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# ON COVERING PROPERTIES BY REGULAR CLOSED SETS

Dragan **Janković**

*Department of Mathematics, East Central University, Ada, Oklahoma 74820, USA*

Chariklia **Konstadilaki**

*Department of Mathematics, Faculty of Sciences, Aristotle University of Thessaloniki, 54006 Thessaloniki, Greece*

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**Abstract:** A topological space $X$ is [countably] rc-compact (rc-Lindelöf) if every [countable] cover of $X$ by regular closed sets has a finite (countable) subcover. It is established, among other results, that 1. A space is rc-compact iff its semiregularization is an extension of a compact extremally disconnected space; 2. An uncountable $T_3$ first countable crowded space is rc-Lindelöf iff it is a Luzin space, and 3. A countably rc-compact $T_3$ first countable or generalized ordered space is finite.

## 0. Introduction

In this paper separation axioms are not assumed without explicit mention. A topological space $X$ is defined to be *rc-compact (rc-Lindelöf)* if every cover of $X$ by regular closed sets has a finite (countable) subcover. In [22] rc-compact spaces were introduced and studied under the name of S-closed spaces. In order to have a uniform terminology for cov-

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ering properties by regular closed sets we have adopted the prefix rc- for obvious reasons. The concept of rc-compactness is related to extremal disconnectedness since regular or Hausdorff rc-compact spaces are extremally disconnected. These results due to T. Thompson [22] were strengthened by R. Hermann [10] who showed that rc-compact spaces with regular or $T_1$ semiregularzations are extremally disconnected. In Section 2 we exhibit the "real" nature of rc-compact spaces by showing that rc-compact spaces are precisely spaces whose semiregularizations possess a dense compact extremally disconnected subspace, or equivalently, a dense compact set consisting of points of extremal disconnectedness. Now it is not surprising that imposing a very weak separation property $R_0$ (a common generalization of regularity and $T_1$) on the semiregularization of an rc-compact space will force extremal disconnectedness. On the other hand, there are $T_1$ rc-compact spaces which are not extremally disconnected as shown by D. Cameron [4].

In Section 3 rc-discrete sets are introduced as an important tool in investigating rc-Lindelöf spaces. We point out that rc-discrete sets are independent interest. A collection of rc-discrete sets in a $T_3$ (i.e., regular $T_1$) space includes a collection of discrete sets and in $T_1$ crowded spaces (i.e., without isolated points) is included in the collection of nowhere dense sets. We show that in $T_3$ first countable spaces nowhere dense sets are rc-discrete and use this result to establish, via the existence of Luzin spaces, that the statement that there exists an uncountable $T_3$ first countable rc-Lindelöf space with countably many isolated points is independent of ZFC. The same conclusion is true in case of uncountable rc-Lindelöf generalized ordered spaces.

In the last section we study countably rc-compact spaces which are defined as spaces whose countable covers by regular closed sets have finite subcovers. It turns out that the intersection of this class of spaces whith the class of $T_3$ first countable spaces as well as the class of generalized ordered spaces is percisely the class of finite spaces.

## 1. Definitions and notations

Throughout, $X$ or $(X, \tau)$ will denote a topological space and ClA ($\operatorname{Int} A, \operatorname{Bd} A$) will denote the closure (interior, boundary) of a subset $A$ of a space. A set $A$ in a space is *regular open* (*regular closed*) if $A = \operatorname{Int} \mathrm{Cl} A$ ($A = \operatorname{Cl} \operatorname{Int} A$). We denote by $\mathrm{RO}(X)(\mathrm{RC}(X))$ the family of regular open (regular closed) sets in a space $X$. The family of regular open sets in $(X, \tau)$ is a base for a topology $\tau_S$ on $X$ coarser than $\tau$. The space $(X, \tau_S)$ is called the *semiregularization* of $(X, \tau)$ and $(X, \tau)$ is called *semiregular* if $\tau = \tau_S$. A topological property $\mathcal{P}$ is said to be semiregular if a space $(X, \tau)$ has $\mathcal{P}$ iff the space $(X, \tau_S)$ has $\mathcal{P}$. Since $\mathrm{RC}\,(X, \tau) = \mathrm{RC}\,(X, \tau_S)$, it is clear that both rc-compactness and rc-Lindelöfness are semiregular properties. Another example of a semiregular property is extremal disconnectedness. A space $X$ is *extremally disconnected* if every open set in $X$ has an open closure, or equivalently if $\mathrm{RO}\,(X) = \mathrm{RC}\,(X)$. A "pointed" version of extremal disconnectedness introduced in [7] may be described in the following way: A point $x$ in a space $X$ is a *point of extremal disconnectedness* (shortly, *an e.d. point*) if $x \notin \mathrm{Bd}\,U$ for every $U \in \mathrm{RO}\,(X)$. Note that boundaries of regular open sets in $(X, \tau)$ and $(X, \tau_S)$ coincide. Therefore, $x \in X$ is an e.d. point in $(X, \tau)$ iff it is an e.d. point in $(X, \tau_S)$. We denote the set of e.d. points of a space $X$ or its semiregularization by $\mathrm{ED}\,(X)$. Interesting examples of extremally disconnected spaces are obtained by absolutes of spaces. Recall that with every $\mathrm{T}_3$ space $X$ we associate the space $\mathrm{E}X$, called the Iliadis absolute, which is unique (up to homeomorphism) with respect to having these properties: $\mathrm{E}X$ is Tychonoff extremally disconnected and there exists a perfect continuous irreducible surjection $k_X : \mathrm{E}X \to X$.

