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A SANDWICH WITH CONVEXITY FOR SET–VALUED FUNCTIONS

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Abstract: We present necessary and sufficient conditions under which for given set-valued functions F and G there exists a convex set-valued function H such that $F(x) \subset H(x) \subset G(x)$, $x \in D$. Some applications of these results are also given.

1. Introduction

In this note we give conditions under which for given set-valued functions F, G defined on a convex set D and satisfying $F(x) \subset G(x)$, $x \in D$, there exists a convex set-valued function H such that $F(x) \subset H(x) \subset G(x)$, $x \in D$. This problem leads us to the following condition:

(1) $\quad tF(x) + (1-t)F(y) \subset G(tx + (1-t)y), \quad x, y \in D, t \in [0,1].$

It generalizes some conditions defining known classes of set-valued functions. For instance, a set-valued function F, defined on a convex set, is said to be *convex* (*K-convex, ϵ-convex, hull-convex*) if it satisfies (1) for all $x, y \in D$ and $t \in [0,1]$ with G defined by $G(x) = F(x)$ ($G(x) = F(x) + K$, $G(x) = F(x) + (-\epsilon, \epsilon)$, $G(x) = \operatorname{conv} F(x)$, respectively).

Given a set Y we denote by $\mathrm{n}(Y)$ the family of all non-empty subsets of Y. By the graph of a set-valued function $F : D \to \mathrm{n}(Y)$ we mean the set

$$\operatorname{Gr} F := \{(x,y) \in D \times Y : y \in F(x)\}.$$

It is known that $F : D \to \mathrm{n}(Y)$ is convex if and only if its graph is a convex subset of $D \times Y$.

2. Sandwich theorems

We start with the following result.

Theorem 1. *Let I be a real interval and $F, G : I \to \mathrm{n}(\mathbb{R})$ be given set-valued functions such that $\operatorname{Gr} F$ is the union of two connected subsets of $\mathbb{R}^2$. Then F and G satisfy (1) for all $x, y \in I$ and $t \in [0,1]$, if and only if there exists a convex set-valued function $H : I \to \mathrm{n}(\mathbb{R})$ such that*

$$F(x) \subset H(x) \subset G(x), x \in I. \tag{2}$$

Proof. Assume that F and G satisfy (1) and consider the set-valued function $H : I \to \mathrm{n}(\mathbb{R})$ defined by

$$H(x) := \{y \in \mathbb{R} : (x,y) \in \operatorname{conv} \operatorname{Gr} F\}.$$

It is easy to verify $F(x) \subset H(x), x \in I$. Indeed, if $y \in F(x)$, then $(x,y) \in \operatorname{Gr} F \subset \operatorname{conv} \operatorname{Gr} F$, which means that $y \in H(x)$. Moreover, H is a convex set-valued function because $\operatorname{Gr} H = \operatorname{conv} \operatorname{Gr} F$ is a convex subset of $\mathbb{R}^2$. To prove that $H(x) \subset G(x)$, $x \in I$, fix an $x \in I$ and take $y \in H(x)$. Then $(x,y) \in \operatorname{conv} \operatorname{Gr} F$. Since $\operatorname{Gr} F$ is the union of two connected subsets of $\mathbb{R}^2$, each element of its convex hull is a convex combination of two elements of $\operatorname{Gr} F$ (cf. p.169, Prop. 3.3]). Therefore there exist $(x_1,y_1),(x_2,y_2) \in \operatorname{Gr} F$ and a $t \in [0,1]$ such that $(x,y) = t(x_1,y_1) + (1-t)(x_2,y_2)$. Hence, using (1), we get

$$y = ty_1 + (1-t)y_2 \in tF(x_1) + (1-t)F(x_2) \subset$$
$$\subset G(t_1x_1 + (1-t)x_2) = G(x),$$

which shows that $H(x) \subset G(x)$. The converse implication is clear (and the condition of connectness is not needed here). $\Diamond$

Remark 1. Recently K. Baron, J. Matkowski and K. Nikodem [1] proved that real functions f, g defined on a real interval I, satisfy

$$f(tx + (1-t)y) \le tg(x) + (1-t)g(y), \quad x,y \in I, t \in [0,1],$$

if and only if there exists a convex function $h : I \to \mathbb{R}$ such that

$$f(x) \le h(x) \le g(x), \quad x \in I.$$

Th. 1 is a set-valued analogue of this result. It can be also obtained by use of a remark on separation of sets on the plane given by Zs. Páles (cf. [7, p. 296, Remark 23]). The following examples show that the assumptions that $\operatorname{Gr} F$ is the union of two connected sets as well as that F and G are defined on a real interval and have values in $\mathbb{R}$ are essential.

