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ON THE HOMOGENEOUS IDEAL OF UNREDUCED PROJECTIVE SCHEMES

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Abstract: Here we give upper bounds for the degrees of generators of the homogeneous ideal of a suitable unreduced scheme $Z \subset \boldsymbol{P}^n$. Here we take as Z a disjoint union of fat points or of multiple structures on linear subspaces or we take as Z_{red} a subscheme of a smooth curve or a scroll.

The actors in this paper are unreduced subschemes (or "fattening") of $\boldsymbol{P}^n$ with connected components which are fattening of very simple building blocks: points or rational normal curves or linear spaces. For these subschemes of $\boldsymbol{P}^n$ in this paper we consider the postulation (i.e. the Hilbert function), the degree of the generators of the homogeneous ideal and of higher syzygies. Most of the methods used

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are refinements of [5] (with [8] and [13] very essential here and for [5]).

Section 0 contains basic notations, background material and a few conventions which we will use. Section 1 contains the general set up needed to handle the minimal free resolution of fat points with support contained in a fixed subscheme of $\boldsymbol{P}^n$. In Section 2 we consider as 0-dimensional subschemes the curvilinear ones (i.e. the one with Zariski tangent space of dimension ≤ 1 at each point of their support) and some fattening of them; these unreduced schemes are the easiest ones to handle and obtain, say, bounds on their postulation. This section depends heavily on the proofs in [8] and [13]. Here we stress also the interest of fattening of higher dimensional disconnected schemes.

In Section 3 we prove upper bounds for the regularity index of fattening of disjoint union of certain positive dimensional subvarieties of $\boldsymbol{P}^n$, i.e. linear subspaces (see Th. 0.1 stated below), lines (see Th. 3.1) and rational normal curves (see Th. 3.2).

Then in Section 4 we consider the case in which the support of the fat points is on a rational normal curve and (very briefly) the case in which it is on an elliptic linearly normal curve (see 4.9 and Prop. 4.10). In 4.11 and Prop. 4.12 we consider briefly the case of fat points with support on a rational norinal scroll (which may be singular, i.e. a cone over a lower dimensional rational normal scroll).

In Section 5 we give a very general result on the behaviour of the cohomology of unreduced 0-dimensional subschemes of $\boldsymbol{P}^n$ with support on a fixed curve.

Just to give to the reader a feeling of the results proven in this paper, now we give the statement of the following theorem which will be proved in Section 3; in the body of the paper the reader will find more details on the notions appearing in its statement.

Theorem 0.1. *Fix s disjoint proper linear subspaces A_i $1 \leq i \leq s$, of $\boldsymbol{P}^n$ of any dimension, linear subspaces (of any dimension) M_i, $1 \leq i \leq s$ with $A_i \subseteq M_i$ for every i, integers m_i, $1 \leq i \leq s$ with $m_i > 0$ and let U be the union of the $(m_i - 1)$-th infinitesimal neighborhoods of A_i in M_i. Set $m := \max\{m_i\}$. Then $h^1(\boldsymbol{P}^n, \boldsymbol{I}_U(t)) = 0$ for every integer $t \geq nm + m_1 + \dots + m_s$. Furthermore, the homogeneous ideal of U is generated by forms of degree $\leq nm + m_1 + \dots + m_s + 1$.*

Here are a few motivations. Unreduced schemes arise often and for several different purposes. Fat points in $\boldsymbol{P}^n$ arise in the problem of interpolation for homogeneous polynomials; if the fat points are fattening in natural varieties (e.g. scrolls) this is related to the interpolation of polynomials modulo interesting ideals and (in the case of rational scrolls) to the interpolation of weighted homogeneous polynomials. Unreduced subschemes (and very often disjoint union or general disjoint union) are used (thanks to Serre correspondence) to the construction of bundles and the cohomological properties of the unreduced schemes reflects cohomological properties (and even the stability) of the corresponding bundles. Unreduced schemes arise as limits of a flat family of reduced ones and by semicontinuity cohomological properties of the unreduced schemes give cohomological conditions of the general member of the flat family; from this point of view, unreduced structures with support a curve arise often (e.g. ribbons) and their 0-dimensional subschemes are linked to the cohomolgy of their line bundles. For an example in which disjoint unions of lines arise as a technical tool (and a partial motivation for Section 3) see [7] and several related papers.

0. Notations and preliminaries

We work over an algebraically closed field $\boldsymbol{K}$; we assume $\mathrm{char}\,(\boldsymbol{K}) = 0$ because some results (e.g. on the conormal bundle of a linearly normal elliptic curve) are not true (in the way we will state them) in positive characteristic. Let $R := \boldsymbol{K}[x_0, \ldots, x_n] = \bigoplus_{t\geq 0} R_t$ be the homogeneous coordinate ring of $\boldsymbol{P}^n$. Set $\boldsymbol{P} := \boldsymbol{P}^n$; let Ω^i be the sheaf of alternating i-differential forms on $\boldsymbol{P}$. If I is a homogeneous ideal of $\boldsymbol{R}$, $\mathrm{r}(I)$ will denote its regularity index; recall that if $A := \bigoplus_{t\geq 0} A_t := R/I := \bigoplus_{t\geq 0}(R_t/I_t)$ has dimension 0, then $\mathrm{r}(I)$ is the first integer u such that $I_u = R_u$, while if A has dimension 1 $\mathrm{r}(I)$ is the first integer such $\dim(A)$ reaches its maximum (which is the multiplicity of A). If Z is a subscheme of $\boldsymbol{P}^n$, $I(Z)$ will denote its saturated homogeneous ideal; set $\mathrm{r}(Z) := r(I(Z))$. For a closed subscheme B of a scheme A, let $\boldsymbol{I}_{B,A}$ be the ideal sheaf of B as subscheme of A; if A is an ambient projective space we will often write $\boldsymbol{I}_B$ instead of $\boldsymbol{I}_{B,A}$. For all non negative integers a, b, set $((a;b)) := a!/(b!(a-b)!)$ (the binomial coefficient).