We denote by $\mathrm{Z}(X)(\mathrm{Coz}\,(X))$ the family of zero sets (cozero sets) in a space $X$. $\beta X$ is the Stone-Čech compactification of a Tychonoff space $X$, $X^*$ denotes the Stone-Čech remainder $\beta X - X$, and $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbf{I}$ denote the sets of natural numbers, rationals, reals and the unit interval usually equipped with the euclidean (subspace) topology. Cardinals are initial ordinals, $\omega$ is the first infinite ordinal and the cardinality of a set $A$ is denoted by $|A|$. We refer the reader to [16] for undefined terms and notation.

## 2. rc-compact spaces

A set $A$ in a space $X$ is said to be *locally dense* if it is dense in an open set in $X$, or equivalently, if $A \subseteq \mathrm{Int}\,\mathrm{Cl A}$. The proof of our first result is left to the reader.

**Lemma 2.1.** *Let $(X, \tau)$ be a space and $Y$ a locally dense set in $(X, \tau_S)$. Then*

(a) $\mathrm{RO}\,(Y) = \{A \cap Y : A \in \mathrm{RO}\,(X)\}$;
(b) $\mathrm{RC}\,(Y) = \{A \cap Y : A \in \mathrm{RC}\,(X)\}$;

(c) $(\tau|Y)_S = \tau_S|Y$;
(d) $\mathrm{ED}\,(Y) = Y \cap \mathrm{ED}\,(X)$;
(e) $Y$ *is extremally disconnected iff* $Y \subseteq \mathrm{ED}\,(X)$.

Note that locally dense sets in $(X, \tau)$ are locally dense in $(X, \tau_S)$ while the converse is not true in general.

The following characterizations of rc-compact spaces are obtained in a standard way so we omit the proofs. A regular open filter base (filter, ultrafilter) in a space $X$ is a filter base (filter, ultrafilter) in the lattice of regular open subsets of $X$.

**Theorem 2.2.** *A space $X$ is rc-compact iff every regular open filter base (filter, ultrafilter) in $X$ has a nonempty intersection.*

The fact that regular open ultrafilters in an rc-compact space have nonempty intersection shows how strong this property must be. This observation is better reflected in our main result in this section.

**Theorem 2.3.** *A space $(X, \tau)$ is rc-compact iff $(X, \tau_S)$ is an extension of a compact extremally disconnected space.*

**Proof.** The "if" part follows from the following facts: (1) compact extremally disconnected spaces are rc-compact; (2) if a space has a dense rc-compact subspace, then it is rc-compact; and (3) rc-compactness is a semiregular property. Now, in showing the "only if" part we shall assume that $(X, \tau)$ is a semiregular in order to simplify the notation. There is no loss of generality in light of Lemma 2.1 and the fact that the semiregularization of a space is semiregular. Let $Y = \cup\{\cap\mathcal{U} : \mathcal{U} \in \in \mathrm{S}(X)\}$, where $\mathrm{S}(X)$ denotes the set of regular open ultrafilters on $X$. We first show tha $Y$ is dense in $X$. Let $G \in \mathrm{RO}\,(X)$ and $G \neq \emptyset$. Then $\{U : U \in \mathrm{RO}\,(X)$ and $G \subseteq U\}$ is regular open filter and hence is contained in some $\mathcal{U} \in \mathrm{S}(X)$. This gives $G \in \mathcal{U}$, and consequently $\cap\mathcal{U} \subseteq G$. By Th. 2.2 $\cap\mathcal{U} \neq \emptyset$. So, $G \cap Y \neq \emptyset$ and $Y$ is dense in $X$. We now show that $Y \subseteq \mathrm{ED}\,(X)$. Let $x \in Y$ and suppose that $x \in \mathrm{Bd}\,G$ for some $G \in \mathrm{RO}\,(X)$. Then $x \in \cap\mathcal{U}$ for some $\mathcal{U} \in \mathrm{S}(X)$. Since $x \in \mathrm{Cl}\,G$, $U \cap G \neq \emptyset$ for every $U \in \mathcal{U}$. Hence $G \in \mathcal{U}$ and $\cap\mathcal{U} \subseteq G$. This contradicts $x \notin G$ and establishes that $x$ is an e.d. point. By Lemma 2.1, $Y$ is extremally disconnected subspace of $X$. To show that $Y$ is compact, let $Y \subseteq \cup\{V_\alpha : V_\alpha \in \mathrm{RO}\,(X)$ and $\alpha \in A\}$. We claim that $X = \cup\{\mathrm{Cl}\,V_\alpha : : \alpha \in A\}$. Let $x \in X$ and $\mathcal{U} \in \mathrm{S}(X)$ such that $\mathcal{U}$ converges to $x$. Clearly, there is an $\alpha \in A$ such that $\cap\mathcal{U} \cap V_\alpha \neq \emptyset$. Since $V_\alpha$ meets every member of $\mathcal{U}$, $V_\alpha \in \mathcal{U}$ and hence $x \in \mathrm{Cl}\,V_\alpha$ as $\mathcal{U}$ converges to $x$. So, we have that $\{\mathrm{Cl}\,V_\alpha : \alpha \in A\}$ is a cover of $X$ by regular closed sets. Since $X$ is rc-compact there exist $\alpha(1), \alpha(2), \ldots, \alpha(n) \in A$ such that $X =$

