the proof since it can be found in Schipp [13] or Weisz [19].

4. Predictable martingales and atomic decomposition

The predictable martingales can be defined in the tree case, too. A martingale difference sequence $df = (d_t f, t \in \overline{\mathbf{T}})$ is said to be *predictable* by a sequence $\lambda = (\lambda_t, t \in \mathbf{T})$ if λ is increasing and each λ_t is $\mathcal{F}_{t^-}$-measurable with

$$f_t^* \leq \lambda_t \qquad (t \in \mathbf{T}).$$

Denote by $\mathbf{P}_p^q$ $(0 < p < \infty, 0 < q \leq \infty)$ the collection of such sequences df for which $\lambda \in \mathbf{H}_p^q$ and set

$$\|df\|_{\mathbf{P}_p^q} := \inf \|\lambda\|_{\mathbf{H}_p^q}$$

where the infimum is taken over all predictions $\lambda \in \mathbf{H}_p^q$ belonging to df.

Now we introduce the concept of tree atoms: a martingale difference sequence $da = (d_t a, t \in \overline{\mathbf{T}})$ with $a^* \in L_\infty$ is a (tree) *atom* if there exists a non-increasing sequence $(A_t, t \in \mathbf{T})$ of adapted sets such that $v^+ = t^+$ implies $A_v = A_t$ and

(i) $$d_{u^-} a = 0 \quad (u \in \mathbf{T}_0)$$

(ii) $$d_t a 1_{A_t} = 0 \quad (t \in \mathbf{T}).$$

Let

$$\alpha_t := \alpha_t^a := 1_{A_{t^-}^c} \prod_{u<t} 1_{A_{u^-}}$$

where $A_{t^-} := A_u$ whenever $u^+ = t$.

The atomic decomposition of $\mathbf{P}_p^q$ in the tree case reads as follows.

Theorem 2. *If the martingale difference sequence $df = (d_t f, t \in \overline{\mathbf{T}})$ is in $\mathbf{P}_p^q$ $(0 < p < \infty, 0 < q \leq \infty)$ then there exist $da^k = (d_t a^k, t \in \overline{\mathbf{T}})$ $(k \in \mathbb{Z})$ atoms such that*

(9) $$\|a^{k*}\|_\infty \leq 3 \cdot 2^k, \qquad d_t f = \sum_{k=-\infty}^{\infty} d_t a^k \quad (t \in \mathbf{T})$$

and

(10) $$\sup_{k \in \mathbb{Z}} 2^k \Big(E\Big[\Big(\sum_{t \in \mathbf{T}} \alpha_t^k \Big)^{p/q} \Big] \Big)^{1/p} \leq C_{p,q} \|df\|_{\mathbf{P}_p^q}$$

where $\alpha_t^k := \alpha_t^{a^k}$.

Proof. Let λ be a prediction of df and let

$$d_t a^k := 1_{\{2^k < \lambda_{t^+} \leq 2^{k+1}\}} d_t f \quad (t \in \mathbf{T}), \qquad d_{u^-} a^k := 0 \quad (u \in \mathbf{T}_0).$$

Since $\{2^k < \lambda_{t^+} \leq 2^{k+1}\} \in \mathcal{F}_t$ for all $t \in \mathbf{T}$, $da^k := (d_t a^k, t \in \overline{\mathbf{T}})$ $(k \in \mathbb{Z})$ is a martingale difference sequence. Setting

$$A_t^k := \{\lambda_{t^+} \leq 2^k\} \qquad (t \in \mathbf{T})$$

we can see that da^k is an atom. Moreover, it is easy to show that the second equality of (9) holds.

