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DIRECT KINEMATICS OF DOUBLE-TRIANGULAR PARALLEL MANIPULATORS

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Abstract: Three types of double-triangular mechanism are introduced. These contain planar, spherical or spatial triangle pairs that move with respect to each other while remaining in line contact. Using planar, spherical and spatial trigonometry, the three pertinent direct kinematic problems are solved and typical numerical examples are presented.

1. Introduction

Parallel manipulators are known to offer some advantages over conventional serial manipulators, namely, higher accuracy, superior structural stiffness and lower inertia. However, existing parallel robots, like their serial counterparts, have slender legs that are undesirably flexible, which compromises manipulator accuracy. Three versions of a new class of parallel manipulators, namely, the planar, spherical and spatial double-triangular (DT) manipulators, as shown in Figs. 1a, 1b and 2b, respectively, have been designed to overcome this drawback. Each of these manipulators consists of two rigid bodies, namely, two planar, spherical or spatial triangles, connected by three multi-dof legs. An interesting feature of this class of manipulators is the presence of virtually zero-length legs, to avoid the objectionable flexibility of long-legged conventional parallel robots, while retaining desirable features like high load-carrying capacity and speed.

Fig. 1: a) Planar 3-dof DT manipulator;
b) Spherical 3-dof DT manipulator

As the name indicates, DT manipulators are based on two triangles supplied with actuators to provide motion. The strokes of these actuators produce a desired relative position and orientation, *pose*, of the two triangles. The problem to determine the relative pose for given actuator strokes is called *direct kinematics* (DK). In general, the DK of parallel manipulators is a challenging problem [2, 3, 5–8], the mechanisms under study not escaping to this rule. The kinematics of the planar and spherical mechanisms are solved with planar and spherical trigonometry. This has inductively led us to expand the solution concept to three dimensions by invoking the well-established, although somehow less familiar principles of *spatial trigonometry* to satisfactorily treat the direct kinematics of the spatial DT mechanisms. Spatial trigonometric relationships are expressed with *dual-number algebra* [1, 10, 11]. This tool is used to describe the geometric relations among lines in space by treating them as relations among points on the surface of a sphere centred at the origin of the 4-dimensional *dual space*. This sphere is thus called the *dual sphere*. We give below a brief review of spatial trigonometry.

2. The spatial triangle

A spatial triangle consists of three skew lines and their three common perpendiculars, as depicted in Fig. 2a. Here, the three lines are labelled $\{\mathcal{L}_i\}_1^3$, their corresponding normals being $\{\mathcal{N}_i\}_1^3$, where $\mathcal{N}_1$ is the common normal between lines $\mathcal{L}_2$ and $\mathcal{L}_3$, $\mathcal{N}_2$ is that between $\mathcal{L}_1$ and $\mathcal{L}_3$, with a similar definition for $\mathcal{N}_3$. The lines are given by the three dual vectors $\{\lambda_i^*\}_1^3$, defined as

$$\lambda_i^* \equiv \lambda_i + \epsilon\lambda_{i0}, \quad i = 1, 2, 3 \tag{1}$$

where λ_i and λ_{i0} are the direction and the moment vectors of $\mathcal{L}_i$ about an origin, respectively, while ϵ is the *dual unit*.

Fig. 2: a) Spatial triangle;
b) Spatial DT manipulator

Moreover, the three common perpendiculars of the foregoing lines, $\{\mathcal{N}_i\}_1^3$, are given by the three dual vectors $\{\nu_i^*\}_1^3$, defined as

$$\nu_i^* \equiv \nu_i + \epsilon \nu_{i0}, \quad i = 1, 2, 3 \tag{2}$$

where ν_i and ν_{i0} represent the direction and the moment vectors of line $\mathcal{N}_i$ about the same origin, respectively.

Similar to planar and spherical trigonometries, one may define three *sides* of the triangle by the associated dual angles, namely, $\hat{\alpha}_i = \alpha_i + \epsilon \nu_i$, for $i = 1, 2, 3$, where ν_i is the distance and α_i is the twist angle between $\mathcal{L}_{i+1}$ and $\mathcal{L}_{i-1}$, the sum and the difference in the subscripts throughout this paper being understood as *modulo 3*. The three *angles* of the triangle, similarly, are defined as $\hat{\theta}_i = \theta_i + \epsilon \lambda_i$, for $i = 1, 2, 3$, where λ_i is the distance and θ_i is the twist angle between $\mathcal{N}_{i+1}$ and $\mathcal{N}_{i-1}$.