1. General set up

Here we explain the simplest case (the only one used in this paper) of the so called Horace' method. For recent expositions of the method and its use see e.g. [2], Prop. 1.6, and [1].

Remark 1.1. (Horace' method). Let A be a projective scheme, B a closed subscheme of A, L a line bundle on A and Z an effective Cartier divisor of A (even not reduced). Set $B' := Z \cap B$ and let B'' be the residual scheme of B with respect to Z. By the exact sequence

$$0 \to L(-Z) \otimes \boldsymbol{I}_{B'',A} \to L \otimes \boldsymbol{I}_{B,A} \to (L|Z) \otimes \boldsymbol{I}_{B',Z} \to 0 \tag{1}$$

to prove that $H^1(A, L \otimes \boldsymbol{I}_{B,A}) = 0$ it is sufficient to prove that $H^1(A, L(-Z) \otimes \boldsymbol{I}_{B'',A}) = H^1(Z, (L|Z) \otimes \boldsymbol{I}_{B',Z}) = 0$.

For later sections we need more notations. If t and j are integers with $j > 0$ and B is a closed subscheme of the subscheme A of $\boldsymbol{P} := \boldsymbol{P}^n$, let

$$\mathrm{r}_{A,B,j}(t) : H^0(A, (\Omega^j|A) \otimes \boldsymbol{O}_A(t)) \to H^0(B, (\Omega^j|B) \otimes \boldsymbol{O}_B(t))$$

be the restriction map; if $A = \boldsymbol{P}$, set $\mathrm{r}_{B,j}(t) := \mathrm{r}_{\boldsymbol{P},B,j}(t)$; set $\mathrm{r}_{A,B}(t) :$ $:= \mathrm{r}_{A,B,0}(t)$. If $C \subset \boldsymbol{P}^n$ is a reduced scheme, kC will be the closed subscheme of $\boldsymbol{P}$ with ideal sheaf $(\boldsymbol{I}_C)^k$; hence $(k+1)C := C(k)$ (the k^{th}-infinitesimal neighborhood of C in $\boldsymbol{P}^n$). For all non negative integers j, k we have the following exact sequence:

$$\begin{aligned} 0 \to (\Omega^j|A) \otimes \boldsymbol{I}^k/\boldsymbol{I}^{k+1}(t) \to (\Omega^j|(k+1)C) \otimes \boldsymbol{O}_{(k+1)C}(t) \to \\ \to (\Omega^j|kC) \otimes \boldsymbol{O}_{kC}(t) \to 0. \end{aligned} \tag{2}$$

If C is locally a complete intersection we have $\boldsymbol{I}^k/\boldsymbol{I}^{k+1} \simeq S^k(\boldsymbol{I}|\boldsymbol{I}^2)$ (a suitable symmetric power of the conormal bundle).

From now on in this section we assume that C is a curve. Fix a point $P \in C_{\text{reg}}$ and take formal coordinates in $\boldsymbol{P}^n$ around P in such a way that C has equation $u_1 = u_2 = \cdots = u_{n-1} = 0$ near P and P has equation $u = u_1 = u_2 = \cdots = u_{n-1} = 0$. Set $a(x,k) :=$ $:= \text{length}\,(xP \cap kC)$; note that $a(x,k)$ depends only on x and k. If $x \le k$, we have $xP \subseteq kC$; thus $a(x,k) = ((x+n-1;n))$ if $x \le k$. If $x \ge k$, $a(x,k)$ is given by the sum of the number of monomials in n variables $u, u_1, u_2, \ldots, u_{n-1}$ of total degree $\le k-1$ and the monomials of total degree $\le x-1$ whose total degree with respect to the last $n-1$ variables is $k-1$; hence

(3) $a(x,k) = ((n+k-1;n)) + (x-1-k)((n+k-2;n-1))$ if $x \geq k$.
If $Z = m_1P_1 + \cdots + m_sP_s$, set $a(Z,k) := \sum_i a(m_i,k)$.

2. Curvilinear subschemes

The unifying theme of this section is the use of the methods and ideas contained in [8] and [13]. First we give a lemma (see 2.1) which extends the key elementary lemmas in [8] and [13] from bounds on the regularity index to bounds on the postulation. Then (see Prop. 2.3) we consider the same matter for the minimal degree of a set of generators of the homogeneous ideal. Here we stress the fact that the same considerations work for disjoint union of unreduced schemes of any dimension (see Remark 2.2 and Th. 2.3). Then we consider curvilinear 0-dimensional schemes (see Th. 2.4). The proof of Th. 2.4 gives easily results for fattening of curvilinear 0-dimensional schemes; toy examples for which we do not have any applications are available from the authors.

