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SOME WEIGHTED MULTIDIMENSIONAL BERWALD, THUNSDORFF AND BORELL INEQUALITIES

Josip **Pečarić**[1]

Faculty of Textile Technology, University of Zagreb, Pierottijeva 6, 10000 Zagreb, Croatia

Lars Erik **Persson**[2]

Department of Mathematics, Luleå University, S–971 87 Luleå, Sweden

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Abstract: Some weighted versions of the Berwald, Thunsdorff and Borell inequalities for several variables are stated, proved and discussed. The sharpness of the results and the relations to other generalizations of these inequalities are pointed out.

1. Introduction

Let f be a nonnegative concave function on the finite interval $[a, b]$. If $0 < r < s$, then, according to the well-known Berwald inequality (see [1]),

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$$(1.1)\qquad \left(\frac{s+1}{b-a}\int_a^b f(x)^s\,dx\right)^{1/s} \leq \left(\frac{r+1}{b-a}\int_a^b f(x)^r\,dx\right)^{1/r}.$$

Moreover, if f is convex and $f(a)=0$, then (1.1) holds in the reversed direction. This fact is due to Thunsdorff [11]. For some recent weighted versions of (1.1) we refer to [5] and [8]. Some multidimensional versions of the Berwald and Thunsdorff inequalities have been proved in [2] and [3], respectively. For some additional references and results see also the recent books [7] and [10].

This paper is organized as follows: In Sections 2 and 3 we will prove some weighted multidimensional versions of the Berwald and Thunsdorff inequalities. The key arguments are to use suitable versions of the Chebyshev inequality and the power mean inequality in this connection. In Section 4 we use in particular these results to also obtain a new weighted multidimensional version of the Borell inequality (see [2]). A complement of this result is proved in Section 5. This result may also be regarded as a weighted multidimensional version of a (Grüss–Barnes type) inequality recently proved in [6].

Some notations and preliminaries. We say that the multidimensional function, $f(x)$, $x \in X$,

$$X = \left\{x = (x_1, x_2, \ldots, x_n) \middle|\ \ a_i \leq x_i \leq b_i, \quad i = 1, 2, \ldots, n\right\},$$

is nondecreasing (nonincreasing) if, for each fixed i, $1 \leq i \leq n$, the function $x_i \to f(x)$ is nondecreasing (nonincreasing). Moreover, we let Y denote the class of all nonnegative functions $P(x)$ of the form

$$P(x) = p_1(x_1)p_2(x_2)\ldots p_n(x_n).$$

For later purposes we note that e.g. the following functions obviously belong to Y:

$$L_1(x) = \prod_{i=1}^n (x_i - a_i) \quad \text{and} \quad L_2(x) = \prod_{i=1}^n (b_i - x_i).$$

We need the following generalization of the Chebyshev inequality (see [9]):

(C) If F(x) and $G(x)$ are monotone in the same sense on X and $P \in Y$ is an integrable function, then

$$\int_X P(x)dx \int_X P(x)F(x)G(x)dx \geq$$

$$\geq \int_X P(x)F(x)dx \int_X P(x)G(x)dx,$$

where $dx = dx_1 dx_2 \dots dx_n$. If $F(x)$ and $G(x)$ are monotone in the opposite sense, then (2.1) holds in the reversed direction.

We also need the power mean inequality in the following form (c.f. [4, formula 192.]):

(PM) If $r \leq s$ $r, s \neq 0$, $p(x), h(x) \geq 0$ and the involved integrals are positive, then

$$\left(\int_X p(x)(h(x))^r dx \Big/ \int_X p(x)dx\right)^{1/r} \leq$$

$$\leq \left(\int_X p(x)(h(x))^s dx \Big/ \int_X p(x)dx\right)^{1/s}.$$

2. A weighted multidimensional Berwald inequality

In the sequel we let f denote a nonnegative function on X. Our weighted multidimensional Berwald inequality reads:

Theorem 1. *Let $\omega \in Y$ be an integrable function on X and let r, s be real numbers such that $0 < r < s$.*

(i) *If $f(x)$ is nondecreasing and $f(x)/L_1(x)$ is nonincreasing, then*

$$\left(\int_X \omega(x)(f(x))^s dx \Big/ \int_X \omega(x)(L_1(x))^s dx\right)^{1/s} \leq$$

$$\leq \left(\int_X \omega(x)(f(x))^r dx \Big/ \int_X \omega(x)(L_1(x))^r dx\right)^{1/r}.$$

The inequality is sharp and equality holds for $f(x) = L_1(x)$.