$= \cup\{\mathrm{Cl}\, V_{\alpha(i)} : i = 1, 2, \ldots, n\}$. Now each point of $Y$ is an e.d. point so it must belong to some $V_{\alpha(i)}$. Therefore, $Y \subseteq \cup\{V_{\alpha(i)} : i = 1, 2, \ldots, n\}$ and the proof is complete. $\Diamond$

By Th. 2.3 it follows at once that rc-compact Hausdorff spaces are extremally disconnected since Hausdorffness is a semiregular property. To see how regularity type separation axioms produce the same effect and at the same time to extend the previous result we need the following common generalization of regularity and $\mathrm{T}_1$. A space $X$ is called $\mathrm{R}_0$ [5] if every open set in $X$ contains the closure of each of its points. A useful characterization of spaces having $\mathrm{R}_0$ semiregularizations is obtained by use of rc-closure. For a set $A$ in a space $X$, the *rc-closure* of $A$, denote by $\mathrm{Cl}_{\mathrm{rc}}\, A$, is $\cap\{U \in \mathrm{RO}\,(X) | A \subset U\}$. This concept was inroduced in [6] under the name of s-closure. Note that for a space $(X, \tau)$, $(X, \tau_S)$ is $\mathrm{R}_0$ iff for every $F \in \mathrm{RC}\,(X)$ whenever $x \in F$ then $\mathrm{Cl}_{\mathrm{rc}}\,\{x\} \subset F$ and that extremally disconnected spaces have regular semiregularizations.

**Theorem 2.4.** *An rc-compact space $(X, \tau)$ is extremally disconnected iff $(X, \tau_S)$ is $\mathrm{R}_0$.*

**Proof.** We show that $(X, \tau_S)$ is extremally disconnected. By Th. 2.3 there exists a dense compact set $D$ in $(X, \tau_S)$ consisting of e.d. points. Let $x \in X - D$ and suppose that $x$ is not an e.d. point. Then there exists a $V \in \mathrm{RO}\,(X)$ with $x \in \mathrm{Bd}\, V$. (Note that $\tau_S - \mathrm{Bd}\, V = \tau - - \mathrm{Bd}\, V$ and $\tau_S - \mathrm{Cl}\, V = \tau - \mathrm{Cl}\, V$ for $V \in \mathrm{RO}\,(X)$). Since $(X, \tau_S)$ is $\mathrm{R}_0$, $\mathrm{Cl}_{\mathrm{rc}}\,\{x\} \subset \mathrm{Bd}\, V$. Also, $\mathrm{Bd}\, V \cap D = \emptyset$ since $D$ consists of e.d. points. Therefore, $\mathrm{Cl}_{\mathrm{rc}}\,\{x\} \cap D = \emptyset$. On the other hand, $U \cap D \neq \emptyset$ for every $U \in \mathrm{RO}\,(X)$ with $x \in U$. Since $\mathrm{Cl}\, U \cap D = U \cap D$, $\{U \cap D | U \in \in \mathrm{RO}\,(X)$ and $x \in U\}$ is a closed filter base in $D$. The compactness of $D$ implies $\cap\{U \cap D | U \in \mathrm{RO}\,(X)$ and $x \in U\} \neq \emptyset$, and consequently $\mathrm{Cl}_{\mathrm{rc}}\,\{x\} \cap D \neq \emptyset$. This contradiction completes the proof. $\Diamond$

**Remark 2.5.** In [14] T. Noiri defined locally S-closed spaces in a way which differs from a usual way of localizing a global property. A space $X$ is *locally S-closed* if every $x \in X$ has an open S-closed neighbourhood. It is left to the reader to show tha in Th. 2.4 "rc-compactness" may be replaced by "locally S-closed".

## 3. rc-Lindelöf spaces

In this section we study the class of rc-Lindelöf spaces. The fact that this natural generalization of rc-compactness implies perfect $\kappa$-normality, or equivalently, that rc-Lindelöf spaces are in the class Oz, and the following observation have motivated our study of rc-Lindelöf spaces.

In [22] it was observed that $\mathbb{N}^*$ is not rc-compact since under the Continuum Hypothesis there exist P-points in $\mathbb{N}^*$. Clearly, the fact that $\mathbb{N}^*$ is not extremally disconnected suffices. On the other hand, one may exhibit a regular closed cover of $\mathbb{N}$ having no finite, moreover countable, subcover. We employ the well known facts that there is a continuous surjection from $\mathbb{N}^*$ to the unit interval $\mathbf{I}$ and that $\mathbb{N}^*$ is a $\mathrm{P}'$-space (i.e., its zero sets are regular closed). Now, consider the partition $\mathcal{P} = \{f^{-1}(x) | x \in \mathbf{I}\}$ of $\mathbb{N}^*$. Obviously, $\mathcal{P}$ does not have a countable subcover, otherwise $|\mathbf{I}| \leq \omega$.