For fixed $\omega \in \Omega$, $u \in \mathbf{T}_0^-$ and $u < t \in \mathbf{T}$ let v_1 be the minimum element such that $u \leq v_1 < t$ and $2^k < \lambda_{v_1^+} \leq 2^{k+1}$ and let v_2 be the minimum element such that $u \leq v_2$ and $\lambda_{v_2^+} > 2^{k+1}$. If such a v_2 does not exist then let $v_2 = \infty$.

$$\sum_{u \leq s < t} d_t a^k = \sum_{u \leq s < t} 1_{\{2^k < \lambda_{s+} \leq 2^{k+1}\}} d_s f =$$

$$= \sum_{v_1 \leq s < t \wedge v_2} d_s f = \sum_{u \leq s < t \wedge v_2} d_s f - \sum_{u \leq s < v_1} d_s f.$$

From this it follows that

$$\Big| \sum_{u \leq s < t} d_t a^k \Big| \leq f^*_{t \wedge v_2} + f^*_{v_1} \leq \lambda_{t \wedge v_2} + \lambda_{v_1} \leq 2^{k+1} + 2^k$$

which proves the first inequality of (9). Since $A^k_{t-} := \{\lambda_t \leq 2^k\}$, we have

$$\alpha_t^k = 1_{(A^k_{t-})^c} \prod_{u<t} 1_{A^k_{u-}} = 1_{\{\lambda_t > 2^k, \ \lambda_u \leq 2^k \ (\forall u < t)\}}.$$

Thus (10) follows from the definition of the $\mathbf{H}_p^q$ norm. ◊

The next theorem says that for the boundedness of a sublinear operator from $\mathbf{P}_p^q$ to the weak L_p space, it is enough to check the operators on atoms.

Theorem 3. *Let $p < q < \infty$ and U be a sub-linear operator which maps the collection of martingale difference sequences into the set of non-negative measurable functions. Suppose that $U(d_{u-}f) = 0$ ($u \in$ $\in \mathbf{T}_0$) and*

(11) $$U(\xi d_t f) = \xi U(d_t f)$$

provided that ξ is $\mathcal{F}_t$ measurable and $t \in \mathbf{T}$. If there exists a constant $R > 0$ such that

(12) $$P\big(U(da) > z\big) \leq C_{p,q}(z - 2R)^{-q} E\Big[\Big(\sum_{v \in \mathbf{T}} \alpha_v\Big)^{p/q}\Big]$$

for all $z > 2R$ and all atoms da with $\|a^*\|_\infty \leq 1$, then

$$\sup_{y>0} yP\big(U(df) > y\big)^{1/p} \leq C_{p,q} \|df\|_{\mathbf{P}_p^q} \qquad (df \in \mathbf{P}_p^q).$$

Proof. The sub-linearity of U implies

$$U(df) \leq \sum_{k \in \mathbb{Z}} U(da^k)$$

where the atoms da^k are defined in the proof of Th. 2. Choosing $j \in \mathbb{Z}$ such that $2^j < y \leq 2^{j+1}$ we get for $k \geq j+1$ that

$$1_{\{\lambda^*\le y\}}U(da^k)\le \sum_{t\in\mathbf{T}} U(d_ta^k)1_{\{\lambda^*\le y\}}=$$
$$=\sum_{t\in\mathbf{T}} U(d_tf)1_{\{2^k<\lambda_{t+}\le 2^{k+1}\}}1_{\{\lambda^*\le y\}}=0.$$

Thus

$$1_{\{\lambda^*\le y\}}U(df)\le \sum_{k=-\infty}^{\infty} 1_{\{\lambda^*\le y\}}U(da^k)\le \sum_{k\le j} U(da^k).$$

