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ON SOME JORDAN–HÖLDER–DEDEKIND TYPE THEOREMS IN LATTICES

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Abstract: In this paper we will prove some Jordan–Hölder–Dedekind type theorems in general lattices. All of these theorems work in lattices more general then the modular one. We will give a significant example, too.

Applying results established in [4], Gh. Fărcaş proved in [2] a nice Schreier type theorem for general lattices, using chains of standard elements. There were also deduced two Jordan–Hölder–Dedekind type theorems, like its consequences.

Let us recall the definition of standard element. Suppose $(L, \vee, \wedge)$ denotes a lattice having 0 and 1. An element $s \in L$ is called *standard*, if for any $x, y \in L$, $x \wedge (s \vee y) = (x \wedge s) \vee (x \wedge y)$. The theorem proved in [4] says, if

$$0 = a_0 < a_1 < \ldots < a_{k-1} < a_k = 1$$

and

$$0 = b_0 < b_1 < \ldots < b_{l-1} < b_l = 1$$

are to chains of L, where the second chain is built up from standard elements, then they admit refinements of the same length.

In the following we will find weaker conditions than the standardness of the chain's elements, and we will obtain even some stronger consequences. The only price paid for them is, the conditions have to be claimed on both chains.

Definition 1. Let L be a lattice, and $(a]$ a principal ideal of it. We will say an element $b \in (a]$ is *a-standard* if for every $c \in L, b \vee (a \wedge c) = a \wedge \wedge (b \vee c)$. We also call $[b, a]$ *standard* interval.

Let us notice that for every $a \in L$, the intervals $[0, a]$ and $[a, 1]$ are standard. Therefore in N_5, the nonmodular lattice of 5 elements, $(0 < b < a < 1, 0 < c < 1)$ only $[b, a]$ is not a standard interval. It is easy to see that if b is standard, then b is a-standard for every a, $a \geq b$, but not conversely. Indeed, in M_5, the nondistributive lattice of 5 elements $(0 < a, b, c < 1)$, the element $a \in L$ is x-standard for every $x \geq a$, but a is not standard, as $b \wedge (a \vee c) \neq (b \wedge a) \vee \vee (b \wedge c)$.

Definition 2. A chain in L, $0 = a_0 < a_1 < \cdots < a_{k-1} < a_k = 1$ is called *standard* chain, if all the intervals $[a_i, a_{i+1}]$ are standard, $i = = 0, 1, \ldots, k-1$.

Theorem 1. *Let*

(1) $$0 = a_0 < a_1 < \cdots < a_{k-1} < a_k = 1 \quad \textit{and}$$

(2) $$0 = b_0 < b_1 < \cdots < b_{l-1} < b_l = 1$$

two chains in which $[a_i, a_{i+1}]$, $i = 0, 1, \ldots, k-1$ *and* $[b_j, b_{j+1}]$, $j = = 0, 1, \ldots, l-1$ *are standard intervals. Then they admit refinements of the same length.*

Proof. Let us define

(3) $$a_{ij} = a_i \vee (a_{i+1} \wedge b_j), \quad i = 0, 1, \ldots, k-1, \quad j = 0, 1, \ldots, l$$

(4) $$b_{ji} = b_j \vee (b_{j+1} \wedge a_i), \quad j = 0, 1, \ldots, l-1, \quad i = 0, 1, \ldots, k.$$

We have by this definitions $a_{i0} = a_i$ and $a_{il} = a_{i+1}$, $i = 0, 1, \ldots, k-1$ and $b_{j0} = b_j$, $b_{jk} = b_{j+1}$, $j = 0, 1, \ldots, l-1$, and also

$$a_{ij} \leq a_{i,j+1}, \quad j = 0, 1, \ldots, l-1, \quad i = 0, 1, \ldots, k$$
$$b_{ji} \leq b_{j,i+1}, \quad i = 0, 1, \ldots, k-1, \quad j = 0, 1, \ldots, l.$$