Lemma 2.1. *Fix an ideal J of finite colength in R and a point P not on the support of R/J and let W be the homogeneous ideal of a scheme supported at P. Set $d :=$ colenght (W) and $a := \mathrm{r}(R/W)$. Fix an integer $t \geq a-1$. Then:*

(a) $\mathrm{r}(R/J \cap W) = \max\{a, \mathrm{r}(R/J), \mathrm{r}(R/(J+W))\}$;

(b) $\mathrm{r}(R/(J+W)) \leq t$ *if and only if there exist d forms of degree t in J whose residue classes in R/W are linearly independent;*

(c) *fix an integer $w > 0$; assume the existence of a form of degree w in J not vanishing on P; then* $\mathrm{r}(R/(J+W)) \leq w+a$.

Proof. The reader can easily check that, just changing a few notations, the proofs of all parts are respectively the proof of [8], Lemma 1, [13], Lemma 1.3, [13], Cor. 1.4. $\Diamond$

Now we show how to use the methods of [8] and [13] to handle the postulation of disjoint union of unreduced schemes, even if the schemes have positive dimension; the same remarks give also bounds for the generators of the homogeneous ideal (see Prop. 2.3).

Here J and W will be homogeneous ideals of R such that $J+$ $+W$ has as radical the maximal irrelevant ideal of R; set $I := J \cap W$. When J and W are saturated ideals, they are the ideal associated to disjoint subschemes (say $J := I(A)$ and $W := I(B)$) of $\boldsymbol{P}^n$ and then I is the saturated ideal $I(A \cup B)$ associated to the union of these two subschemes.

Remark 2.2. Fix an integer $t \geq 0$. Consider the exact sequence of finite dimensional vector spaces:

(4) $$0 \to I_t \to J_t \oplus W_t \to (J+W)_t \to 0.$$

Thus for the corresponding Hilbert functions we have:

(5) $$H_{R/I}(t) = H_{R/J}(t) + H_{R/W}(t) - \dim(R_t/(J+W)_t).$$

In particular if $J = I(A)$, $W = I(B)$, we have:

(6) $$h^1(\boldsymbol{I}_{A\cup B}(t)) \leq h^1(\boldsymbol{I}_A(t)) + h^1(\boldsymbol{I}_B(t)) + \dim(R_t/(J+W)_t).$$

Here we give the bound for the degree of the generators of the homogeneous ideal I.

Proposition 2.3. *Fix an integer $t \geq 2$ and assume that J_{t+1} and W_{t+1} are generated by J_t and W_t and that $(J+W)_{t-1} = R_{t-1}$. Then I_{t+1} is generated by I_t.*

Proof. Choose homogeneous coordinates x_i. Fix $f \in I_{t+1}$. By assumption there exist $g_i \in J_t$ and $h_i \in W_t$ with $\sum_i x_i g_i = \sum_i x_i h_i = f$. Since $\sum_i x_i(g_i - h_i) = 0$, by the Koszul complex (which is a resolution of the empty subscheme) there are forms m_{ij} with $m_{ii} = 0$, $m_{ij} = -m_{ji}$ and such that $g_i - h_i = \sum x_j m_{ij}$. Since $(J+W)_{t-1} = R_{t-1}$, we may write $m_{ij} = a_{ij} + w_{ij}$ with $a_{ij} \in J_{t-1}$, $w_{ij} \in W_{t-1}$ for $i < j$ and set $a_{ij} = -a_{ji}$, $a_{ii} = 0$, $w_{ji} = -w_{ij}$, $w_{ii} = 0$. Then $f_i := g_i - \sum_j x_j a_{ij} = h_i - \sum_j x_j w_{ij} \in J_t \cap W_t$ is such that $\sum_i x_i f_i = f$. $\lozenge$

We recall that a 0-dimensional scheme Z is called *curvilinear* if for every $x \in Z_{\mathrm{red}}$ the Zariski tangent space $T_x Z$ has dimension ≤ 1; equivalently, Z is curvilinear if and only if it is contained in a smooth (affine) curve; if Z is a curvilinear subscheme of $\boldsymbol{P}$, then Z is even contained in a smooth projective curve contained in $\boldsymbol{P}$. We will say that the curvilinear scheme Z is *of type* C if C is a curve with $Z \subset C_{\mathrm{reg}}$ (hence we assume only that C is smooth near Z). Now we consider certain fattening of a curvilinear scheme. Fix a curvilinear scheme X of type C and a hypersurface M with $X = C \cap M$; M exists because X is a Cartier divisor on C; set $W := kC \cap M$; W will be called the k-th fattening of X along C and M; usually we will use kX to denote such a scheme W (when C is clear and the choice of M does not matter in the statements/proofs).