Since $y>2^j$ we can conclude that

$$y^pP\Big(U(df)>(3+12R)y\Big)\le$$
$$\le y^pP(\lambda^*>y)+y^pP\Big(U(df)>(3+12R)y,\lambda^*\le y\Big)\le$$
$$\le \|\lambda\|_{\mathbf{H}_p^\infty}^p+y^pP\Big(\sum_{k\le j}U(da^k)>(3+12R)y\Big)\le$$
$$\le C_{p,q}\|df\|_{\mathbf{P}_p^q}^p+y^pP\Big(\sum_{k\le j}U(da^k)>\sum_{k\le j}(3z_k2^k+6R2^k)\Big)$$

where $z_k:=c_\beta 2^{(\beta-1)(k-j)}$, $c_\beta=1-2^{-\beta}$ and $\beta>0$. Observe that $\sum_{k\le j}2^kz_k=2^j$. Then for $\beta=(q-p)/(2q)$ we get

$$z_k^{-q}2^{-pk}\le C_{p,q}2^{-pj}2^{p(j-k)+q(\beta-1)(j-k)}\le C_{p,q}y^{-p}2^{(q-p)(k-j)/2}.$$

Consequently, by (12),

$$P\Big(\sum_{k\le j}U(da^k)>\sum_{k\le j}(3z_k2^k+6R2^k)\Big)\le$$
$$\le \sum_{k\le j}P\Big(U\left(\frac{da^k}{3\cdot 2^k}\right)>(z_k+2R)\Big)\le$$
$$\le C_{p,q}\sum_{k\le j}z_k^{-q}E\Big[\Big(\sum_{v\in\mathbf{T}}\alpha_v^k\Big)^{p/q}\Big]\le$$
$$\le C_{p,q}\sum_{k\le j}z_k^{-q}2^{-kp}\|df\|_{\mathbf{P}_p^q}^p\le C_{p,q}y^{-p}\|df\|_{\mathbf{P}_p^q}^p,$$

which proves the theorem. ◊

5. Bounded operators on $\mathbf{P}_p^q$

We investigate the following tree martingale transform. Suppose that the sequence $T=(T^t,t\in\mathbf{T})$ of linear operators satisfies the following conditions for all $t \in \mathbf{T}$:

(i) $P_{t^+}(T^t(d_t f)) = T^t(d_t f)$,

(ii) $P_t(T^t(d_t f)) = 0$,

(iii) for every $\mathcal{F}_t$ measurable function ξ one has $T^t\big(\xi(d_t f)\big) = = \xi T^t(d_t f)$,

(iv) $|T^t(d_t f)|^2 \le R^2 E_t |d_t f|^2$ where the constant R is independent of t and of df.

The *maximal function of a tree martingale transform* is defined by

$$T_t^{**}(df) := \sup_{r \in \mathbf{T}_t} \Big| \sum_{t \le u < r} T^u(d_u f) \Big|, \qquad T^{**}(df) := \sup_{t \in \mathbf{T}} T_t^{**}(df).$$

We introduce the *quadratic variation* and the *conditional quadratic variation* for tree martingale difference sequences by

$$S(df) := \sup_{t \in \mathbf{T}} \Big(\sum_{u \in \mathbf{T}_t} |d_u f|^2 \Big)^{1/2} \quad \text{and} \quad s(df) := \sup_{t \in \mathbf{T}} \Big(\sum_{u \in \mathbf{T}_t} E_u |d_u f|^2 \Big)^{1/2}.$$

Now we are ready to prove the boundedness of the martingale transform and the quadratic variations.

Theorem 4. *If $df = (d_t f, t \in \overline{\mathbf{T}}) \in \mathbf{P}_p^q$ is a tree martingale difference sequence and $(p \vee \sqrt{2p}) < q < \infty$ then*

$$\sup_{y>0} y P(T^{**}(df) > y)^{1/p} \le C_{p,q} \|df\|_{\mathbf{P}_p^q},$$

$$\sup_{y>0} y P(S(df) > y)^{1/p} \le C_{p,q} \|df\|_{\mathbf{P}_p^q},$$

$$\sup_{y>0} y P(s(df) > y)^{1/p} \le C_{p,q} \|df\|_{\mathbf{P}_p^q}.$$

Proof. We are going to verify only the first statement because the others are similar. It is enough to prove the inequality (12) for the operator T^{**}. So let da be an atom with $\|a^*\|_\infty \le 1$. Let us fix $t < u$ in $\mathbf{T}$ and ω in Ω. If the set