Consequently, the chain consisting from a_{ij} is a refinement of (1) while that of b_{ji} is a refinement of (2). Their formal length are kl, so we have to prove there is a one-to-one correspondence between their repetitions. Let therefore suppose $a_{ij} = a_{i,j+1}$ for some i and j. Then we have

$$\begin{aligned} a_{i+1} \wedge b_{j+1} &= a_{i+1} \wedge b_{j+1} \wedge a_{i,j+1} = a_{i+1} \wedge b_{j+1} \wedge a_{ij} = \\ &= a_{i+1} \wedge b_{j+1} \wedge (a_i \vee (a_{i+1} \wedge b_j)) = \\ &= a_{i+1} \wedge b_{j+1} \wedge (a_{i+1} \wedge (a_i \vee b_j)) = \\ &= b_{j+1} \wedge (a_{i+1} \wedge (a_i \vee b_j)) = \\ &= b_{j+1} \wedge (a_i \vee (b_j \wedge a_{i+1})) \end{aligned}$$

$$\begin{aligned} b_{j,i+1} &= b_j \vee (b_{j+1} \wedge a_{i+1}) = b_j \vee (b_{j+1} \wedge (a_i \vee (a_{i+1} \wedge b_j))) = \\ &= b_{j+1} \wedge (b_j \vee a_i \vee (b_j \wedge a_{i+1})) = b_{j+1} \wedge (b_j \vee a_i) = \\ &= b_j \vee (b_{j+1} \wedge a_i) = b_{ji}. \end{aligned}$$

It means that $a_{ij} = a_{i,j+1}$ force $b_{ji} = b_{j,i+1}$. Owing to the symmetry between (1) and (2), as well as between (3) and (4), we have also $a_{ij} = a_{i,j+1}$ if $b_{ji} = b_{j,i+1}$. $\Diamond$

We can now prove two corollaries, analogous to which were proved in [2], actually Jordan–Hölder–Dedekind type theorems.

Corollary 1. *If* (1) *and* (2) *are standard chains, and they are maximal like chains, then they have the same length.*

Corollary 2. *If L contains a maximal chain with length n, which is standard, then the length of any other standard chain is less than n, and moreover this last one can be refined to a chain of length n.*

It is natural to ask now if a standard chain still remain standard by applying a proper refinement. We will show in the followings that a simple compatibility condition of the standard intervals with the lattice operations assures an affirmative answer. Before the next definitions, let us notice a failure of duality which occurs shifting a standard interval through an element $c \in L$ using the first or the second lattice operation. More precisely, a standard interval $[b, a]$ shifted by $c \in L$ using $\wedge$ still remain in the principal ideal $(a]$, i.e. $[b \wedge c, a \wedge c] \subset (a]$, for every $c \in L$, while trough the $\vee$-shift by $c \in L$ this is not true: the interval $[b \vee c, a \vee c]$ may not be included in $(a]$. According to this, we give the next definitions.

Definition 3. We will say a standard interval $[b, a]$ is $\wedge$-*shift compatible* if $[b \wedge c, a \wedge c]$ is standard for every $c \in L$.

Definition 4. Suppose $[b, a]$ is standard interval. We will say $[b, a]$ is $\vee$-*shift compatible* if for every $d \in L$ satisfying $[b, a] \subset (d]$ and for every $c \in (d]$, where c is d-standard, the interval $[b \vee c, a \vee c]$ is standard.

Definition 5. A standard interval is *normal* if it is both $\wedge$-shift and $\vee$-shift compatible. Also a chain is *normal* if any interval of it is normal.