Theorem 2.4. *Fix integers n, s, $m(i)$, $1 \leq i \leq s$, with $m(1) \geq \ldots \geq m(s) > 0$, and let w be the first positive integer with $((n+w;n)) > m(2) + \cdots + m(s)$. Set $a := ((n+w;n)) - (m(2) + \cdots + m(s)) - 1$; let b' be the maximal integer with $((n+b'-1;n)) \leq a$ and set $b := \min(b', [m(1)/2])$. Then for the union, Z, of s general curvilinear* *schemes* $Z(i)$ *of* $\boldsymbol{P}^n$, $1 \le i \le s$, *with* $\mathrm{length}\,(Z(i)) = m(i)$, *we have* $\mathrm{r}(Z) \le w + m(1) - 1 - b$.

Proof. (a) Fix $P \in \boldsymbol{P}^n$ and take a general hypersurface, Q, of degree w with a point of multiplicity b at P. Let Z' be the general union of $s-1$ general curvilinear subschemes $Z(i)$, $2 \le i \le s$, of Q with $\mathrm{length}\,(Z(i)) = m(i)$ for all i.

(b) By the condition on the integer a, for any union, Z'', of $s-1$ curvilinear subschemes $Z(i)$, $2 \le i \le s$, of $\boldsymbol{P}^n$ with $\mathrm{length}\,(Z''(i)) = m(i)$ for all i, there is a degree w hypersurface, Q'', containing them and with a point of multiplicity b at P. Hence Z' may be considered as a general union of $s-1$ general curvilinear subschemes of $\boldsymbol{P}^n$.

(c) We take P as support of $Z(1)$ and assume that $Z(1)$ has tangent not in the tangent cone of Q at P. Let m be the maximal ideal of $Z(1)$. The condition on the tangent line is equivalent to the condition that for any formal smooth curve F in $\boldsymbol{P}^n$ around P containing $Z(1)$, the intersection multiplicity of F and Q is b, i.e. Q cuts the Cartier divisor bP on F. This implies that the product of the equation of Q and all forms of degree $m(1) - 1 - b$ generates the kernel of the surjection $t(b) : \boldsymbol{O}_{Z(1)} \to \boldsymbol{O}_{Z(1)}/m^b$; vice versa, any such form is contained in $\ker(t(b))$.

(d) By part (b) it is easy to check that we may assume the existence of a degree w hypersurface, U, containing Z' but not P. Hence the space of forms which are the product of the equation of U and the forms of degree $m(1) - b - 1$ are sent by the quotient map $t(b)$ surjectively onto $\boldsymbol{O}_{Z(1)}/m^b$. By part (c), we conclude using induction on s (as in [13], proof of Th. 2.4). $\diamondsuit$

Note that we have $b > 0$ if and only if $a > 0$; if $a = 0$, Th. 2.4 is not interesting since it follows easily from [8], Th. 6.

3. Fattening of a disjoint union of rational normal curves and of linear spaces

The aim of this section is the proof of upper bounds for the regularity index of fattening of suitable disjoint unions of suitable positive dimensional subvarieties of $\boldsymbol{P}^n$, i.e. linear subspaces for Th. 0.1, lines for Th. 3.1 and rational normal curves for Th. 3.2. Note that the schemes called U in the statements of Ths. 0.1, 3.1 and 3.2 have usually degree much bigger than the upper bound obtained for the regularity index. Now we will prove Th. 0.1 stated in the introduction.

Proof of Theorem 0.1. We divide the proof into 3 parts.

Part (A) Fix an integer $t \geq nm + m_1 + \cdots + m_s$. For the vanishing of the cohomology groups we will use induction on n. If $n = 1$ the result is obvious. If $n > 1$, take a hyperplane H containing a linear subspace A_i (say A_1) appearing with multiplicity m and containing M_i unless $M_i = \boldsymbol{P}^n$.

Part (B) Let mH be the degree m hypersurface with support H. Set $U'' := U \cap mH$ and let U' be the residual scheme of U with respect to mH. By Horace' method (Remark 1.1) it is sufficient to prove that $H^1(\boldsymbol{I}_{U'}(t-m)) = H^1(mH, \boldsymbol{I}_{U'',mH}(t)) = 0$. The former cohomology group vanishes because $U' \cap A_1 = \emptyset$. To prove that the latter cohomology group vanishes we will use the peeling method, which we will explain now. Note that for all integers $k > 0$ we have $\boldsymbol{I}_{(k-1)H,kH} \simeq (\boldsymbol{I}_{(k-1)H})/ /(\boldsymbol{I}_{kH}) \simeq \boldsymbol{O}_H(-k)$. For all integers k and w with $1 \leq k \leq m$ consider the exact sequences

$$0 \to \boldsymbol{I}_{(U''|H),H}(w-k) \to \boldsymbol{I}_{(U''|kH),kH}(w) \to \\ \to \boldsymbol{I}_{(U''|(k-1)H),(k-1)H}(w) \to 0. \tag{7}$$

By the exact sequences (7) we are reduced to check that $H^1(H, \boldsymbol{I}_{(U''|H),H}(t-m)) = 0$. This is true by the inductive assumption on n and the fact that on H we apply this inductive statement with the addendum "$(n-1)m$" instead of "nm".