$$\big\{ v \in \mathbf{T} : t \le v < u, 1_{A_v^c}(\omega) = 1 \big\}$$

is empty then

$$T_{t,u}(da)(\omega) := \sum_{t \le r < u} T^r(d_r a)(\omega) = \sum_{t \le r < u} 1_{A_r^c}(\omega) T^r(d_r a)(\omega) = 0$$

or else let t_1 be its minimum element. Moreover, denote by t_0 a minimum element of the set

$$\big\{ v \in \mathbf{T} : v \le t^+{}_1, 1_{A_{v^-}^c}(\omega) = 1 \big\}.$$

Thus $\alpha_{t_0}(\omega) = 1$ and $T_{t,u}(da)$ can be written in the form

$$T_{t,u}(da)(\omega)=T_{t_1,u}(da)(\omega)=T^{t_1}(d_{t_1}a)(\omega)+T_{t_1^+,u}(da)(\omega)$$
$$=T^{t_1}(d_{t_1}a)(\omega)+\alpha_{t_0}(\omega)\big(T_{t_0,u}(da)(\omega)-T_{t_0,t^+_1}(da)(\omega)\big).$$

By predictability and by (iv) of the definition of the martingale transform we have

$$\big|T^{t_1}(d_{t_1}a)\big|\le R\big[E_{t_1}(|d_{t_1}a|^2)\big]^{1/2}\le R\big[E_{t_1}(4a^{*2}_{t_1^+})\big]^{1/2}\le 2R.$$

On the other hand,

$$\big|T_{t_0,u}(da)(\omega)-T_{t_0,t_1^+}(da)(\omega)\big|\le 2T^{**}_{t_0}(da)(\omega).$$

Taking the supremum over all $u\in\mathbf{T}_t$ and $t\in\mathbf{T}$ we get

$$T^{**}(da)\le 2\sup_{v\in\mathbf{T}}\alpha_v T_v^{**}(da)+2R. \tag{13}$$

By (i) of the definition of the martingale transform one can see that

$$T^u(d_ua)=P_{u^+}\big(T^u(d_ua)\big)=\phi_vE_{u^+}\big[T^u(d_ua)\overline{\phi}_v\big]\qquad(u\ge v),$$

hence $\overline{\phi}_vT^u(d_ua)$ is $\mathcal{F}_{u^+}$ measurable. On the other hand, by (ii),

$$0=P_u\big(T^u(d_ua)\big)=\phi_vE_u\big[T^u(d_ua)\overline{\phi}_v\big]\qquad(u\ge v),$$

consequently, $\Big(T^u(d_ua)\overline{\phi}_v\Big)_{u\ge v}$ is a one-parameter martingale difference sequence relative to $(\mathcal{F}_{u^+})_{u\ge v}$. By $|\phi_v|=1$, one has for each $v\in\mathbf{T}$

$$T_v^{**}(da)=\sup_{r\ge v}\Big|\sum_{v\le u<r}T^u(d_ua)\overline{\phi}_v\Big|.$$

Using the inequalities (1) and (2), (iv) and the convexity theorem in (4) we obtain

$$E_v[(T_v^{**}(da))^{p_0}]\le C_{p_0}E_v\Big[\Big(\sum_{u\ge v}|T^u(d_ua)\overline{\phi}_v|^2\Big)^{p_0/2}\Big]\le$$
$$\le C_{p_0}E_v\Big[\Big(\sum_{u\ge v}E_u|d_ua|^2\Big)^{p_0/2}\Big]\le C_{p_0}E_v\Big[\Big(\sum_{u\ge v}|d_ua|^2\Big)^{p_0/2}\Big],$$

provided that $p_0\ge 2$. Since $\Big(d_ua\overline{\phi}_v\Big)_{u\ge v}$ is again a one-parameter martingale difference sequence and $\|a^*\|_\infty\le 1$, (1) and (2) implies

$$E_v[(T_v^{**}(da))^{p_0}]\le C_{p_0}E_v\Big[\Big|\sum_{u\ge v}d_ua\Big|^{p_0}\Big]\le C_{p_0}. \tag{14}$$