The previous observation that $\mathbb{N}^*$ is not rc-Lindelöf may be generalized by showing in the same way that the remainder of the Stone-Čech compactification of a $\mathrm{T}_2$ locally compact Lindelöf non countably compact space is not rc-Lindelöf. We will see later that this result generalizes further.

It is clear that regular rc-Lindelöf spaces are Lindelöf and also that countable as well as extremally disconnected Lindelöf spaces are rc-Lindelöf. As we have already seen a compact space is not necessarily rc-Lindelöf. Th. 3.8 implies that the unit interval is not rc-Lindelöf.

We first give a characterization of rc-Lindelöf spaces. A standard proof is omitted. Recall that a filter in a space has the *countable intersection property* if every countable family of elements of the filter has a nonempty intersection.

**Theorem 3.1.** *A space $X$ is rc-Lindelöf iff every regular open filter in $X$ with the countable intersection property has a nonempty intersection.*

We show next that rc-Lindelöfness is inherited by certain subspaces.

**Proposition 3.2.** *In a regular rc-Lindelöf space both regular open and regular closed sets are rc-Lindelöf.*

**Proof.** Let $U \in \mathrm{RO}\,(X)$ and $\mathcal{U} = \{F_\alpha | \alpha \in A\}$ be a cover of $U$ by regular closed sets in $U$. It is easy to see that there exists an $F'_\alpha \in \mathrm{RC}\,(X)$ such that $F_\alpha = U \cap F'_\alpha$ for each $\alpha \in A$. Now, $\{F_\alpha | \alpha \in A\} \cup \{X - U\}$ is a cover of $X$ by regular closed sets and the result follows. Note that we did not use the assumption that $X$ is regular. Now let $F \in \mathrm{RC}\,(X)$ and $\mathcal{U} = \{F_\alpha | \alpha \in A\}$ be a cover of $F$ by regular closed sets in $F$. Then $\mathcal{U}$ is a cover of $F$ by regular closed sets in $X$. For each $x \in X - F$ let $V_x$ be an open set with $x \in V_x \subset \mathrm{Cl}\, V_x \subset X - F$. Then $\mathcal{U} \cup \{\mathrm{Cl}\, V_x | x \in X - F\}$ is a cover of $X$ by regular closed sets in $X$ and the result follows. $\Diamond$

Since $\mathbb{N}^*$ is closed in the rc-compact space $\beta\mathbb{N}$, rc-Lindelöfness is not inherited by closed subspaces. Also, an uncountable discrete set

$D$ is open and dense in the rc-compact space $\beta D$ but not rc-Lindelöf. Therefore rc-Lindelöfness is not inherited neither by open nor dense subspaces. Finally, rc-Lindelöfness is not contagious, namely there are non rc-Lindelöf spaces with rc-Lindelöf dense subspaces. A non rc-Lindelöf extension of $\mathbb{N}$ with the discrete topology suffices and also witnesses that there is no result analogous to Th. 2.3. So called anti-Michael line, $\mathbb{R}$ with the usual topology and rational points declared open, is a non rc-Lindelöf (by Th. 3.8) metric separable, moreover hereditarily Lindelöf, space where $\mathbb{Q}$ is a dense rc-Lindelöf subspace.

The following important class of spaces was introduced independently by R. Blair and E. Ščepin. A space $X$ is said to be in *the class* Oz [3] or $X$ is called *perfectly* $\kappa$*-normal* [19] if regular closed sets in $X$ are zero sets, or equivalently, if (i) disjoint regular closed sets are separated by open sets, and (ii) regular closed sets are intersections of countably many regular open sets. Perfectly $\kappa$-normal spaces generalize perfectly normal spaces, extremally disconnected spaces and products of metric separable spaces. Also, compact Hausdorff topological groups are perfectly $\kappa$-normal and perfectly $\kappa$-normal uncountable products of compact Hausdorff spaces satisfy countable chain condition [18]. Several useful characterizations of spaces in the class Oz are given in [3]. R. Blair also showed that many important classes of spaces are not included in the class Oz. The following result enables us to use the known facts about perfectly $\kappa$-normal spaces. In the terminology of [3] it states that regular rc-Lindelöf spaces are regularly normal i.e., normal with regular closed sets being $G_\delta$-sets.

**Theorem 3.3.** *Regular rc-Lindelöf spaces are perfectly* $\kappa$*-normal.*

**Proof.** Let $X$ be a regular rc-Lindelöf space. Then $X$ is Lindelöf and hence normal. Let $U \in \mathrm{RO}(X)$. For each $x \in U$ there is an $U_x \in \mathrm{RO}(X)$ such that $x \in U_x \subset \mathrm{Cl}\, U_x \subset U$. Clearly, $\mathrm{Cl}\, U_x \in \mathrm{RC}(U)$. By Prop. 3.2, $U$ is rc-Lindelöf and hence a countable union of regular closed sets in $X$. Therefore $U$ is an open $F_\sigma$-set in the normal space $X$. So $U \in \mathrm{Coz}(X)$ and the result follows. $\Diamond$

By Th. 3.8, $\mathbb{R}$ with the usual topology is an example of perfectly $\kappa$-normal but not rc-Lindelöf space.

