Mathematica Pannonica
7/2 (1996), 177 – 183

COLOURED KNOTS AND PERMUTATIONS REPRESENTING 3-MANIFOLDS

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Received: October 1994

MSC 1991: 57 Q 15, 57 M 12

Keywords: 3-manifolds, branched coverings, coloured knots, coloured graphs.

Abstract: It is known that every closed orientable 3-manifold can be represented by coloured knots ([4], [5]), edge-coloured graphs ([2]) or transitive permutation pairs ([6]). The present paper describes some relations between these representation theories: in particular, it is shown how to obtain a transitive permutation pair representing a 3-manifold M^3, starting from a coloured knot representing M^3.

1. Introduction

Throughout this paper, all spaces and maps are piecewise-linear; manifolds are always supposed to be closed and connected.

In [6], Montesinos proves that every orientable 3-manifold M^3 can be represented by a transitive pair of permutations (σ, τ) of $\sum_h$, the symmetric group on $N_h = \{1, 2, \dots, h\}$. In fact, M^3 is a covering of the 3-sphere S^3, branched over the graph G of Fig. 1; thus, M^3 is

Work performed under the auspices of the G.N.S.A.G.A. of the C.N.R. (National Research Council of Italy) and financially supported by MURST.

Fig. 1.

determined by a monodromy map $\delta : \prod_1(S^3 - G) \longmapsto \sum_h$ defined by sending the meridians m_1 and m_2 to σ and τ respectively. If M^3 is represented by (σ, τ), we write $M^3 = M(\sigma, \tau)$.

A well-known theorem of Hilden [4] and Montesinos [5] states that every orientable 3-manifold M^3 is a simple 3-fold covering of S^3, branched over a knot $\mathcal{K} \subset S^3$ (where "simple" means that the associated monodromy $\omega : \prod_1(S^3 - \mathcal{K}) \longmapsto \sum_3$ sends meridians to transpositions). In other words, M^3 is representable by a pair $(\mathcal{K}, \omega)$, where $\mathcal{K}$ is a knot and $\omega : \prod_1(S^3 - \mathcal{K}) \longmapsto \sum_3$ is a simple monodromy map. Such a pair is called a *coloured knot*, since it can be visualized by a planar *coloured diagram* $\mathcal{D}$ of $\mathcal{K}$, in which each arc is coloured by $k \in Z_3 = \{0, 1, 2\}$ if and only if the transposition associated to the corresponding meridian fixes k. Moreover, we can always suppose (see, for example, [1]) that such a diagram is 3-*coloured*, i.e. at each crossing, the three incident arcs have distinct colours. If the 3-manifold M^3 is represented by the 3-coloured diagram $\mathcal{D}$, we write $M^3 = M(\mathcal{D})$.

In the present paper, by making use of results contained in [1] and [3], we show how to obtain a transitive permutation pair (σ, τ) representing an orientable 3-manifold M^3, starting from a 3-coloured diagram $\mathcal{D}$ representing M^3.

2. Main results

Let $\mathcal{D}$ be a 3-coloured diagram of a knot, draw in the oriented euclidean plane $\mathcal{E}^2$. Let $\mathcal{C} = \{c_1, \ldots, c_n\}$ denote the set of the $n \geq 3$ crossings of $\mathcal{D}$ and, for each $k \in Z_3 = \{0, 1, 2\}$, let $\alpha_1^k, \ldots, \alpha_{s_k}^k$ be the k-coloured arcs of $\mathcal{D}$, with $s_k \geq 1$ and $s_0 + s_1 + s_2 = n$. For each $k \in Z_3$ and each $r \in N_{s_k}$, we are going to define a cyclic permutation on the set $\mathcal{C} \times Z_3$, associated to the k-coloured arc α_r^k. Denote the endpoints of α_r^k by the corresponding crossings c_h, c_j of $\mathcal{D}$ and suppose that α_r^k has $t = t_r^k$ undercrossings (where t can eventually be zero). If S^1 denotes the standard circle and d is a diameter of S^1, consider an embedding $\psi_r^k : S^1 \cup d \longmapsto \mathcal{E}^2$ such that

a) $\psi_r^k(d) = \alpha_r^k$ (and hence $\psi_r^k(\delta d) = \{c_h, c_j\}$);

b) the intersection of $\psi_r^k(S^1 - \delta d)$ with the diagram $\mathcal{D}$ is given by $2t$ points, one for each arc incident with α_r^k.