Part (C) The assertion on the homogeneous ideal follows (by induction on n) from the vanishing of cohomology just proved (used for all integers $n' \leq n$) as in the usual inductive proof of Castelnuovo–Mumford Lemma ([11]). $\Diamond$

There are several numerical data on $\dim(A_i)$ such that the inductive proof of Th. 0.1 works for these data with a better bound. Here we consider the case in which $\dim(A_i) \leq 1$ for every i (disjoint fat lines plus possibly disjoint fat points) and there is a very strong linear general position assumption.

Theorem 3.1. *Fix s disjoint linear subspaces A_i, $I \leq i \leq s$, of $\boldsymbol{P}^n$ with* $\dim(A_i) < 1$, *linear subspaces (of any dimension) M_i, $1 \leq i \leq s$, with $A_i \subseteq M_i$ for every i, integers m_i, $1 \leq i \leq s$, with $m_i > 0$, and let U be the union of the $(m_i - 1)$-th infinitesimal neighborhoods of A_i in M_i. Assume $s \geq n+2$. Assume that for all non negative integers u, v*

the union of u components of U_{red} of dimension 0 and v components of U_{red} of dimension 1 spans a linear space of dimension $\min(n, u + 2v - 1)$. *Set $\mu := m_1 + \cdots + m_s$. Then $h^1(\boldsymbol{P}^n, \boldsymbol{I}_U(t)) = 0$ for every integer $t \geq \mu$. Furthermore, the homogeneous ideal of U is generated by forms of degree $\leq \mu + 1$.*

Proof. The last assertion follows from the first one as in part (C) of the proof of Th. 0.1. For the first part, we use again induction on n. Set $m := \max\{m_i\}$. Take again a hyperplane H containing a linear subspace A_i appearing with multiplicity m and containing M_i unless $M_i = \boldsymbol{P}^n$. Then repeat part (B) of the proof of Th. 0.1. Now the inductive game with H (and integers $t' \geq t - m + 1$) works essentially because H does not contains U_{red}; if it does not contain a zero dimensional A_j, no further discussion. Assume H is compelled to contains all A_i's with $\dim(A_i) = 0$ (if any). Here we use that a fat line with support not in H intersects H in a fat point. Let j be the number of lines of U_{red} contained in H. Hence $H \cap U$ is the union of j fat lines and $s - j$ fat points. By the linear normality assumption we have $2j + (s-j) \leq n+1$, contradicting the assumption on s. $\lozenge$

Theorem 3.2. *Fix integers $s \geq n$, $m_i > 0$, $1 \leq i \leq s$, proper linear subspaces N_i and M_i, $1 \leq i \leq s$, of $\boldsymbol{P}^n$ with $N_i \subseteq M_i$, rational normal curves C_i of N_i (hence $\deg(C_i) = \dim(N_i)$) with $C_i \cap C_j = \emptyset$ if $i \neq j$. Fix integers m_i, $1 \leq i \leq s$, with $m_1 \geq m_2 \geq \ldots \geq m_s > 0$. Set $e_i = 1$ if $\deg(C_i) \geq 2$, $e_i = 2$ if C_i is a line. Let U be the union of all $(m_i - 1)$-th infinitesimal neighborhoods Z_i of C_i in M_i. Set $\delta := 2 + 3m_1 + e_1 m_1 + \cdots + e_s m_s$. Assume that for a general hyperplane H the set $H \cap U_{\mathrm{red}}$ is in linear general position; assume that for every index i with $N_i \neq \boldsymbol{P}^n$ and for a general hyperplane $H(i)$ containing N_i the set $H(i) \cap (U_{\mathrm{red}} \setminus C_i)$ is finite and in general linear position. Then $H^1(\boldsymbol{P}^n, \boldsymbol{I}_U(t)) = 0$ for every integer $t \geq \delta$. Furthermore the homogeneous ideal $I(U)$ of U is generated by forms of degree $\leq \delta + 1$.*

Proof. The proof is by triple induction on n, δ and $\deg(U_{\mathrm{red}})$ (the starting case being $n = 1$, hence trivial). Fix an integer $t \geq \delta$. The proof is divided into 6 steps. We assume the case $s = 1$ which will be considered in step (6).

(1) Let Π be a projective space and Y a hyperplane of Π. Let D be a rational normal curve of Π. By [6], Prop. 2.5, or [7], Th. 0, there is a family $\{h_t\}_{t \in \Delta \setminus \{o\}}$, Δ smooth affine integral curve with $o \in \Delta$, of projective transformations of Π such that the family of projectively equivalent curves $\{h_t(D)\}$ has as flat limit at o the curve $J \cup K$

with K rational normal curve in Y, J a line not in Y and $J \cap K \neq \neq \emptyset$.

(2) Take the notations of step (1) and fix an integer $r > 0$. In this step we will check that the family $\{h_t(rD)\}$ has $Q := rJ \cup rK$ as flat limit. Since the Hilbert scheme $\mathrm{Hilb}\,(\Pi)$ of Π is proper, and Δ is a smooth curve, there is a flat limit, Q'', and, taking hyperplane sections (and help from the proof of the result stated as step (1), e.g. the picture in [6], or [3], §7, or [7], Th. 0), we see easily that Q'' is the scheme union of Q with a nilpotent with support on $\{P\} := J \cap K$. Hence ([10], Th. III.9.9) it is sufficient to check that Q and Q'' have the same Hilbert polynomial, p_Q and $p_{Q''} := p_{rD}$. Since the general hyperplane sections of Q and Q'' are projectively equivalent, $p_Q - pr_D$ is a constant. It is easy to check that $p_Q(0) = 1$ and $p_{rD}(0) = 1$. Hence $p_Q = p_{Q''}$, as wanted.