Example 1. Let us take the set-valued functions $F, G : [0,1] \to \mathrm{n}\,([0,1])$ defined by

$$F(x) = \begin{cases} \{0,1\}, & x \in \{0,1\} \\ \{1\}, & x \in (0,1)\ . \end{cases}$$

$$G(x) = \begin{cases} \{0\} \cup [x,1], & x \in \left[0, \frac{1}{2}\right] \\ \{0\} \cup [1-x,1], & x \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

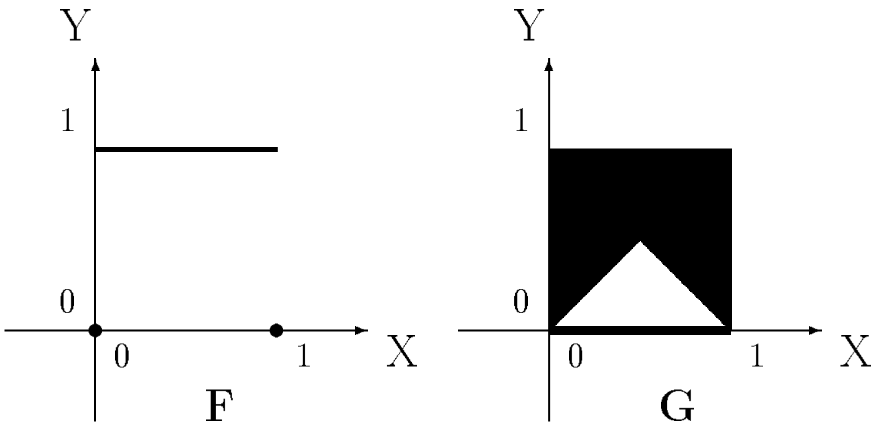

Fig. 1

It is easy to see that F and G satisfy (1) but there is not any convex set-valued function $H : [0,1] \to \mathrm{n}\,(\mathbb{R})$ satisfying (2). Clearly, $\mathrm{Gr}\, F$ can not be represented as the union of two connected sets.

Example 2. Consider the set-valued functions $F, G : [0,1] \times [0,1] \to$ $\to \mathrm{n}\,(\mathbb{R})$ defined by

$$F(x_1,x_2) := \begin{cases} [0,1], & (x_1,x_2) \in [0,1] \times (0,1] \\ \{0,1\}, & (x_1,x_2) \in \{0,1\} \times \{0\} \\ \{1\}, & (x_1,x_2) \in (0,1) \times \{0\} \end{cases}$$

$$G(x_1,x_2) := \begin{cases} [0,1], & (x_1,x_2) \in [0,1] \times (0,1] \\ \{0\} \cup [x_1,1], & (x_1,x_2) \in \left[0, \frac{1}{2}\right] \times \{0\} \\ \{0\} \cup [1-x_1,1], & (x_1,x_2) \in \left(\frac{1}{2}, 1\right] \times \{0\}. \end{cases}$$

These set-valued functions satisfy (1) and the graph of F is connected. However there is no convex set-valued function $H : [0,1] \times [0,1] \to \mathrm{n}\,(\mathbb{R})$

satisfying (2).

Example 3. Let $F, G : [0,1] \to \mathrm{n}\left(\mathbb{R}^2\right)$ be defined by the formulas

$$F(x) := \begin{cases} \{0\} \times \{0,1\} \cup (0,1] \times [0,1], & x \in \{0,1\} \\ \{0\} \times \{1\} \cup (0,1] \times [0,1], & x \in (0,1) \end{cases}$$

$$G(x) := \begin{cases} \{0\} \times (\{0\} \cup [x,1]) \cup (0,1] \times [0,1], & x \in \left[0, \frac{1}{2}\right] \\ \{0\} \times (\{0\} \cup [1-x,1]) \cup (0,1] \times [0,1], & x \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Similarly as in the previous example F and G satisfy (1) and $\mathrm{Gr}\, F$ is connected. However, there does not exist any convex set-valued function $H : [0,1] \to n(\mathbb{R}^2)$ for which (2) holds.