In [2] it is shown that $X^* \notin$ Oz if $X$ is a Tychonoff locally compact non pseudocompact space. This generalizes our observation from Introduction that $X^*$ is not rc-Lindelöf if $X$ is a Tychonoff locally compact Lindelöf non pseudocompact space.

Note also that none of the familiar spaces of ordinals is in the class Oz [3]. R. Blair [3] also improved the well known result that $\beta\mathbb{N}\times\beta\mathbb{N}$ is not extremally disconnected by showing that $\beta\mathbb{N}\times\beta\mathbb{N}\notin$ $\notin$ Oz. This shows that rc-Lindelöfness is not productive and also that the inverse image of rc-Lindelöf, moreover rc-compact, spaces under continuous open perfect surjections are not necessaily rc-Lindelöf as the projection $p: \ \beta\mathbb{N}\times\beta\mathbb{N}\to\beta\mathbb{N}$ witnesses. The last observation also answers a question posed by T. Noiri [15] whether rc-compactness is inversely preserved under continuous open perfect surjections. A different example answering Noiri's question is given in [21]. It is easily seen that rc-Lindelöfness is preserved under continuous open surjections since under these mappings the preimages of regular closed sets are regular closed. On the other hand, rc-Lindelöfness is not preserved by continuous surjections as the following example shows. Let $D$ be a discrete space with $|D|=2^{\omega}$. Then there exists a continuous surjection from $\beta D$ to $D^*$. As we have already observed, $D^*$ is not rc-Lindelöf.

We turn now our attention to rc-discrete sets. A point $x$ in a subset $A$ of a space $X$ is called an *rc-discrete* point of $A$ if there exists an $F\in \mathrm{RC}(X)$ such that $F\cap A=\{x\}$. A set $A$ is *rc-discrete* if each of its points is rc-discrete. It is clear that in a regular space an isolated point of $A$ is an rc-discrete point of $A$. Therefore, discrete subspaces of regular spaces are rc-discrete. On the other hand, in an extremally disconnected space rc-discrete points of a set are isolated and rc-discrete sets are discrete. Prop. 3.4 below shows that the crowded Cantor set $C$ in $\mathbb{R}$ with the usual topology is rc-discrete. Moreover, by Prop. 3.4 and Lemma 4.7 of [8] in any crowded $\mathrm{T}_3$ first countable space there exist a countably infinite crowded rc-discrete subset. Our next result generalizes, in case of $\mathrm{T}_3$ spaces, the well known fact that in $\mathrm{T}_1$ crowded spaces discrete sets are nowhere dense.

**Proposition 3.4.** *In a crowded* $\mathrm{T}_1$ *space rc-discrete sets are nowhere dense.*

**Proof.** Let $A$ be an rc-discrete set in a crowded $\mathrm{T}_1$ space $X$ and $U$ be a nonempty open set in $X$. We show that there exists a nonempty set $V$ such that $V\subset U$ and $V\cap A=\emptyset$. Assume that $U\cap A\neq\emptyset$ and let $x\in U\cap A$. Since $x$ is an rc-discrete point of $A$ there exists an open set $W$ such that $\mathrm{Cl}\,W\cap A=\{x\}$. Now $V=W\cap U-\{x\}\neq\emptyset$, otherwise $x$ would be an isolated point in $X$. Clearly, $V$ is the desired open set. $\Diamond$

To see that nowhere dense sets in crowded $\mathrm{T}_1$ spaces are not necessarily rc-discrete consider the following example.

**Example 3.5.** Let $\mathbf{I}$ be the unit interval, $C$ the Cantor set in $\mathbf{I}$, and

$X = \mathrm{E}\mathbf{I}$ the Iliadis absolute of $\mathbf{I}$. Since $k_X : X \to \mathbf{I}$ is continuous closed irreducible surjection, $f^{-1}(C)$ is a closed nowhere dense set in the crowded compact Hausdorff extremally disconnected space $X$. Obviously, $f^{-1}(C)$ is not rc-discrete, otherwise it would be discrete and finite, implying $|C| < \omega$.

Our next result is not only important in the present setting, but also in relation to cardinal functions.