(3) Assume there is an integer $u \leq s$ such that $N_u \neq \boldsymbol{P}^n$. Take a general hyperplane H containing N_u. We apply Horace' method (Remark 1.1) with respect to $m_u H$. The residual scheme, U'', of U with respect to $m_u H$ is $U \setminus m_u C_u$. Hence by the inductive assumption on δ we have $h^1(\boldsymbol{P}^n, \boldsymbol{I}_{U''}(t - m_u)) = 0$. Set $U' := U \cap m_u H$. Note that $U \cap H$ is the disjoint union, T, of C_u and $U'' \cap H$ and that we assumed that the points of $(U'' \cap H)_{\mathrm{red}}$ are in linear general position. To conclude by Horace' method it is sufficient to prove that $H^1(m_u H, \boldsymbol{I}_{u', m_u H}(t)) = 0$. Exactly as in part (B) of the proof of Th. 0.1, by the peeling method this vanishing holds if $H^1(H, \boldsymbol{I}_{U \cap H, H}(b)) = 0$ for every $b \geq t - m_u$. Fix such an integer b. Take a hyperplane M of H. If $N_u \neq H$ we take M containing N_u. Note that the residual scheme of U' in H with respect to $N_u M$ consists of fat points with support in linear general position and that $e_j \dim(H) \geq \deg(C_j)$ for every j. Now we apply Horace' method (Remark 1.1) with H as ambient space and with M as divisor. To conclude it is sufficient to apply [8], Th. 6, to the residual scheme of $U \cap H$ with respect to $m_u M$ and the peeling method as in part (B) of the proof of Th. 0.1.

(4) Now assume $N_u = H$. We take as M a general hyperplane of H. By steps (1) and (2) there is a flat family of projective transformations $\{h_t\}$ of H such that $\{h_t(m_u C_u)\}$ has as flat limit a multiple of the union of a rational normal curve K of M and a line. By the properness of the Hilbert scheme, the generality of H and the proof of the result stated as step (1) (see in particular the picture in [6] and/or [3], §7,

and [7], Th. 0) we may assume that the family $\{h_t((U'' \setminus C_u) \cap H)\}$ has a limit formed by fat points with support in linear general position and disjoint from M and with the same multiplicities as $(U'' \setminus C_u) \cap H$. By semicontinuity it is sufficient to check the vanishing for the limit scheme. We apply again Horace and the peeling method as in step (4). Now the only difference is that as residual scheme we have also a line with multiplicity m_u. We may apply induction on the degree by the choice of the integer e_u.

(5) Now we assume $N_i = \boldsymbol{P}^n$ for every integer i. We apply steps (1) and (2) to m_1C_1 as in step (4) and then repeat the same proof.

(6) Here we consider the case $s = 1$. The case of a multiple mD (i.e. the vanishing of $H^1(\boldsymbol{I}_{mD}(t))$ for every $t \geq 2m-1$) of a rational normal curve was proven in [5], 3.4; alternatively a proof can easily be given using steps (1), (2) and the proof of step (4). The same vanishing for the general case with arbitrary $\dim(N_1)$ and $\dim(M_1)$ may easily be proved using steps (1), (2) and the proof of step (4). $\Diamond$

4. Fat points on linearly normal smooth rational or elliptic curves

In this section we take as curve either a rational normal curve of $\boldsymbol{P}^n$, D, or (but only briefly in 4.9 and 4.10) a linearly normal elliptic curve, E. At the end of the section we consider briefly the case in which the supporting variety is a higher dimensional rational normal scroll (see 4.11 and 4.12). Let $r_{mD,kD,j}(t)$ and $r_{mD,kD}(t)$ be the restriction maps. Recall (see e.g. [12]) that (in any characteristic) $\boldsymbol{I}/\boldsymbol{I}^2$ is the direct sum of $n-1$ line bundles of degree $-n-2$. Thus $\boldsymbol{I}^s/\boldsymbol{I}^{s+1} \simeq S^s(\boldsymbol{I}|\boldsymbol{I}^2)$ is isomorphic to the direct sum of $((n+s-2;s))$ line bundles of degree $-2s-ns$ on $\boldsymbol{P}^1$. Recall (see e.g. [4], Lemma 1.3) that $\Omega^1|D$ is the direct sum of n line bundles of degree $-n-1$. Hence $\Omega^j|D$ is the direct sum of $((n;j))$ line bundles of degree $-j(n+1)$. Hence from (2) and the cohomology of line bundles on $\boldsymbol{P}^1$ we obtain at once the following 2 lemmas.