Applying Tsebisev's inequality, the concavity theorem in (5), (13) and (14) one can see that

$$
\begin{aligned}
P(T^{**}(da) > z) &\le P(2 \sup_{v\in \mathbf{T}} \alpha_v T_v^{**}(da) + 2R > z) \le \\
&\le \left(\frac{z-2R}{2}\right)^{-q} E\Big[(\sup_{v\in \mathbf{T}} \alpha_v (T_v^{**}(da))^q\Big] \le \\
&\le C_q (z-2R)^{-q} E\Big[\Big(\sum_{v\in \mathbf{T}} \alpha_v (T_v^{**}(da))^{p_0}\Big)^{q/p_0}\Big] \le \\
&\le C_{p_0,q} (z-2R)^{-q} E\Big[\Big(\sum_{v\in \mathbf{T}} \alpha_v E_v[(T_v^{**}(da))^{p_0}]\Big)^{q/p_0}\Big] \le \\
&\le C_{p,q} (z-2R)^{-q} E\Big[\Big(\sum_{v\in \mathbf{T}} \alpha_v\Big)^{p/q}\Big]
\end{aligned}
$$

where $p_0 := q^2/p$. Th. 3 completes the proof. $\Diamond$

The concept of regularity is going to be introduced. $\mathcal{F}$ is said to be *regular* if there exists a constant $C > 0$ such that for all $f \in L_1$

$$ |E_t f| \le C E_{t-} |E_t f| \qquad (t \in \mathbf{T}) \tag{15} $$

where E_{t-} denotes the conditional expectation operator with respect to $\mathcal{F}_{t-}$.

Corollary 1. *Suppose that $1 < p, q < \infty$ with $(1/p, 1/q) \in \Delta$ and $f = (f_t, t \in \mathbf{T}) \in L_p$ is a tree martingale. If $\mathcal{F}$ is regular then*

$$ \|f\|_{\mathbf{P}_p^q} \le C_{p,q} \|f\|_p. $$

Proof. By (15), $|f_t| \le C E_{t-} |f_t|$ $(t \in \mathbf{T})$ and so the sequence

$$ \lambda_t := C \sup_{u \le t} E_{u-} |f_u| \qquad (t \in \mathbf{T}) $$

is a prediction of f. For each $t \in \mathbf{T}$ we choose one $u \in \mathbf{T}$ for which $u^+ = t$ and set $\delta(t) := u$. The corollary can be derived from Th. 1. $\Diamond$

The next consequence, that is a generalization of the one-parameter results, comes immediately from Th. 4 and Cor. 1.

Corollary 2. *If $\mathcal{F}$ is regular, $1 < p < \infty$ and $f = (f_t, t \in \mathbf{T}) \in L_p$ then*

$$ \sup_{y>0} y P(T^{**}(f) > y)^{1/p} \le C_p \|f\|_p, \quad \sup_{y>0} y P(S(f) > y)^{1/p} \le C_p \|f\|_p, $$

$$ \sup_{y>0} y P(s(f) > y)^{1/p} \le C_p \|f\|_p. $$

The strong version of Cor. 2 can also be shown which includes Burkholder–Gundy inequality.

Corollary 3. *If $\mathcal{F}$ is regular, $1 < p < \infty$ and $f = (f_t, t \in \mathbf{T}) \in L_p$ then*

$$\|T^{**}f\|_p \le C_p\|f\|_p, \quad c_p\|f\|_p \le \|S(f)\|_p \le C_p\|f\|_p,$$
$$c_p\|f\|_p \le \|s(f)\|_p \le C_p\|f\|_p.$$