If c_p is an undercrossing of α_r^k, denote by the same symbol c_p the intersection point of $\psi_r^k(S^1)$ with the $(k+1)$-coloured arc incident to α_r^k in c_p. Denote by $\mathcal{C}_r^k$ the set given by the union of these intersection points together with the endpoints c_h, c_j of α_r^k; hence, $\mathcal{C}_r^k \subset \psi_r^k(S^1)$ and $\operatorname{Card} \mathcal{C}_r^k = t + 2$. Finally, let $(c_{q_1}, \ldots, c_{q_{t+2}})$ be the ordered sequence obtained by reading the elements of $\mathcal{C}_r^k$ while walking along $\psi_r^k(S^1)$, coherently with the orientation induced on it by the fixed orientation of $\mathcal{E}^2$; the cyclic permutation σ_r^k on $\mathcal{C} \times Z_3$ is defined in the following way:

$$\sigma_r^k = \Big((c_{q_1}, k), \ldots, (c_{q_{t+2}}, k)\Big).$$

With these notations, we have the following result.

Main Theorem. *Let $\mathcal{C} = \{c_1, \ldots, c_n\}$ be the set of the n crossings of a 3-coloured diagram $\mathcal{D}$ of a knot and denote by s_k, for each $k \in Z_3$, the number of the k-coloured arcs of $\mathcal{D}$. If σ, τ are the permutations on $\mathcal{C} \times Z_3$ defined by*

$$\sigma = \prod_{k \in Z_3} \sigma_1^k \cdot \ldots \cdot \sigma_{s_k}^k,$$

$$\tau = \prod_{i \in N} \Big((c_1, 0), (c_1, 1), (c_1, 2)\Big),$$

then (σ, τ) is a transitive permutation pair such that $M(\sigma, \tau) = M(\mathcal{D})$.

The proof of this theorem, given in Section 3, makes use of the possibility of representing manifolds by means of edge-coloured graphs and is performed by joining two constructions, respectively described in [1] and [3].

Fig. 2

Fig. 2 contains a 3-coloured diagram representing $S^1 \times S^2$ (see [7], Fig. 21); the application of the above algorithm produces the following transitive permutation pair (σ, τ) on $N_9 \times Z_3$ such that $M(\sigma, \tau) = S^1 \times \times S^2$:

$$\sigma = \Big((c_1,0)(c_2,0)(c_7,0)(c_8,0)\Big)\Big((c_3,0)(c_9,0)\Big)\Big((c_4,0)(c_6,0)(c_5,0)\Big)\cdot$$
$$\cdot\Big((c_1,1)(c_2,1)(c_3,1)\Big)\Big((c_4,1)(c_8,1)(c_9,1)(c_5,1)\Big)\Big((c_6,1)(c_7,1)\Big)\cdot$$
$$\cdot\Big((c_1,2)(c_6,2)(c_5,2)\Big)\Big((c_2,2)(c_3,2)(c_4,2)\Big)\Big((c_7,2)(c_8,2)(c_9,2)\Big);$$
$$\tau = \prod_{i \in N_9} \Big((c_i,0)(c_i,1)(c_i,2)\Big).$$

Corollary. *Let M^3 be an orientable 3-manifold which is a simple 3-fold covering of S^3 branched over a knot $\mathcal{K}$ and suppose that a 3-coloured diagram $\mathcal{D}$ of $\mathcal{K}$ representing M^3 has n crossings. Then M^3 is also a $(3n)$-fold covering of S^3 branched over the graph G.*

Proof. The Main Theorem states that $M^3 = M(\mathcal{D})$ can be represented by a permutation pair (σ, τ) acting on $\mathcal{C} \times Z_3$. Since Card $(\mathcal{C} \times Z_3) = 3n$, M^3 is determined by a monodromy map $\delta : \prod_1(S^3 - G) \longmapsto \sum_{3n}$ and hence M^3 is a $(3n)$-fold covering of S^3 branched over G. $\Diamond$

3. The proof

Let us recall some notations and results about the manifold representation theory by means of edge-coloured graphs; for a general survey on it, see [2] or [9].