Lemma 4.1. *For every $m \geq 2$ and $j \geq 0$ we have*

$$\chi(\boldsymbol{O}_{mD} \otimes \Omega^j) = \chi(\boldsymbol{O}_{(m-1)D} \otimes \Omega^j) - \\ - ((n+m-2;m)) \cdot (m(n+2)+j(n+1)-1)((n;j)).$$

Lemma 4.2. *If $j \geq 0$, $m > k \geq 1$ and $tn \geq m(n+2)+j(n+1)-1$, then the restriction map $r_{mD,kD,j}(t)$ is surjective.*

It is known ([5], Prop. 3.4), that mD is $2m$-regular; the following more precise vanishings are known. We have (see step (6) of the proof of Th. 3.2) $h^1(\boldsymbol{I}_{mD}(t)) = 0$ for every $t \geq 2m$; we have (see the exact sequences (2) for $j = 0$) $h^2(\boldsymbol{I}_{mD}(t)) = h^1(mD, \boldsymbol{O}_{mD}(t)) = 0$ if $tn \geq \geq m(n-1)-1$; we have $h^u(I_{mD}(t)) = h^u(\boldsymbol{P}^n, \boldsymbol{O}(t)) = 0$ for all integers u, t with $u \geq 3$, $t \geq -n$. By the interpretation via Koszul cohomology of the minimal free resolution and the regularity index we have the following result which in case $j = 0$ was the first part of the statement of [5] , Prop. 3.4.

Proposition 4.3. *The restriction map*

$$\mathrm{r}_{mD,j}(t) : H^0(\boldsymbol{P}^n, \boldsymbol{O}_{\boldsymbol{P}}(t) \otimes \Omega^j) \to H^0(mD, O_{mD}(t) \otimes \Omega^j)$$

is surjective for every $t \geq 2m + j$.

We fix $Z := m_1P_1 + \cdots + m_sP_s$ with $m_1 \geq \ldots \geq m_s$ and $Z_{\mathrm{red}} \subset \subset D$; set $m := m_1$. Set $Z(k) := Z \cap (kD)$. The next result follows from Prop. 4.3 and the exact sequence (2).

Proposition 4.4. *If* $k < m$ *and* $t \geq 2k + 2 + j$, *we have:*

$$h^1(\boldsymbol{P}^n, \boldsymbol{I}_{Z(k+1)}(t) \otimes \Omega^j) = h^1(\boldsymbol{P}^n, \boldsymbol{I}_{Z(k)}(t) \otimes \Omega^j) + \\ + \max(0, ((n;j))(jn + j - nt - 1 + a(Z, k+1) - \\ -a(Z, k)) \cdot ((n + k; n - 1))).$$

Applying $m - k$ times the previous result we obtain the following two propositions.

Proposition 4.5. *If* $k < m$ *and* $t \geq 2m + j$, *we have:*

$$h^1(\boldsymbol{P}^n, \boldsymbol{I}_{Z(m)}(t) \otimes \Omega^j) = h^1(\boldsymbol{P}^n, \boldsymbol{I}_{Z(k)}(t) \otimes \Omega^j) + \\ + \sum_{k \leq a < m} \max(0, ((n;j))(jn + j - nt - 1 + a(Z, k+1) - \\ -a(Z, k)) \cdot ((n + a; n - 1))).$$

Proposition 4.6. *Assume* $1 < t < 2m - 2 + j$ *and take* $k < m$ *with* $2k - 1 \leq t \leq 2k$. *Then we have:*

$$h^1(\boldsymbol{P}^n, \boldsymbol{I}_{Z(k)}(t) \otimes \Omega^j) + \\ + \sum_{k \leq a < m} \max(0, ((n;j))(jn + j - nt - 1 + a(Z, k+1) - \\ -a(Z, k)) \cdot ((n + a; n - 1))) \leq \\ \leq h^1(\boldsymbol{P}^n, \boldsymbol{I}_Z(t)) \leq h^1(\boldsymbol{P}^n, \boldsymbol{I}_{Z(k)}(t) \otimes \Omega^j) +$$

$$+\sum_{k\le a<m}\max(0,((n;j))(jn+j-nt-1+a(Z,k+1)-$$

$$-a(Z,k))\cdot((n+a;n-1)))+\dim(\operatorname{coker}(r_{mD,j}(t))).$$

Remark 4.7. Note that using inductively Prop. 4.4 we get the value of $h^1(\boldsymbol{P}^n, \boldsymbol{I}_{Z(k)}(t)\otimes\Omega^j)$ in Prop. 4.5.

Remark 4.8. If $t\ge 2m+j$ with $m\ge 2$, we have $tn\ge 2nm-n+j+1$. Hence by 4.3 and 4.5 $h^0(\boldsymbol{P}^n, \boldsymbol{I}_Z(t)\otimes\Omega^j)$ is known, computed and independent of the positions of the points P_i on D.