Proof. The right hand sides of the inequalities come by interpolation and from Cor. 2. To prove the left hand side of the second inequality observe that, by the one-parameter Burkholder–Gundy inequality (see (1) and (2)),

$$\|f\|_p = \|f\overline{\phi}_{v_0}\|_p \le C_p\|\Big(\sum_{t\ge v_0} |d_t f|^2\Big)^{1/2}\|_p \le C_p\|S(f)\|_p$$

where v_0 is the distinguished minimal element of $\mathbf{T}_0$. The left hand side of the third inequality can be shown in the same way by using the regularity (cf. Weisz [19]). $\diamondsuit$

6. Convergence of Vilenkin–Fourier series

First we introduce the Vilenkin systems. In this section $\Omega = [0,1)$, $\mathcal{A}$ is the σ-algebra of the Borel sets and P is the Lebesgue measure. Let $(p_n, n \in \mathbb{N})$ be a sequence of natural numbers with entries at least 2. Introduce the notations $P_0 = 1$ and $P_{n+1} := \prod_{k=0}^{n} p_k$ $(n \in \mathbb{N})$. Every point $x \in [0,1)$ can be written in the following way:

$$x = \sum_{k=0}^{\infty} \frac{x_k}{P_{k+1}}, \qquad 0 \le x_k < p_k,\ x_k \in \mathbb{N}.$$

In case there are two different forms, we choose the one for which $\lim_{k\to\infty} x_k = 0$.

The functions

$$r_n(x) := \exp\frac{2\pi\imath x_n}{p_n} \qquad (n \in \mathbb{N})$$

are called *generalized Rademacher functions* where $\imath := \sqrt{-1}$. The product system generated by these functions is said to be a *Vilenkin system:*

$$w_n(x) := \prod_{k=0}^{\infty} r_k(x)^{n_k}$$

where $n = \sum_{k=0}^{\infty} n_k P_k$, $0 \le n_k < p_k$ and $n_k \in \mathbb{N}$.

Let $\mathcal{F}_n$ be the σ-algebra generated by $\{r_0, \ldots, r_{n-1}\}$. It is easy to see that

$$\mathcal{F}_n = \sigma\left\{\left[kP_n^{-1}, (k+1)P_n^{-1}\right) : 0 \le k < P_n\right\}.$$

Denote by $R_n f$ the nth partial sum of the Vilenkin-Fourier series of $f \in L_1$, i.e.,

$$R_n f := \sum_{k=0}^{n-1} \hat{f}(k) w_k$$

where $\hat{f}(k) := E(f\overline{w}_k)$ are the *Vilenkin-Fourier coefficients* of f. Notice that $R_{P_n} f = E_n f$.

A Vilenkin system generated by a bounded sequence (p_n) is said to be *bounded*. It is easy to show that in this case the sequence $(\mathcal{F}_n, n \in \mathbb{N})$ of σ-algebras is regular (see e.g. Weisz [19]).

Special trees generated by Vilenkin systems are considered. Let us introduce the index set

$$\mathbf{T} := \mathcal{I} := \Big\{ \big[kP_n, (k+1)P_n\big) \cap \mathbb{N} : k, n \in \mathbb{N} \Big\}.$$

The ordering in $\mathcal{I}$ is defined by set inclusion. Obviously, $\mathcal{I}_0 = \mathbb{N}$. For $I = [kP_n, (k+1)P_n) \cap \mathbb{N} \in \mathcal{I}$ we set $\mathcal{F}_I := \mathcal{F}_n$. Therefore, $(\mathcal{F}_I, I \in \mathcal{I})$ can linearly be ordered. The projections

$$P_I f := \sum_{j \in I} \hat{f}(j) w_j \qquad (I \in \mathcal{I})$$

are to be investigated. For a function $f \in L_1$ we suppose that $\hat{f}(0) = 0$. It is easy to see that

$$\begin{aligned} P_I f &= \sum_{j \in [kP_n, (k+1)P_n)} E(f\overline{w}_j) w_j = \\ &= \sum_{i=0}^{P_n - 1} E\big[(f\overline{w}_{kP_n})\overline{w}_i\big] w_{kP_n} w_i = w_{kP_n} E_n(f\overline{w}_{kP_n}) \end{aligned}$$

whenever $I = [kP_n, (k+1)P_n) \cap \mathbb{N} \in \mathcal{I}$. It can be proved in the same way that, for an arbitrary $m \in I$,

$$P_I f = w_m E_n(f\overline{w}_m).$$

This implies that $(\mathcal{F}_I, P_I; I \in \mathcal{I})$ is a tree basis, indeed.