**Lemma 3.6.** *In* $\mathrm{T}_3$ *first countable spaces nowhere dense sets are rc-discrete.*

**Proof.** Let $A$ be a closed nowhere dense set in a $\mathrm{T}_3$ first countable space $X$ and let $x \in A$. We show that $x$ is an rc-discrete point of $A$. Obviously, we may assume that $x$ is a limit point of $A$. Choose a decreasing open base $\mathcal{U}_x = \{U_n | n \in \mathbb{N}\}$ at $x$. Since $X$ is $\mathrm{T}_3$ and $A$ is nowhere dense there exists a nonempty open set $V_1$ such that $\mathrm{Cl}\, V_1 \subset U_1$ and $\mathrm{Cl}\, V_1 \cap A = \emptyset$. Let $U_{n(2)} \in \mathcal{U}_x$ and $U_{n(2)} \subset U_1 - \mathrm{Cl}\, V_1$. It is clear that with recursion on $n \in \mathbb{N}$ we can construct a decreasing sequence $\{U_{n(k)} | U_{n(k)} \in \mathcal{U}_x \text{ and } k \in \mathbb{N}\}$ ($n(1) = 1$ and $n(k)$ is increasing) and a sequence of nonempty open sets $\{V_k | k \in \mathbb{N}\}$ so that $U_{n(k+1)} \subset U_{n(k)} - \mathrm{Cl}\, V_k$, $\mathrm{Cl}\, V_k \subset U_{n(k)}$ and $\mathrm{Cl}\, V_k \cap A = \emptyset$. Clearly, $x \in \mathrm{Cl}\, V$ where $V = \cup\{V_k | k \in \mathbb{N}\}$. We claim that $\mathrm{Cl}\, V \cap A = \{x\}$. Let $y \in A$ and $y \neq x$. There are disjoint open sets $W$ and $U_n \in \mathcal{U}_x$ such that $y \in W$. Let $k \in \mathbb{N}$ with $U_{n(k)} \subset U_n$ and let $W' = W - \bigcup_{i=1}^{k-1} \mathrm{Cl}\, V_i$. Clearly, $W'$ is open and $W' \cap V = \emptyset$. So, $y \notin \mathrm{Cl}\, V$. $\Diamond$

Recall that a $\mathrm{T}_3$ sace $X$ is called an *accessibility space* [23] if for every limit point $x$ of a set $A$ there exists a closed set $C$ such that $x$ is a limit point of $C$ and $C \subset A \cup \{x\}$. Note that $\mathrm{T}_3$ first countable, moreover Frechét-Urysohn spaces, are accessibility spaces.

**Lemma 3.7.** *Nowhere dense sets in normal* $\mathrm{T}_1$ *accessibility spaces with* $G_\delta$*-points are rc-discrete.*

**Proof.** Let $A$ be a closed nowhere dense set in a space $X$ satisfying the conditions and let $x$ be a limit point of $A$. Then $x$ is a limit point of $(X - A) \cup \{x\}$, otherwise there would exist an open neighbourhood $U$ of $x$ contained in $A$ contradicting the assumption that $A$ is nowhere dense. Since $X$ is an accessibility space, there exists a closed set $C$ such that $x$ is a limit point of $C$ and $C \cap A = \{x\}$. Therefore, $A - \{x\}$ and $C - \{x\}$ are disjoint closed sets in the subspace $X - \{x\}$. Since $\{x\}$ is a $G_\delta$-set, $X - \{x\}$ is an $F_\sigma$-set and hence normal as a subspace of a normal space. There exist disjoint open sets $U$ and $V$ in $X - \{x\}$

having disjoint closures in $X - \{x\}$ such that $A - \{x\} \subset U$ and $C - \{x\} \subset V$. Clearly $U$ and $V$ are open in $X$ and $\mathrm{Cl}\,U \cap \mathrm{Cl}\,V = \{x\}$. So, $\mathrm{Cl}\,V \cap A = \{x\}$ and the result follows. $\Diamond$

We are now ready for main results in this section. Recall that a Hausdorff space $X$ is called a *Luzin space* [12] if (a) Every nowhere dense set in $X$ is countable, (b) $X$ has at most countably many isolated points, and (c) $X$ is uncountable. As shown in [12], Luzin spaces are zero dimensional and hereditarily Lindelöf. It is well known that the Continuum Hypothesis (CH) implies that $\mathbb{R}$ with the usual topology is an extension of a Luzin space. On the other hand, Martin's axiom plus $\neg$CH implies that there are no Luzin spaces [12].

**Theorem 3.8.** *Let $X$ be an uncountable first countable* $\mathrm{T}_3$ *space with at most countably many isolated points. Then $X$ is rc-Lindelöf iff $X$ is a Luzin space.*

**Proof.** The necessity follows from Lemma 3.6 and the simple fact that closed rc-discrete sets in $\mathrm{T}_3$ rc-Lindelöf spaces are countable. To establish sufficiency let $\mathcal{U} = \{F_\alpha \in \mathrm{RC}(X) | \alpha \in A\}$ be a cover of a Luzin space $X$. By Zorn's lemma there exists a pairwise disjoint open refinement $\mathcal{V}$ of $\mathrm{Int}\,\mathcal{U} = \{\mathrm{Int}\,F_\alpha | \alpha \in A\}$ such that $\cap\mathcal{V}$ is dense in $\cup\mathrm{Int}\,\mathcal{U}$ and consequently dense in $X$. Hence $X - \cup\mathcal{V}$ is nowhere dense and $|X - \cup\mathcal{V}| \leq \omega$. Also, since $X$ is hereditarily Lindelöf, $|\mathcal{V}| \leq \omega$. Now, it is easy to see that there is a countable subcover of $\mathcal{U}$. $\Diamond$

Since $\mathrm{T}_3$ rc-Lindelöf spaces are normal, by Lemma 3.7 we have the following generalization of the previous result.