If a set $A \subset \mathbb{R}^n$ is the union of n connected sets, then each element of its convex hull is a convex combination of n or fewer points of A (cf. [p. 169, Prop. 3.3]). It is also known that every convex set-valued function $H : D \to \mathrm{n}(Y)$, where D is a convex subset of a vector space and Y is a vector space, satisfies

$$t_1 H(x_1) + \cdots + t_n H(x_n) \subset H(t_1 + \cdots + t_n x_n)$$

for all $n \in \mathbb{N}, x_1, \ldots, x_n \in D$ and $t_1, \ldots, t_n \in [0,1]$ summing up to 1 ([4, Th. 2.3]). Using these facts and arguing as in the proof of Th. 1 we get the following extension of this theorem.

Theorem 1.1. *Let D be a convex subset of $\mathbb{R}^k$ and $F, G : D \to \mathrm{n}(\mathbb{R}^l)$ be given set-valued functions such that $\mathrm{Gr}\, F$ is the union of $k+l$ connected subsets of $\mathbb{R}^{k+l}$. Then F and G satisfy*

$$\sum_{i=1}^{k+l} t_i F(x_i) \subset G\left(\sum_{i=1}^{k+l} t_i x_i\right) \tag{3}$$

for every $x_1, \ldots, x_{k+l} \in D$ and for every $t_1, \ldots, t_{k+l} \in [0,1]$ summing up to 1 if and only if there exists a convex set-valued function $H : D \to$ $\to \mathrm{n}(\mathbb{R}^l)$ satisfying (2) for all $x \in D$.

Remark 2. According to the Carathéodory theorem (cf. [6, Theorems 1.20 and 1.21]) every element of the convex hull of a set $A \subset \mathbb{R}^n$ is a convex combination of $n+1$ (or fewer) elements of A. Therefore we can omit in Th. 1.1 the assumption that $\mathrm{Gr}\, F$ is the union of $k+l$ connected sets, taking in (3) all convex combination of $k+l+1$ elements of D.

Using a similar method as in the proof of Th. 1 we can obtain also the following result.

Theorem 1.2. *Let X, Y be real vector spaces and D be a convex subset of X. set-valued functions $F, G : D \to \mathrm{n}(Y)$ satisfy*

$$\sum_{i=1}^{n} t_i F(x_i) \subset G\left(\sum_{i=1}^{n} t_i x_i\right) \tag{4}$$

for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in D$ *and* $t_1, \ldots, t_n \in [0,1]$ *summing up to* 1, *if and only if there exists a convex set-valued function* $H : D \to \mathrm{n}(Y)$ *satisfying* (2) *for all* $x \in D$.

3. Applications

Let ϵ be a positive constant. Recall that set-valued function $F :$ $: I \to \mathrm{n}(\mathbb{R})$ is said to be *ϵ-convex* if

$$tF(x) + (1-t)F(y) \subset F(tx + (1-t)y) + (-\epsilon, \epsilon)$$

for all $x, y \in I, t \in [0,1]$. As an immediate consequence of Th. 1 (taking $G(x) = F(x) + (-\epsilon, \epsilon)$) we get the following Hyers-Ulam stability-type result. Similar corollaries we can obtain by Theorems 1.1 and 1.2 (cf. [2, Th. 2]).

Corollary 1. *If a set-valued function* $F : I \to \mathrm{n}(\mathbb{R})$ *is ϵ-convex and* $\mathrm{Gr}\, F$ *is the union of two connected sets, then there exists a convex set-valued function* $H : I \to \mathrm{n}(\mathbb{R})$ *such that*

$$F(x) \subset H(x) \subset F(x) + (-\epsilon, \epsilon), x \in I.$$

Now, denote by J either $[0, +\infty)$ or $(0, +\infty)$. Given $T > 0$ and $F : J \to \mathrm{n}(\mathbb{R})$ we define the set-valued function $F_T : J \to \mathbb{R}$ by the formula

$$F_T(x) = T^{-1} F(Tx).$$

Using a similar method as in [1] we get the following result.