The partial sums $R_m f$ of the Vilenkin–Fourier series of $f \in L_1$ can be expressed as a martingale transform of the tree martingale $(E_I f, I \in \in \mathcal{I})$. For this set

$$m(n) := \sum_{k=n}^{\infty} m_k P_k, \qquad I_n(m) := \big[m(n), m(n) + P_n\big) \qquad (n \in \mathbb{N})$$

for $m \in \mathbb{N}$ with the expansion $m = \sum_{k=0}^{\infty} m_k P_k$ $(0 \le m_k < p_k)$. Notice that m is contained in $I_n(m)$ and $I_n(m) \subset I_{n+1}(m)$. For $I = I_n(m)$, set

$$(16) \qquad T^I := T^{I_n(m)} := \sum_{\substack{[m(n+1),m(n))\supset J\in\mathcal{I}\\ |J|=P_n}} P_J.$$

Since $I_n(m) = I_n(\tilde{m})$ implies $m(n+1) = \tilde{m}(n+1)$, the sequence of operators $T = (T^I, I \in \mathcal{I})$ is well defined. Note that in (16) there are m_n summands. In case the Vilenkin system is bounded, it is easy to show that these operators T^I $(I \in \mathcal{I})$ satisfy the conditions in definition of the tree martingale transform. Note that for the Walsh system, $T^{I_n(m)}(d_{I_n(m)}f) = m_n d_{I_n(m)}f$. Observe that

$$[0, m) = \bigcup_{n=0}^{\infty} [m(n+1), m(n))$$

which implies

$$R_m f = \sum_{n=0}^{\infty} \sum_{k\in[m(n+1),m(n))} \hat{f}(k) w_k = \sum_{n=0}^{\infty} T^{I_n(m)} f = \sum_{\{m\}\le I} T^I(d_I f)$$

where $(d_I f, I \in \mathcal{I})$ is the martingale difference sequence of the tree martingale $f = (E_I f, I \in \mathcal{I})$. Thus the maximal function of the partial sums of the Vilenkin-Fourier series of $f \in L_1$ can be estimated by the maximal function of the martingale transform of the tree martingale f, namely,

$$R^* f := \sup_{m\in\mathbb{N}} |R_m f| \le T^{**} f.$$

If the Vilenkin system is bounded then, by the regularity of $(\mathcal{F}_n, n \in \mathbb{N})$, we get immediately that $\mathcal{F} := (\mathcal{F}_I, I \in \mathcal{I})$ is regular, too.

Applying a usual density argument due to Marczinkiewicz and Zygmund [9] and Corollaries 2 and 3 we obtain our main result.

Corollary 4. *Suppose that the Vilenkin system is bounded,* $1 < p < \infty$ *and* $f \in L_p$. *Then*

$$\sup_{y>0} y P(R^* f > y)^{1/p} \le \|R^* f\|_p \le C_p \|f\|_p.$$

Consequently, for every $f \in L_p$ *with* $p > 1$ *we have*

$$R_m f \to f \quad a.e. \text{ as } \quad m \to \infty.$$

Of course, Cor. 4 implies that $R_m f$ converges to f also in L_p norm $(1 < p < \infty)$ as $m \to \infty$ whenever the Vilenkin system is bounded. In what follows we give a new proof this result for arbitrary Vilenkin systems. Note that for an unbounded Vilenkin system the operators T^I $(I \in \mathcal{I})$ defined in (16) do not satisfy the condition (iv). However, the weaker inequality

