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ON THE AREA SUM OF A CONVEX SET AND ITS POLAR RECIPROCAL

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Abstract: In the Euclidean plane let C be a closed convex set contained in the closed unit circle K, and let $C^\star$ be the polar reciprocal of C with respect to K. In a preceding paper [1] it was proved that the area sum of C and $C^\star$ is greater than or equal to 6. In this paper we show that equality occurs only if C is a square inscribed in K.

Let K be the unit circle centred at the origin O. The polar reciprocal $C^\star$ of a plane convex set C with respect to K is defined as the set of all points x with

$$< x, y > \leq 1$$

for every $y \in C$. We denote the area of a set M by $a(M)$. The subject of this paper is the proof of the following theorem.

Theorem. *Let C be a closed convex set contained in the unit circle K, and let $C^\star$ be the polar reciprocal of C with respect to K. Then*

$$S(C) \equiv a(C) + a(C^\star) \geq 6 \tag{1}$$

with equality if and only if C is a square inscribed in K.

In [1], the theorem was proved in the case when C is a convex polygon. This result immediately implies that inequality (1) is satisfied

On page 78 line 25 of the paper [1] there is a typo. The correct version appears in (3) of this paper.

for any closed convex set C contained in K. However, the question of equality remained open in this case and will be answered in the present paper.

Proof of the theorem. Let us recall two points, partially in extended form, of the proof of the theorem established in [1].

Let $P = A_1 A_2 \dots A_n$ be a convex polygon contained in K such that O is an interior point of P and A_1 an interior point of K. Then A_1 can be moved to a new position A_1' satisfying the following conditions:

(i) The polygon $P' = A_1' A_2 \dots A_n$ is convex and contains O in the interior,

(ii) the vertex A_1' is either on the boundary of K, or A_1' is in the interior of K and at least one of the triples (A_{n-1}, A_n, A_1') and (A_1', A_2, A_3) is collinear,

(iii) $S(P) = a(P) + a(P^\star) > S(P')$,

(iv) $a(P) \leq a(P')$.

(A similar procedure was used in the proof of Satz 1 and Satz 2 in [3]). The vertices of P on the boundary of K are not moved.

By (ii), the interior of K contains fewer vertices of P' than vertices of P. Repeated application of the process described leads to a convex polygon $\overline{P}$ inscribed in K, containing O in the interior and satisfying

$$S(P) > S(\overline{P}). \tag{2}$$

More generally, let D be a closed convex set such that $P \subset D \subset K$, and let us assume that some vertex of P is an interior point of D. Then there exists a convex polygon $\overline{P}$ inscribed in D, containing O in the interior and satisfying (2).

We shall refer to the transition from P to $\overline{P}$ by saying that $\overline{P}$ is obtained from P by translation of vertices.

Let $\overline{P}$ be a convex polygon inscribed in K and containing O in its interior. We denote the central angles spanned by the sides of $\overline{P}$ by $2x_1, \dots, 2x_n$, where $0 < x_j < \pi/2$, for $j = 1, \dots n$, and $x_1 + \dots + x_n = \pi$. Let us assume that $x_1 \leq x_2 < x_0$, where the constant x_0 is defined by

$$x_0 = \arccos(1/\sqrt[4]{2}) = 32.765\dots^\circ \tag{3}$$

(see [1]). We replace x_1 and x_2 by x_1' and x_2' such that

$$0 \leq x_1' < x_1 \leq x_2 < x_2' \leq x_0\,,$$
$$x_1' + x_2' = x_1 + x_2$$

and $x_1' = 0$, or $x_2' = x_0$, or both. The polygon $\overline{P}'$ inscribed in K

and determined by the central angles $2x_1', 2x_2', 2x_3, \dots, 2x_n$ of its sides satisfies

$$S(\overline{P}) > S(\overline{P}')$$

(see [1]). We shall refer to the (possibly repeated) application of this process as reduction. If the polygon Q inscribed in K is obtained from $\overline{P}$ by reduction, then O is an interior point of Q and

$$S(\overline{P}) \geq S(Q). \tag{4}$$

Let us now proceed to the proof of the theorem. Since inequality (1) was proved in [1], it is sufficient to show that a closed convex set C contained in K and satisfying

$$a(C) + a(C^*) = 6 \tag{5}$$

is a square inscribed in K. Note that such a set C necessarily contains O in its interior. By the corollary in [1], at least one point of C, say A, is on the boundary of K. If B, A', B' are the other vertices of a square $ABA'B'$ inscribed in K, we have to show that

$$C = ABA'B'. \tag{6}$$

The proof of (6) consists of four parts.