4.9. Here we consider the case of an elliptic normal curve E. The discussion made in [5], §4, works without any change, thanks to the general set up, because we know exactly $\Omega^1|E$. Indeed, by [9], Prop. 3.2, $\Omega^1|E$ is stable and hence it is the unique rank n stable vector bundle on E with $\boldsymbol{O}_E(-n-1)$ as determinant. Since $\Omega^1|E$ is known (and we have char $(\boldsymbol{K})=0$), the bundle $\Omega^j|E$ is in principle computable using the known multiplicative structure of the ring of vector bundles on E. However, often life is easier; indeed by a very particular case of [9], Corollary in the introduction with $i=0$, this bundle is semistable with rank $((n;j))$ and determinant $\boldsymbol{O}_E(-(n+1)((n;j))-(n+1)((n-1;j-1)))$; if the integers rank $(\Omega^j|E)$ and $\deg(\Omega^j|E)$ are coprime, this forces $\Omega^j|E$ to be indecomposable, hence uniquely determined by Atiyah's classification of vector bundles on an elliptic curve. Using this information one obtains upper and lower bounds for $h^1(\boldsymbol{I}_{Z'(k+1)}(t)\otimes\Omega^j)-h^1(\boldsymbol{I}_{Z'(k)}(t)\otimes\Omega^j)$ and all (k,t) with $k<m$ and $t\ge 2k+1$. To pass from the cohomology of $\boldsymbol{I}_{Z,mE}(t)\otimes\Omega^j$ to the cohomology of $\boldsymbol{I}_{Z,\boldsymbol{P}}(t)\otimes\Omega^j$ we need a good bound (or the exact value) for the regularity index of mE; everything would be obvious if we know the minimal free resolution of mE. If E has general moduli, this was obtained in the proof of [5], Prop. 4.2. Hence we have the following result

Proposition 4.10. *Let $E^\wedge$ be an elliptic curve with general moduli. For all integers n, m, t with $n\ge 2$, $m\ge 2$, $t\ge 2m-1+j$, and every elliptic normal curve $E\subset\boldsymbol{P}^n$ with $E\simeq E^\wedge$ (as abstract curves) the restriction map $r_{mE,j}(t)$ is surjective.*

Thus we have a good understanding of the minimal free resolution of a union X of fat points of order $\le m$ and supported by $E^\wedge$ in the level $t\ge 2m+1$. If (for fixed m) length (X) is sufficiently high, then all generators of $I(X)$ will appear in degree $\ge 2m-1$. Hence in this case our understanding of the minimal free resolution would be quite good.

4.11. We are interested also in the case in which the fat points are supported on a higher dimensional rational normal scroll, S, say of dimension u (hence of degree $n-u+1$). We do not exclude the case in which S is singular, i.e. a cone over a lower dimensional rational normal scroll, but in this case we assume that no fat point is supported on the linear space $\mathrm{Sing}\,(S)$. Call V'' the cone of dimension u over a rational normal curve (hence with as vertex a projective space of dimension $u-{}-2$). There is an integral family of projective transformations $\{g_t\}$ such that $\{g_t(S)\}$ has V'' as flat limit. Hence, as in steps (1) and (2) of the proof of Prop. 3.2 by semicontinuity we get the following result (with obvious notations).

Proposition 4.12. *The restriction map $r_{mS,j}(t)$ is surjective for every $t \geq 2m+j$.*

The postulation of fat points supported on S depends very much from their position (after all, S contains curves of arbitrarily large degree).

5. General nonsense bounds for the cohomology

The aim of this section is to show why the methods of [5] allow one to obtain a very general result on the possible behaviour of the cohomology of unreduced 0-dimensional subschemes of $\boldsymbol{P}^n$ with support on a fixed curve. The result (i.e. Th. 5.1) is very general, but also very vague (just a simple application of Serre Theorem B and semicontinuity). However the method of proof gives obviously that, if we have enough informations on the curve and the "numerical data" (defined below), in the statement of Th. 5.1 we may give explicit (although not sharp) bounds.

Fix an integral variety C (say a curve) and a finite set $S \subset C_{\mathrm{reg}}$; let Y be a scheme with $S = Y_{\mathrm{red}}$; the numerical data of Y (with respect to C) is just the sequence of all integers $\mathrm{length}\,(\boldsymbol{O}_Y \cap \boldsymbol{I}_C^k)$; if S is not connected, this sequence should be called the total numerical data to distinguish it from the set of all numerical data of the connected components of Y.

Theorem 5.1. *Fix an integer $m > 0$ and an integral curve C of $\boldsymbol{P}^n$. Then there exists an integer u such that for all total numerical data of multiplicity at most m with support on C, and for every pair of 0-dimensional schemes, X and W, supported on C_{reg} and with that total* *numerical data, the cohomology of $\boldsymbol{I}_X(t)$ and $\boldsymbol{I}_W(t)$ is the same except at most for u integers t.*

Proof. First fix an integer, a, (depending only on C and m) such that for every integer k with $1 \leq k \leq m$, the scheme kC is a-regular. Now we have just to use the exact sequences (2) for $j = 0$, and the peeling method explained in part (B) of the proof of Th. 0.1 as in Section 3. $\diamondsuit$

Note that in the statement of Th. 5.1 we required only that X and W have the same total numerical data, independently on the number and the numerical data of the connected components of X and W (not only independently of the position of the supports of X and W on C_{reg}). It is easy to modify Th. 5.1 and its proof to avoid the condition that the support of X and W is disjoint from the singular locus of C. We do not claim that in the statement of Th. 5.1 the $\leq u$ "exceptional integers t" are consecutive or at least that the difference between the minimal and the maximal one is bounded. Indeed the proof gives only that there are at most m intervals of possible exceptional integers and each of these intervals has bounded measure, but the distribution of the intervals may, a priori, depends on the choice not only of the total numerical data, but also of the numerical data of the 0-dimensional schemes.

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