An $(m+1)$-*coloured graph* is a pair (Γ, γ), where $\Gamma = (V(\Gamma), E(\Gamma))$ is a finite regular multigraph of degree $m+1$ and $\gamma : E(\Gamma) \longmapsto Z_{m+1} = \{0, 1, \ldots, m\}$ is a map such that $\gamma(e) \neq \gamma(f)$, for each pair e, f of adjacent edges. For every $\mathfrak{F} \subset Z_{m+1}$, set $\Gamma_{\mathfrak{F}} = (V(\Gamma), \gamma^{-1}(\mathfrak{F}))$; each connected component of $\Gamma_{\mathfrak{F}}$ is often called an $\mathfrak{F}$-*residue*. Note that, for every distinct $i, j \in Z_{m+1}$, the $\{i, j\}$-residues of (Γ, γ) are cycles alternatively coloured by i and j.

An m-dimensional ball complex $K(\Gamma)$, triangulating an m-pesudo-manifold, can be associated to a given $(m+1)$-coloured graph (Γ, γ) by the following rules:

— take an m-simplex $\alpha(v)$ for each $v \in V(\Gamma)$ and label its vertices by Z_{m+1};

— if $v, w \in V(\Gamma)$ are joined by a k-coloured edge, identify the $(m-1)$-faces of $\alpha(v)$ and $\alpha(w)$ opposite to the vertices labelled by k, so that equally labelled vertices are identified together.

Even if its balls are simplexes, the resulting complex $K(\Gamma)$ is not in general a simplicial one, since the intersection of two simplexes may be the union of more than one maximal face; nevertheless, it is a *pseudocomplex* ([2], p. 122). The graph (Γ, γ) is said to *represent* $K(\Gamma)$, $|K(\Gamma)|$ and every homomorphic space. Note that every h-simplex of $K(\Gamma)$, whose vertices are labelled by the distinct colours $k_0, \ldots, k_h$, corresponds to a unique $(Z_{m+1} - \{k_0, \ldots, k_h\})$-residue of (Γ, γ) and viceversa.

Every m-manifold M^m is representable by $(m+1)$-coloured graphs ([8]). *If (Γ, γ) represents M^m, then M^m is orientable if and only if Γ is bipartite.*

In [3], the following method is described for producing, starting from a 4-coloured graph (Γ, γ) representing an orientable 3-manifold M^3, a transitive permutation pair (σ, τ) such that $M(\sigma, \tau) = M^3$. Set $\mathrm{Card}(V(\Gamma)) = 2n$. Let V', V'' be the two bipartition classes of $V(\Gamma)$ and identify V' with N_n. If $v \in V'$ and $k \in Z_3$, denote by $\overline{\sigma}_k(v)$ the vertex of V'' which is k-adjacent to v and denote by $\sigma_k(v)$ the vertex of V' which is 3-adjacent to $\overline{\sigma}_k(v)$. With these notations, define the two permutations on $N_n \times Z_3$ in the following way:

$\tau = \prod_{v \in N_n} \Big((v,0),(v,1),(v,2)\Big)$; $\forall v \in N_n,\ \forall k \in Z_3,\ \sigma(v,k) = (\sigma_k(v),k)$.

Roughly speaking, the permutation σ is the product of the disjoint cycles obtained by "reading" the vertices of V' in a suitable orientation of the $\{k,3\}$-residues of (Γ,γ). The pair (σ,τ) is said to be *associated* to (Γ,γ)

Proposition 1. [3] *Let (Γ,γ) be a 4-coloured graph representing the orientable 3-manifold M^3 and let (σ,τ) be the pair of permutations on $N_n \times Z_3$ $(2n = \mathrm{Card}\, V(\Gamma))$ associated to (Γ,γ). Then $M(\sigma,\tau) = M^3$.*