**Theorem 3.9.** *Let $X$ be an uncountable* $\mathrm{T}_3$ *accessibility space with $G_\delta$-points and at most countably many isolated points. Then $X$ is rc-Lindelöf iff $X$ is a Luzin space.*

It would be of interest to find an example of a $\mathrm{T}_3$ uncountable crowded rc-Lindelöf space with $G_\delta$-points.

We next consider linearly ordered topological spaces (LOTS) and generalized ordered spaces (GO-spaces). Recall that a GO-space is a space which can be embedded in a LOTS. We assume that GO-spaces are $\mathrm{T}_1$. In proving that the statement that there exists an uncountable rc-Lindelöf GO-space is independent of ZFC the crucial part is played by a result due to H. Bennett and D. Lutzer [1], namely, perfectly $\kappa$-normal GO-spaces are perfectly normal.

**Theorem 3.10.** *Let $X$ be an uncountable GO-space. Then $X$ is rc-Lindelöf iff $X$ is a Luzin space.*

**Proof.** Since $X$ is perfectly $\kappa$-normal, $X$ is perfectly normal and hence first countable [1]. Being perfect and Lindelöf, $X$ is hereditarily Lindelöf and thus has at most countably many isolated points. By Th. 3.8, $X$ is a Luzin space. $\Diamond$

Our next result gives a sufficient condition for a closed subset of a $T_3$ rc-Lindelöf space to be rc-Lindelöf. In the proof we use the following interesting characterization of hereditarily normal spaces [17]. A space $X$ is hereditarily normal iff whenever $U$ is an open set in $A \subset X$ there exists an open set $V(U)$ such that $U = A \cap V(U)$ and $\mathrm{Cl}_A U = A \cap \cap \mathrm{Cl}_X V(U)$.

**Proposition 3.11.** *Let $X$ be a $T_1$ hereditarily normal rc-Lindelöf space. If $A \subset X$ is closed, then $A$ is rc-Lindelöf.*

**Proof.** Let $\mathcal{U} = \{F_\alpha \in \mathrm{RC}(A) | \alpha \in A\}$ be a regular closed cover of $A$. Clearly, $F_\alpha = \mathrm{Cl}_A U_\alpha$ for some open $U_\alpha$ in $A$. Since $X$ is hereditarily normal $\mathcal{V} = \{\mathrm{Cl}_X V(U_\alpha) | \alpha \in A\}$ is a regular closed cover of $A$ in $X$. For each $x \in X - A$ let $V_x$ be an open set in $X$ with $x \in V_x$ and $\mathrm{Cl}\, V_x \cap \cap A = \emptyset$. Then $\mathcal{V} \cup \{\mathrm{Cl}\, V_x | x \in X - A\}$ is a cover of $X$ by regular closed sets in $X$ and thus the result follows because $X$ is rc-Lindelöf. $\Diamond$

Since perfect $\kappa$-normality is implied both by $T_3$ rc-Lindelöfness and perfect normality it is of interest to consider $T_3$ rc-Lindelöf perfect spaces. In this direction we have the following theorem.

**Theorem 3.12.** *In $T_1$ perfectly normal rc-Lindelöf spaces compact sets are countable.*

**Proof.** It is enough to show that compact $T_1$ perfectly normal rc-Lindelöf spaces are countable. Let $X$ be such a space. Suppose that $X$ is not scattered (i.e., $X$ possesses a nonempty crowded subset). This implies that there exists a continuous surjection from $X$ to the unit interval $\mathbf{I}$ [20]. Obviously, $f$ is closed. Now there exists a closed subset $A$ of $X$ such that $f(A) = \mathbf{I}$ and $g = f|A$ is irreducible [16]. Let $C$ be the Cantor set in $\mathbf{I}$ and set $B = g^{-1}(C)$. Since $g$ is continuous closed and irreducible, $B$ is closed and nowhere dense in $A$. Note that $A$ is compact and perfectly normal and hence first countable. Therefore, $B$ is rc-discrete by Lemma 3.6. From Prop. 3.11 it follows that $A$ is rc-Lindelöf and hence $|B| \leq \omega$. This contradicts $|C| = 2^\omega$ and we conclude that $X$ is scattered. But it is well known that compact Hausdorff scattered spaces with $G_\delta$-points are countable. $\Diamond$

## 4. Countably rc-compact spaces

In [22] T. Thompson proved that $T_3$ first countable rc-compact spaces are finite. As we will see this result holds if rc-compactness is replaced by countable rc-compactness. Countably rc-compact spaces are also finite in some other important cases. A space $X$ is defined to be *countably rc-compact* if every countable cover of $X$ regular closed sets has a finite subcover. First of all, we observe that countable $T_3$ countably rc-compact spaces are finite. In case that a $T_3$ countable infinite space $X$ is countably rc-compact it would be rc-compact since it is rc-Lindelöf. Therefore, $X$ would be compact and extremally disconnected and consequently would contain a copy of $\beta\mathbb{N}$ contradicting $|X| = \omega$. This observation easily follows from a generalization of the mentioned Thompson's result. Recall now that a space $X$ is *feebly compact* if every countable open cover of $X$ has a finite subfamily whose union is dense in $X$ and that a Tychonoff space is feebly compact iff it is pseudocompact [16]. It is evident that countably rc-compact spaces are feebly compact. The unit interval with the usual topology shows that the converse does not hold. On the other hand, extremally disconnected feebly compact spaces are countably rc-compact but not necessarily countably compact. As shown in [11] the Iliadis absolute of the deleted Tychonoff plank is such a space.