(ii) *If $f(x)$ is nonincreasing and $f(x)/L_2(x)$ is nondecreasing, then*

$$\left(\int_X \omega(x)(f(x))^s dx \Big/ \int_X \omega(x)(L_2(x))^s dx\right)^{1/s} \leq$$

$$\leq \left(\int_X \omega(x)(f(x))^r \, dx \Big/ \int_X \omega(x)(L_2(x))^r \, dx \right)^{1/r}.$$

The inequality is sharp and equality holds for $f(x) = L_2(x)$.

Remark 1. The conditions in (i) are satisfied e.g. if $f(x)$ is nondecreasing and concave in each variable. The conditions in (ii) are satisfied e.g. if $f(x)$ is nonincreasing and concave in each variable.

Proof. According to our assumptions we have that the function $F(x) = (f(x)/L_1(x))^r$ is nonincreasing and the function $G(x) = (f(x))^{s-r}$ is nondecreasing. Therefore we can use the Chebyshev inequality (C) with the weight $P(x) = \omega(x)(L_1(x))^r$ to obtain that

$$\int_X \omega(x)(L_1(x))^r \, dx \int_X \omega(x)(f(x))^s \, dx \leq$$

$$\leq \int_X \omega(x)(f(x))^r \, dx \int_X \omega(x)(L_1(x))^r (f(x))^{s-r} \, dx,$$

i.e.,

$$\begin{aligned} &\left(\int_X \omega(x)(L_1(x))^r \, dx \Big/ \int_X \omega(x)(f(x))^r \, dx \right)^{1/r} \leq \\ &\leq \left(\int_X \omega(x)(L_1(x))^r (f(x))^{s-r} \, dx \Big/ \int_X \omega(x)(f(x))^s \, dx \right)^{1/r}. \end{aligned} \tag{2.1}$$

Moreover, by using the power mean inequality (PM), we find that

$$\begin{aligned} &\left(\int_X \omega(x)(L_1(x))^r (f(x))^{s-r} \, dx \Big/ \int_X \omega(x)(f(x))^s \, dx \right)^{1/r} \leq \\ &\leq \left(\int_X \omega(x)(L_1(x))^s \, dx \Big/ \int_X \omega(x)(f(x))^s \, dx \right)^{1/s}. \end{aligned} \tag{2.2}$$

We combine (2.1) with (2.2) and the inequality in (i) is proved. The sharpness statement is obvious. The proof of (ii) only consists of obvious modifications of the proof of (i) so we omit the details. ◊

3. A weighted multidimensional Thunsdorff inequality

Our weighted Thunsdorff's inequality can be formulated as follows:

Theorem 2. *Let $\omega \in Y$ be an integrable function on X and let r, s be real numbers such that $0 < r < s$. If $f(x)/L_1(x)$ is nondecreasing, then*

$$\left(\int_X \omega(x)(f(x))^r\,dx \Big/ \int_X \omega(x)(L_1(x))^r\,dx\right)^{1/r} \leq$$

$$\left(\int_X \omega(x)(f(x))^s\,dx \Big/ \int_X \omega(x)(L_1(x))^s\,dx\right)^{1/s}.$$

The inequality is sharp and equality holds for $f(x) = L_1(x)$.

Remark 2. The function $f(x)/L_1(x)$ is nondecreasing for example if $f(x)$ is convex in each variable and $f(x_1, \ldots, x_{i-1}, a_i, x_{i+1}, \ldots, x_n) = 0$ for all $i = 1, 2, \ldots, n$.