$$E_n|T^I f|^2 \le C E_n|f|^2 \qquad (f \in L_1, I = I_n(m))$$

follows from Bessel's inequality. This result can be extended to each $1 < p < \infty$. Since

$$m - (m_n - l)P_n \in \big[m(n+1) + lP_n, m(n+1) + (l+1)P_n\big)$$

$(0 \le l < m_n)$, the operator T^I can be written in the following form:

$$T^I f = \sum_{l=0}^{m_n-1} \sum_{k=m(n+1)+lP_n}^{m(n+1)+(l+1)P_n-1} \hat{f}(k) w_k =$$

$$= \sum_{l=0}^{m_n-1} w_{m-(m_n-l)P_n} E_n(f\overline{w}_{m-(m_n-l)P_n}) =$$

$$= w_m \sum_{l=0}^{m_n-1} \overline{w}_{(m_n-l)P_n} E_n[(f\overline{w}_m) w_{(m_n-l)P_n}] =$$

$$= w_m \sum_{l=1}^{m_n} \overline{r}_n^{m_n-l} E_n[(f\overline{w}_m) r_n^{m_n-l}].$$

The inequality

$$\|R_m f\|_p \le C_p \|f\|_p \qquad (1 < p < \infty)$$

for trigonometric Fourier series (see e.g. Zygmund [21], Vol. I. p. 266) and an inequality relative to the discrete Fourier series presented in [21] (Vol. II. p. 28) imply that

$$(17) \qquad E_n|T^I f|^p \le C_p E_n|f|^p \qquad (f \in L_p)$$

where C_p is independent of n and f. Using this we can prove the following norm convergence result due to Schipp [11] and Young [20] (see also Simon [15]).

Theorem 5. *Suppose that the Vilenkin system is arbitrary,* $1 < p < \infty$ *and* $f \in L_p$. *Then*

$$\|R_m f\|_p \le C_p \|f\|_p$$

where C_p *is independent of* m *and* f. *Consequently, for every* $f \in L_p$ *with* $1 < p < \infty$ *we have*

$$R_m f \to f \quad \text{in} \quad L_p \quad \text{norm as} \quad m \to \infty.$$

Proof. First suppose that $2 \le p < \infty$. It follows from (3) that

$$\|R_m f\|_p = \Big\| \sum_{\{m\}\le I} T^I(d_I f)\Big\|_p =$$

$$\le C_p \Big\| \Big(\sum_{\{m\}\le I} E_I |T^I(d_I f)|^2\Big)^{1/2}\Big\|_p + C_p \| \sup_{\{m\}\le I} |T^I(d_I f)| \|_p =$$

$$\le C_p \Big\| \Big(\sum_{\{m\}\le I} E_I |T^I(d_I f)|^2\Big)^{1/2}\Big\|_p + C_p \Big(E\Big[\sum_{\{m\}\le I} |T^I(d_I f)|^p\Big]\Big)^{1/p} =$$

$$\le C_p \Big\| \Big(\sum_{\{m\}\le I} E_I |T^I(d_I f)|^2\Big)^{1/2}\Big\|_p + C_p \Big(E\Big[\sum_{\{m\}\le I} E_I |T^I(d_I f)|^p\Big]\Big)^{1/p}.$$

Using the inequality (17), (4), (1) and (2) we obtain

$$\|R_m f\|_p \le C_p \Big\| \Big(\sum_{\{m\}\le I} E_I |d_I f|^2\Big)^{1/2}\Big\|_p + C_p \Big\| \Big(\sum_{\{m\}\le I} |d_I f|^p\Big)^{1/p}\Big\|_p \le$$

$$\le C_p \Big\| \Big(\sum_{\{m\}\le I} |d_I f|^2\Big)^{1/2}\Big\|_p \le C_p \|f \overline{w}_m\|_p = C_p \|f\|_p.$$

The theorem for $1 < p < 2$ can be proved by a usual duality argument. ◊

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