The proofs of the following two useful results are left to the reader.

**Theorem 4.1.** *A space $X$ is countably rc-compact iff every countable regular open filter base in $X$ has a nonempty intersection.*

**Proposition 4.2.** (a) *In countably rc-compact spaces, regular open sets and regular closed sets are countably rc-compact.*

(b) *Countable rc-compactness is contagious.*

(c) *A finite union of regular open countably rc-compact subspaces is countably rc-compact.*

(d) *Countable rc-compactness is not productive.*

(e) *Countable rc-compactness is preserved under continuous open surjections but not under continuous surjections. Also, countable rc-compactness is not inversely preserved under continuous open perfect surjections.*

Now we extend and prove differently Th. 3 from [22].

**Theorem 4.3.** $T_3$ *countably rc-compact spaces with $G_\delta$-points are finite.*

**Proof.** Let $X$ be a $T_3$ countably rc-compact space with $G_\delta$-points. Since $X$ is $T_3$ feebly compact, $X$ is first countable. We will show that $X$ is extremally disconnected and since $T_2$ extremally disconnected first countable spaces are discrete the result will follow. Suppose that there exist a $U \in \mathrm{RO}(X)$ and $x \in \mathrm{Bd}\,U$. Let $\mathcal{U}_x = \{U_n | n \in \mathbb{N}\}$ be a decreasing base of regular open sets at $x$. Since $\mathcal{F} = \{U_n \cap U | n \in \mathbb{N}\}$ is a countable filter base of regular open sets in $U$ and $U$ is countably rc-compact by Prop. 4.2 (a), it follows from Th. 4.1 that $U \cap \bigcap\{U_n | n \in \in \mathbb{N}\} \neq \emptyset$. This contradicts the assumption that $x \notin U$. $\Diamond$

As a consequence of Th. 4.3 we have the mentioned result that countable $T_3$ countably rc-compact spaces are finite because every subset of a $T_1$ countable space is a $G_\delta$-set.

**Theorem 4.4.** *Let $X$ be a Tychonoff countably rc-compact space. Then $X$ is extremally disconnected iff it is perfectly $\kappa$-normal.*

**Proof.** Assume that $X$ is perfectly $\kappa$-normal and let $U \in \mathrm{RO}\,(X)$. Then $U \in \mathrm{Coz}\,(X)$ and hence $U$ is a countable union of regular closed sets in $X$. Clearly, these sets are also regular closed in $U$. Since $U$ is countably rc-compact by Prop. 4.2 (a), $U$ is a finite union of regular closed sets in $X$. Therefore, $U$ is closed and consequently $X$ is extremally disconnected. $\Diamond$

In Section 3 we have observed that $\mathbb{N}^*$ is not rc-Lindelöf. Since there is a continuous surjection from $\mathbb{N}^*$ to the one point compactification of $\mathbb{N}$ and $\mathbb{N}^*$ is a $\mathrm{P}'$-space, similar arguments show that $\mathbb{N}^*$ is not countably rc-compact. Our next result generalizes this observation.

**Theorem 4.5.** *Tychonoff countably rc-compact $P'$-spaces are finite.*

**Proof.** Let $X$ be a Tychonoff countably rc-compact space and let $U \in \in \mathrm{Coz}\,(X)$. Since $X$ is $\mathrm{P}'$, $U \in \mathrm{RO}\,(X)$ and hence by Prop. 4.2 (a), $U$ is countably rc-compact. By the same argument as in the proof of Th. 4.4, $U$ is closed. Therefore, $X$ is a P-space i.e., cozero sets are closed. But $T_3$ feebly compact P-spaces are finite [16]. $\Diamond$

In order to establish our final result we need a useful concept of a $\pi$-set in a space. A set $A$ in a space $X$ is called a *$\pi$-set* [24] if it is an intersection of finitely many regular closed sets. In [9] V. Fedorčuk showed that closed $G_\delta$-sets in LOTS are $\pi$-sets.

**Theorem 4.6.** *Countably rc-compact GO-spaces are finite.*

**Proof.** First, we establish the result for LOTS. Let $X$ be a countably rc-compact LOTS and let $U \in \mathrm{Coz}\,(X)$. Since $U$ is a complement of a $\pi$-set, $U$ is a finite union of regular open sets and hence $U$ is countably rc-compact by Prop. 4.2 (a) and (c). By the same argument as in the proof of Th. 4.4, $U$ is closed and thus $X$ is a P-space. By Th. 4.5, $X$ is finite. Now, the well known result that a GO-space is densily embedded in a LOTS [13] and Prop. 4.2 (b) imply the result. $\Diamond$

Finally we remark that in [9] it is shown that some other classes of spaces have a property that closed $G_\delta$-sets are $\pi$-sets. The same argument as in the proof of Th. 4.6 shows that these spaces are finite if countably rc-compact.

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