Proof. First we use the power mean inequality (PM) with $p(x) = \omega(x)(L_1(x))^s$ and $h(x) = f(x)/L_1(x)$ to find that

$$\begin{aligned}&\left(\int_X \omega(x)(L_1(x))^{s-r}(f(x))^r\,dx \Big/ \int_X \omega(x)(L_1(x))^s\,dx\right)^{1/r} \leq \\ &\leq \left(\int_X \omega(x)(f(x))^s\,dx \Big/ \int_X \omega(x)(L_1(x))^s\,dx\right)^{1/s}.\end{aligned} \tag{3.1}$$

Moreover, by the Chebyshev inequality (C), applied with $P(x) = \omega(x)(L_1(x))^r$ and the nondecreasing functions $F(x) = (f(x)/L_1(x))^r$ and $G(x) = (L_1(x))^{s-r}$,

$$\begin{aligned}&\int_X \omega(x)(L_1(x))^r\,dx \int_X \omega(x)(L_1(x))^{s-r}(f(x))^r\,dx \geq \\ &\geq \int_X \omega(x)(f(x))^r\,dx \int_X \omega(x)(L_1(x))^s\,dx.\end{aligned} \tag{3.2}$$

The inequality in Th. 2 follows at once from (3.1) and (3.2). The sharpness assertion is obvious. ◊

Remark 3. The proof above shows that Th. 2 can easily be generalized to hold in more general situations e.g. for functions of higher order of convexity.

4. A weighted multidimensional Borell inequality

The following theorem may be regarded as a weighted variant of a well-known Theorem of Borell [3].

Theorem 3. *Let $\omega \in Y$ be an integrable function on X and suppose that a, b, p, q are real numbers satisfying $a > 0$, $b > 0$, $p \geq 1$ and $q \geq 1$. If $f(x)$ is nondecreasing, $f(x)/(L_1(x))^a$ is nonincreasing, $g(x)$ is nonincreasing and $g(x)/(L_2(x))^b$ is nondecreasing , then*

$$\int_X \omega(x)f(x)g(x)dx \geq$$

$$\geq C\left(\int_X \omega(x)(f(x))^p dx\right)^{1/p}\left(\int_X \omega(x)(g(x))^q dx\right)^{1/q},$$

where

$$C = \int_X \omega(x)(L_1(x))^a(L_2(x))^b dx \Big/ \left(\int_X \omega(x)(L_1(x))^{ap} dx\right)^{1/p} \cdot$$

$$\cdot\left(\int_X \omega(x)(L_2(x))^{bq} dx\right)^{1/q}.$$

The inequality is sharp and equality holds for $f(x) = (L_1(x))^a$ and $g(x) = (L_2(x))^b$.

Remark 4. The assumptions in Th. 3 are satisfied e.g. if $f(x)$ is nondecreasing, $g(x)$ is nonincreasing and $f^{1/a}$, $g^{1/b}$ are both concave functions in each variable. Therefore, Th. 3 with $\omega = 1$ is similar but not the same as the original result by Borell [3]. Our proof is completely different and (in our opinion) much simpler than that in [3].

Proof. First we use the Chebyshev inequality (C) with $P(x) = \omega(x)(L_1(x))^a$ and the nonincreasing functions $F(x) = f(x)/(L_1(x))^a$ and $G(x) = g(x)$ to obtain that

$$\int\limits_X \omega(x)(L_1(x))^a dx \int\limits_X \omega(x)f(x)g(x)dx \geq \tag{4.1}$$
$$\geq \int\limits_X \omega(x)f(x)dx \int\limits_X \omega(x)(L_1(x))^a g(x)dx.$$