(a) Let U be a point of the boundary of K other than A, B, A', B'. Then U is outside C.
(b) The point A' belongs to C.
(c) The points B and B' belong to C.
(d) The segments AB, BA', $A'B'$ and $B'A$ are parts of the boundary of C.

We shall prove these statements by showing that $S(C) > 6$ if C fails to satisfy one of them. To avoid tiresome repetitions we remark that the origin O is an interior point of each convex set appearing in this paper.

(a) Suppose that $U \in C$. We can find a sequence (P_k) of convex polygons inscribed in C and convergent to C such that each P_k contains A and U. Hence

$$\lim_{k\to\infty} S(P_k) = S(C). \tag{7}$$

By translation of vertices we obtain from P_k a convex polygon $\overline{P}_k$ inscribed in K and containing A and U. By (2), we have

$$S(P_k) \geq S(\overline{P}_k), \tag{8}$$

for $k = 1, 2, \dots$. We denote the two arcs on the boundary of K with endpoints A and U by b_1 and b_2. The vertices of $\overline{P}_k$ divide b_1 and b_2 into subarcs of lengths $2x_1, \dots, 2x_n$ and $2y_1, \dots, 2y_m$, respectively. By reduction applied to the set $\{x_1, \dots, x_n\}$ we obtain a set $\{x_1', \dots, x_r'\}$, where

$$0 < x_j' < \frac{\pi}{2} \qquad (j = 1, \dots, r),$$

$$x_1' + \dots + x_r' = x_1 + \dots + x_n$$

and at most one of the x_j' is contained in $(0, x_0)$. A similar procedure applied to $\{y_1, \dots, y_m\}$ yields a set $\{y_1', \dots, y_s'\}$, where

$$0 < y_i' < \frac{\pi}{2} \qquad (i = 1, \dots, s),$$

$$y_1' + \dots + y_s' = y_1 + \dots + y_m$$

and at most one of the y_i' is in $(0, x_0)$. Since

$$x_1' + \dots + x_r' + y_1' + \dots + y_s' = \pi,$$

$2x_1', \dots, 2x_r', 2y_1', \dots, 2y_s'$ are the central angles of a convex polygon Q_k inscribed in K and containing A and U. By (4), we have

$$S(\overline{P}_k) \geq S(Q_k), \tag{9}$$

for $k = 1, 2, \dots$. Because no more than two of the x_j' and y_i' are less than x_0 and $x_0 > \pi/6$, we conclude that $r + s \leq 7$, so that Q_k is a polygon with at most seven sides, in short a heptagon. Observe that $Q_1, Q_2, \dots$ have a fixed circle about O in common. Otherwise the sequence $(S(Q_k))$ would be unbounded, which is impossible by (7), (8) and (9). From the sequence (Q_k) we can select a subsequence, again denoted by (Q_k), which is convergent to a heptagon Q inscribed in K. Since Q contains A and U, Q is not a square, so that by the theorem in [1]

$$S(Q) > 6. \tag{10}$$

The desired result $S(C) > 6$ is a consequence of (7) to (10) and

$$\lim_{k \to \infty} S(Q_k) = S(Q). \tag{11}$$

(b) Suppose that $A' \notin C$. Then C can be separated from A' by a support line g parallel to $B \vee B'$. Since O is an interior point of C, g intersects the boundary of K in two points U and U', where U is between B and A', and U' between A' and B'. By the result in (a), U and U' are not in C. Thus, there is a point $X \in C \cap g$ other than U and U'.