Theorem 2. *Let* (1) *and* (2) *normal chains in* L. *Then they admit standard refinements of the same length.*

Proof. It is sufficient to show that for a_{ij} defined like in (3), the interval $[a_{ij}, a_{i,j+1}]$ is standard. But as $[b_j, b_{j+1}]$ is standard and $\wedge$-shift compatible, we conclude $[a_{i+1} \wedge b_j, a_{i+1} \wedge b_{j+1}]$ remain standard and moreover, it is included in $(a_{i+1}]$. As $a_i \in (a_{i+1}]$ too, and a_i is a_{i+1}-standard, it follows

$$[a_i \vee (a_{i+1} \wedge b_j), a_i \vee (a_{i+1} \wedge b_{j+1})] = [a_{ij}, a_{i,j+1}]$$

is standard. $\Diamond$

Let us examine Th. 1 and 2 from the perspective of the modular lattices, in which a stronger version hold. We are forced to begin again with a new definition.

Definition 6. Two standard chains like (1) and (2) will be called *equivalent* chains if $k = l$, and there exists a permutation σ of $\{0, 1, \ldots, k\}$, so that $[a_i, a_{i+1}]$ and $[b_{\sigma(i)}, b_{\sigma(i)+1}]$ are projective intervals (see [6] for the definition of projectives intervals).

Theorem 3. *Let* L *be a lattice in which for every* $x, y \in L$, $[y, x \vee y]$ *and* $[x \wedge y, x]$ *are isomorphic (we will denote by* $\sim$*). Then every two standard (normal) chains admit equivalent (standard) refinement.*

Proof. Let us use the same notation as in (1), (2), (3) and (4) and denote

$$x = a_{i+1} \wedge b_{j+1}, \quad y = a_{ij},$$
$$x' = b_{j+1} \wedge a_{i+1}, \quad y' = b_{ji}.$$

Then we have

$$x \vee y = (a_{i+1} \wedge b_{j+1}) \vee a_{ij} = (a_{i+1} \wedge b_{j+1}) \vee (a_{i+1} \wedge b_j) \vee a_i =$$
$$= a_i \vee (a_{i+1} \wedge b_{j+1}) = a_{i,j+1}$$
$$x \wedge y = a_{i+1} \wedge b_{j+1} \wedge a_{ij} = a_{i+1} \wedge b_{j+1} \wedge a_{i+1} \wedge (a_i \vee b_j) =$$
$$= a_{i+1} \wedge b_{j+1} \wedge (a_i \vee b_j).$$

It follows therefore $[a_{ij}, a_{i,j+1}] = [y, x \vee y] \sim [x \wedge y, x]$. By an analogous way, we have $[b_{ji}, b_{j,i+1}] = [y', x' \vee y'] \sim [x' \wedge y', x']$. Now we just have to notice that $x = x'$, and $x' \wedge y' = x \wedge y$, so $[x \wedge y, x] = [x' \wedge y', x']$, and we can conclude $[a_{ij}, a_{i,j+1}]$ and $[b_{ji}, b_{j,i+1}]$ are projective intervals. $\Diamond$

Remark 1. The assumption of Th. 3 is still weaker than the modularity condition. It becomes, however, equivalent with the modularity in algebraic lattices. Therefore, Theorem 3 still remains a proper extension of the Jordan–Hölder–Dedekind type theorem for the nonalgebraic lattices.

Remark 2. Let $L(G)$ be the subgroup lattice of a finite group G. If $H, N \in L(G)$, and N is normal in G, then $[N, H]$ is a normal interval, according to our definition 5. This is of course a known result, stated now using the new language of our present paper. Also $L(G)$ is not a modular lattice. It is easy to see that a standard interval $[N, H]$ is an accurate correspondent of the factor group H/N, according to the second group isomorphism theorem, too. Actually, Theorem 3 could be viewed as a proper correspondent of the finite group Jordan–Hölder theorem.

Let us mention, that a different approach to this topic, leading to similar results is to be found in [3].

Finally, let enable us just to point out a related problem, which is actually on **open question**. Given a finite lattice having 0 and 1, the question is if there exists or not a group, admitting this lattice as its (normal) subgroup lattice. Moreover, it is unsolved even the following: is every finite lattice isomorphic to an interval in $L(G)$, for an appropriate finite group G, (see [6])?

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