Next by using (C) with $P(x) = \omega(x)(L_2(x))^b$ and the nondecreasing functions $F(x) = (L_1(x))^a$ and $G(x) = g(x)/(L_2(x))^b$ we have that

$$\int\limits_X \omega(x)(L_2(x))^b dx \int\limits_X \omega(x)g(x)(L_1(x))^a dx \geq \tag{4.2}$$
$$\geq \int\limits_X \omega(x)(L_1(x))^a (L_2(x))^b dx \int\limits_X \omega(x)g(x)dx.$$

Now by using (4.1)–(4.2) and Th. 1 we find that

$$\int\limits_X \omega(x)f(x)g(x)dx \geq \frac{\int\limits_X f(x)\omega(x)dx \int\limits_X \omega(x)g(x)(L_1(x))^a dx}{\int\limits_X \omega(x)(L_1(x))^a dx} \geq$$
$$\geq \frac{\int\limits_X f(x)\omega(x)dx \int\limits_X \omega(x)(L_1(x))^a (L_2(x))^b dx \int\limits_X \omega(x)g(x)dx}{\int\limits_X \omega(x)(L_1(x))^a dx \int\limits_X \omega(x)(L_2(x))^b dx} \geq$$
$$\geq \frac{\int\limits_X \omega(x)(L_1(x))^a (L_2(x))^b dx}{\left(\int\limits_X \omega(x)(L_1(x))^{ap} dx\right)^{1/p} \left(\int\limits_X \omega(x)(L_2(x))^{bq} dx\right)^{1/q}} \cdot$$
$$\cdot \left(\int\limits_X \omega(x)(f(x))^p dx\right)^{1/p} \left(\int\limits_X \omega(x)(g(x))^q dx\right)^{1/q}.$$

In the third inequality we have used Th. 1 (i) with $r = 1$, $s = p$, L_1 replaced by L_1^a and Th. 1 (ii) with $r = 1$, $s = 7$ and L_2 replaced by L_2^b.

The sharpness statement is obvious so the proof is complete. ◊

Remark 5. In fact, the proof above shows that the following "interpolated" version of the inequality in Th. 1 holds:

$$\frac{\int\limits_X \omega(x)f(x)g(x)dx}{\int\limits_X \omega(x)(L_1(x))^a (L_2(x))^b dx} \geq$$

$$\geq \left(\frac{\int_X \omega(x)(f(x))^{p_1}dx}{\int_X \omega(x)(L_1(x))^{ap_1}dx}\right)^{1/p_1} \left(\frac{\int_X \omega(x)(g(x))^{q_1}dx}{\int_X \omega(x)(L_2(x))^{bq_1}dx}\right)^{1/q_1} \geq$$

$$\geq \left(\frac{\int_X \omega(x)(f(x))^{p_2}dx}{\int_X \omega(x)(L_1(x))^{ap_2}dx}\right)^{1/p_2} \left(\frac{\int_X \omega(x)(g(x))^{q_2}dx}{\int_X \omega(x)(L_2(x))^{bq_2}dx}\right)^{1/q_2},$$

where $1 \leq p_1 \leq p_2$ and $1 \leq q_1 \leq q_2$.

5. Another weighted multidimensional inequality

The following theorem may be regarded both as a natural complement of our previous weighted multidimensional Borell inequality and as a generalization of a recently obtained (Grüss–Barnes type) inequality [6, Th. 5]:

Theorem 4. *Let $\omega \in Y$ be an integrable function on X and suppose that a_k and p_k are real numbers satisfying $a_k > 0$ and $0 \leq p_k \leq 1$, $k = 1, 2, \ldots, m$.*

(i) *If, for every $k = 1, 2, \ldots, m$, the function g_k satisfies the growth conditions that $g_k(x)$ is nondecreasing and $g_k(x)/(L_1(x))^{a_k}$ is nonincreasing, then*

$$\frac{\int_X \omega(x) \prod_{k=1}^{m} g_k(x)dx}{\int_X \omega(x)(L_1(x))^{\sum_1^m a_k} dx} \leq \prod_{i=1}^{m} \left(\frac{\int_X \omega(x)(g_k(x))^{p_k}dx}{\int_X \omega(x)(L_1(x))^{a_k p_k}dx}\right)^{1/p_k}. \tag{5.1}$$