We can find a sequence (P_k) of convex polygons inscribed in C and convergent to C such that each P_k contains A and X. Hence relation (7) holds. Let D be the intersection of K with the closed halfplane bounded by g and containing C. By translation of vertices we get from P_k a convex polygon $\overline{P}_k$ inscribed in D, containing A and X and satisfying (8), for $k = 1, 2, \ldots$.

Let $A, \ldots, V$ be the vertices of $\overline{P}_k$ on the arc AU, and $A, \ldots, V'$ the vertices of $\overline{P}_k$ on the arc AU', and let W, W' be the vertices of $\overline{P}_k$ on the segment UU'. Note that possibly $V = A$ or $V = U$ and that possibly $W = U$ or $W = X$. Similarly, as described in (a), we apply reduction to the arcs AV and AV' and obtain altogether no more than seven arcs on the boundary of K. The chords of these arcs, together with the segments VW, WW' and $W'V'$, form the boundary of a convex polygon Q_k with at most ten sides, in short a decagon. The polygon Q_k inscribed in D, contains A and X and satisfies (9), for $k = 1, 2, \ldots$. We assume, as we may, that the sequence (Q_k) is convergent to a set Q. Clearly, Q is a decagon inscribed in D and contains A and X. There are two possible cases: (i) Either some vertex of Q on g is different from U and U', or (ii) U and U' are vertices of Q, so that Q is inscribed in K and is not a square. In both cases, the theorem in [1] implies inequality (10). The conclusion that $S(C) > 6$ is the same as in (a).

(c) Suppose that $B \notin C$. Then C can be separated from B by a support line g parallel to $A \vee A'$. Since O is an interior point of C, g intersects the boundary of K in two points U and U', where U is between A and B, and U' between B and A'. By the result in (a), the points U and U' are not in C. Thus there is a point $X \in C \cap g$ other than U and U'. Now the proof proceeds exactly as in (b), so that we need not give the details. In conclusion, we can state that the assumption $B \notin C$ or $B' \notin C$ implies that $S(C) > 6$.

(d) Suppose that the segment AB is not part of the boundary of C. There is a support line g of C which is parallel to $A \vee B$ and intersects the boundary of K in two points U and U' between A and B. Since U and U' are not in C, a point $X \in C \cap g$ is different from U and U'. Repeating the arguments used in the proof of part (b), we come to the conclusion that $S(C) > 6$, as required.

This completes the proof of (6) and the theorem. $\diamondsuit$

A stability problem. Our theorem suggests to consider the following problem. (For a detailed discussion of stability of geometric inequalities see the review paper [2] by H. Groemer): If for some closed convex set C contained in K the left-hand side of inequality (1) is not very different from 6, what can be said about the deviation of this set from the squares inscribed in K? A real-valued function $\theta(x)$ that is defined on $[0, \infty)$ is called a *stability function* if

$$\theta(x) > 0 \quad \text{for} \quad x \neq 0$$

and

$$\lim_{x \to 0+} \theta(x) = \theta(0) = 0.$$

Let ρ be the Hausdorff metric or an equivalent metric defined on the class of all compact convex subsets of the plane with non-empty interior. The stability problem associated with inequality (1) consists of finding a stability function θ such that for any $\varepsilon \geq 0$ the condition

$$S(C) \leq 6 + \varepsilon \tag{12}$$

implies the existence of a square Q_0 inscribed in K such that

$$\rho(C, Q_0) \leq \theta(\varepsilon). \tag{13}$$

Such a function exists; e.g.,

$$\sup(\inf \rho(C, Q)),$$

defined for $x \geq 0$, has the required property. Here the infimum is to be taken over all squares Q inscribed in K, and the supremum extends over all closed convex sets $C \subset K$ with $S(C) \leq 6 + x$.

Can an explicit function θ be given in a way that (13) follows from (12)?

References

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[3] MAHLER, K.: Ein Minimalproblem für konvexe Polygone, *Mathematica (Zutphen)* **B 7** (1938-39), 118–127.