On the other hand, the following algorithm, given in [1], produces a 4-coloured graph representing the orientable 3-manifold M^3 described by a given 3-coloured diagram $\mathcal{D}$ of a knot, draw in the oriented $\mathcal{E}^2$. Suppose, with the same notations of Section 2, that $\mathcal{C} = \{c_1,\dots,c_n\}$ is the set of the $n \geq 3$ crossings of $\mathcal{D}$ and, for each $k \in Z_3$, let $\alpha_1^k,\dots,\alpha_{s_k}^k$ be the k-coloured arcs of $\mathcal{D}$. For every $r \in N_{s_0}$ (resp. $r \in N_{s_1}$), thicken the arc α_r^0 (resp. α_r^1) to a "strip" $\mathcal{R}_r^0$ (resp. $\mathcal{R}_r^1$), so that all these strips have disjoint interiors and, if the arcs $\alpha_{r'}^0$, $\alpha_{r''}^1$ are incident to the same crossing c_i, then $\delta\mathcal{R}_{r'}^0$ and $\delta\mathcal{R}_{r''}^1$ have a common edge, denoted by e_i. For every $i \in N_n$, denote by v_i and v_i' the endpoints of the edge e_i, so that v_i precedes v_i' while walking along $\delta\mathcal{R}_r^0$ $(r \in N_{s_0})$, coherently with the fixed orientation of $\mathcal{E}^2$. Set $E_3 = \bigcup_{i \in N_n} \{e_i\}$ and denote by E_0 (resp. E_1) the set of the edges of $\bigcup_{r \in N_{s_0}} (\delta\mathcal{R}_r^0)$ (resp. $\bigcup_{r \in N_{s_1}} (\delta\mathcal{R}_r^1)$) not belonging to E_3. For every $i \in N_n$, consider the 2-coloured arc $\alpha_{r(i)}^2$ incident to the crossing c_i; then, draw an edge b_i between v_i' and $v_{j(i)}$ if and only if, while walking around $\alpha_{r(i)}^2$ coherently with the fixed orientation of $\mathcal{E}^2$, starting from the 0-coloured arc incident in c_i, the first incident 0-coloured arc is incident in $c_{j(i)}$. Set $E_2 = \bigcup_{i \in N_n} \{b_i\}$.

Let $(\Gamma,\gamma) = \Gamma(\mathcal{D})$ denote the 4-coloured graph defined by:

$$V(\Gamma) = \bigcup_{i \in N_n} \{v_i, v_i'\};$$

$$E(\Gamma) = \bigcup_{k \in Z_4} E_k, \text{ where each edge of } E_k \text{ is coloured by } k.$$

Proposition 2. [1] *For every 3-coloured diagram $\mathcal{D}$ of a knot, the 4-coloured graph $\Gamma(\mathcal{D})$ represents $M(\mathcal{D})$.*

Proof of the Main Theorem. Let $(\Gamma,\gamma) = \Gamma(\mathcal{D})$ be the 4-coloured graph associated to $\mathcal{D}$, so that $|K(\Gamma)| = M(\mathcal{D})$. Orient each $\{0,3\}$-residue of (Γ,γ) coherently with the fixed orientation of $\mathcal{E}^2$ and denote by V' the bipartition class of $V(\Gamma)$ containing the second vertex of each 3-coloured edge e_i. Note that each crossing c_i of $\mathcal{D}$ corresponds to the 3-coloured edge e_i of (Γ,γ); hence, we can identify the sets $\mathcal{C}$ and V' with N_n. By construction, for each $k \in Z_3$, the $\{k,3\}$-residues of (Γ,γ) are in one-to-one correspondence with the k-coloured arcs $\alpha_1^k,\ldots,\alpha_{s_k}^k$ of $\mathcal{D}$; denote them by $\mathcal{R}_1^k,\ldots,\mathcal{R}_{s_k}^k$ respectively. Moreover, for each $k \in Z_3$ and $r \in N_{s_k}$, there exists an orientation-preserving homeomorphism between $\psi_r^k(S^1)$ and $\mathcal{R}_r^k$ sending each point c_p of $\mathcal{C}_r^k$ to the vertex of the 3-coloured edge e_p belonging to V'. This proves that the permutation pair (σ,τ), acting on the set $\mathcal{C} \times Z_3 \simeq N_n \times Z_3$, defined in Section 2, is precisely the transitive permutation pair, acting on $V' \times Z_3 \simeq N_n \times Z_3$, associated to the 4-coloured graph (Γ,γ). Hence, we have $M(\sigma,\tau) =$ $= |K(\Gamma)| = M(\mathcal{D})$. $\diamondsuit$

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