(ii) *If, for every $k = 1, 2, \ldots, m$, the function g_k satisfies that $g_k(x)/(L_1(x))^{a_k}$ is nondecreasing, then* (5.1) *holds in the reversed direction.*

The inequalities in (i) *and* (ii) *are sharp and equality occurs if $g_k(x) = (L_1(x))^{1/a_k}$.*

Remark 6. The assumptions on g_k in (i) are satisfied e.g. if $g_k(x)$ is nondecreasing and $(g_k(x))^{1/a_k}$ is concave in each variable. Moreover, the conditions on g_k in (ii) are satisfied e.g. if $g_k(x_1, \ldots, x_{i-1}, a_i, x_{i+1}, \ldots \ldots, x_n) = 0$ for all $i = 1, 2, \ldots, n$, and $(g_k(x))^{1/a_k}$ is convex in each variable. Therefore, in particular, Th. 4 gives a slight generalization also of the one-dimensional result (of Grüss–Barnes type) recently obtained in [6, Th. 5].

Proof. (i) Here we use (C) successively, first with $P(x)=\omega(x)(L_1(x))^{a_1}$, $F(x) = g_1(x)/(L_1(x))^{a_1}$ and $G(x) = g_2(x)g_3(x)\dots g_m(x)$, after that with $P(x) = \omega(x)(L_1(x))^{a_k}$, $F(x) = g_k(x)/(L_1(x))^{a_k}$ and $G(x) = (L_1(x))^{a_1+a_2+\dots+a_{k-1}}g_{k+1}(x)\dots g_m(x)$, $k = 2,3,\dots,m-1$, and finally, with $P(x) = \omega(x)(L_1(x))^{a_m}$, $F(x) = g_m(x)/(L_1(x))^{a_m}$ and $G(x) = (L_1(x))^{a_1+a_2+\dots+a_{m-1}}$:

$$\int\limits_X \omega(x)(L_1(x))^{a_1}dx \int\limits_X \omega(x)g_1(x)\dots g_m(x)dx \le$$

$$\le \int\limits_X \omega(x)g_1(x)dx \int\limits_X \omega(x)g_2(x)g_3(x)\dots g_m(x)(L_1(x))^{a_1}dx,$$

$$\int\limits_X \omega(x)(L_1(x))^{a_2}dx \int\limits_X \omega(x)g_2(x)g_3(x)\dots g_m(x)(L_1(x))^{a_1}dx \le$$

$$\le \int\limits_X \omega(x)g_2(x)dx \int\limits_X \omega(x)g_3(x)dx\dots g_m(x)(L_1(x))^{a_1+a_2}dx,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\int\limits_X \omega(x)(L_1(x))^{a_m}dx \int\limits_X \omega(x)g_m(x)(L_1(x))^{a_1+a_2+\dots+a_{m-1}}dx \le$$

$$\le \int\limits_X \omega(x)g_m(x)dx \int\limits_X \omega(x)(L_1(x))^{a_1+a_2+\dots+a_m}dx.$$

Using these inequalities we can derive that

$$\frac{\int\limits_X \omega(x)\prod\limits_{k=1}^m g_k(x)dx}{\int\limits_X \omega(x)(L_1(x))^{\sum_1^m a_k}dx} \le \prod_{i=1}^m \frac{\int\limits_X \omega(x)g_k(x)dx}{\int\limits_X \omega(x)(L_1(x))^{a_k}dx}.$$

Finally, we apply Th. 1 and the inequality (5.1) is proved.

(ii) The proof is quite similar to that in (i) (we only need to use (C) in the reversed direction and Th. 2 instead of Th. 1).

The sharpness assertion is easily checked by inspection. ◊

Remark 7. Our proof shows that, in fact, the inequality (5.1) can be replaced by a refined "interpolated" inequality quite analogous to that presented in Remark 5.

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