Theorem 2. *Let T be a positive real number and* $F : J \to \mathrm{n}(\mathbb{R})$ *be a set-valued function such that* $\mathrm{Gr}\, F$ *is union of two connected sets. Then F satisfies*

$$tF(x) + (T-t)F(y) \subset F(tx + (T-t)y)$$

for all $x, y \in J, t \in [0, T]$ *if and only if there exists a convex set-valued function* $\Phi : J \to \mathrm{n}(\mathbb{R})$ *such that*

$$\Phi(x) \subset F(x) \subset \Phi_T(x), x \in J.$$

Proof. Assume that F satisfies

$$tF(x) + (T-t)F(y) \subset F(tx + (T-t)y), \quad x, y \in J, t \in [0, T].$$

Taking $t = \alpha T, \alpha \in [0,1]$, and multiplying by T^{-1}, we receive the inclusion

$$\alpha F(x) + (1-\alpha)F(y) \subset T^{-1} F(T\alpha x + T(1-\alpha)y)$$

for all $x, y \in J$ and $\alpha \in [0,1]$. According to Th. 1, there exists a convex set-valued function $H : J \to \mathrm{n}(\mathbb{R})$ such that $F(x) \subset H(x) \subset F_T(x)$. So let us define the function

$$\Phi(x) := TH(T^{-1}x), x \in J.$$

Because of the convexity of H, Φ is convex. It is also easy to check that the wanted condition holds.

On the other hand, if there exists a convex set-valued function $\Phi : J \to \mathrm{n}(\mathbb{R})$ such that

$$\Phi(x) \subset F(x) \subset \Phi_T(x), x \in J.$$

we can get

$$\alpha F(x) + (1-\alpha)F(y) \subset T^{-1}\Phi(\alpha Tx + (1-\alpha)Ty)$$

for all $x, y \in J$ and $\alpha \in [0,1]$. And finally, taking $t := T\alpha$ and using the inclusion $\Phi(x) \subset F(x)$ we receive

$$tF(x) + (T-t)F(y) \subset F(tx + (T-t)y)$$

for all $x, y \in J, t \in [0,T]$. $\Diamond$

Let A be a subset of a real vector space X. We say that a point x_0 belongs to the algebraic interior of A (and write $x_0 \in \mathrm{core}\, A$) if for every $x \in X$ there exists an $\epsilon > 0$ such that $tx + (1-t)x_0 \in A$ for all $t \in (-\epsilon, \epsilon)$.

In the next theorem we show that if set-valued functions $F, G : D \to \mathrm{n}(Y)$ satisfy (4) and at a point $x_0 \in \mathrm{core}\, D$ the set $G(x_0)$ is a singleton, then F has to be a single-valued affine function (i.e. $F(tx + (1-t)y) = tF(x) + (1-t)F(y)$ for all $x, y \in D, t \in [0,1]$). An analogous result for convex set-valued functions defined on the whole vector space was recently obtained by F. Deutsch and I. Singer [3] (cf. also [5 Th. 3]).

Theorem 3. *Let X, Y be real vector spaces, D be a convex subset of X and $F, G : D \to \mathrm{n}(Y)$ be set-valued functions such that $G(x_0)$ is a singleton for some $x_0 \in \mathrm{core}\, D$. Then F and G satisfy* (4) *if and only if F is a single-valued affine selection of G.*

Proof. The sufficiency is clear. Now, assume that F and G satisfy (4). By Th. 1.2 there exists a convex set-valued function $H : D \to \mathrm{n}(Y)$ such that (2) holds. Fix a point $x \in D$.Since $x_0 \in \mathrm{core}\, D$, there exist a $y \in D$ and a $t \in (0,1)$ such that $x_0 = tx + (1-t)y$. By the convexity of H and (2) we get

$$tH(x) + (1-t)H(y) \subset H(x_0) \subset G(x_0),$$

which implies that $H(x)$ is a singleton. Thus H as a single-valued function satisfying the condition $tH(x) + (1-t)H(y) \subset H(tx + (1-t)y)$, $x, y \in D$, $t \in [0,1]$, is affine. By (2) also F is single-valued and it is an affine selection of G. $\Diamond$

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