If the three lines represented by $\{\lambda_i^*\}_1^3$ intersect at a common point, then the triangle reduces to a spherical triangle. On the other hand, if the three lines are parallel, the triangle becomes planar.

2.1. Trigonometric identities

A unit dual quaternion is a screw operator that transforms a line into another line through a screw motion [10]. With this concept, the relationship between λ_1^*, λ_2^* and λ_3^* can be expressed as

$$\lambda_1^* = \hat{\nu}_2^* \, \lambda_3^* \tag{3}$$

$$\lambda_2^* = \hat{\nu}_3^* \, \lambda_1^* \tag{4}$$

$$\lambda_3^* = \hat{\nu}_1^* \, \lambda_2^* \tag{5}$$

where $\hat{\nu}_i^*$, for $i = 1, 2, 3$, is a unit dual quaternion, given as

$$\hat{\nu}_i^* = \cos \hat{\alpha}_i + \nu_i^* \sin \hat{\alpha}_i. \tag{6}$$

Substituting the value of λ_3^* from eq.(5) into eq.(3) yields

$$\lambda_1^* = \hat{\nu}_2^* \, \hat{\nu}_1^* \, \lambda_2^*. \tag{7}$$

Moreover, substituting the value of λ_2^* from eq.(4) into eq.(7), upon simplification, leads to

$$\hat{\nu}_2^* \, \hat{\nu}_1^* \, \hat{\nu}_3^* = 1 \tag{8}$$

The foregoing identity is called the *angular closure* equation for spatial triangles; it states that the composition of the three consecutive screw motions of λ_1^*, represented by $\hat{\nu}_3^*$, $\hat{\nu}_1^*$ and $\hat{\nu}_2^*$, is an identity transformation of λ_1^*, via the intermediate poses λ_2^* and λ_3^*.

3. Direct kinematics

3.1. Planar manipulator

Consider two planar triangles, $\mathcal{P}$ and $\mathcal{Q}$, with vertices $P_1P_2P_3$ and $Q_1Q_2Q_3$, respectively, as shown in Fig. 1a. Triangle $\mathcal{P}$ is designated the *fixed triangle* (FT), while $\mathcal{Q}$ is the *movable triangle* (MT), such that P_2P_3 intersects Q_2Q_3 at point R_1, P_3P_1 intersects Q_3Q_1 at R_2 and P_1P_2 intersects Q_1Q_2 at R_3. Moreover, R_i, for $i = 1, 2, 3$, cannot lie outside its corresponding vertices. Thus, feasible or admissible motions maintain R_i within edges $Q_{i+1}Q_{i-1}$ and $P_{i+1}P_{i-1}$, for $i = 1, 2, 3$.

The motion of triangle $\mathcal{Q}$ can thus be described through changes in the edge-length variables, ρ_i, which locate R_i along a side of $\mathcal{P}$, measured from P_{i+1}, for $i = 1, 2, 3$. The non-negative displacements ρ_i are assumed to be produced by actuators, and hence, they are termed the *actuator coordinates*. The coordinates of the moving triangle $\mathcal{Q}$, in turn, are the set of variables used to define its pose. Note that the Cartesian coordinates of the three vertices of $\mathcal{Q}$ can be used to define this pose.

The DK problem may be formulated as: *Given the actuator coordinates ρ_i, for $i = 1, 2, 3$, find the Cartesian coordinates of the vertices of triangle $\mathcal{Q}$.*

We solve this problem by *kinematic inversion*, i.e., by fixing the MT $\mathcal{Q}$ and letting the FT $\mathcal{P}$ to accommodate itself to the constraints imposed. To this end, we define points R_i at given distances ρ_i, for $i = 1, 2, 3$, on the edges of $\mathcal{P}$, thereby defining a triangle $R_1R_2R_3$, henceforth termed triangle $\mathcal{R}$, that is fixed to $\mathcal{P}$. Next, we let d, e and f be the lengths of the sides of this triangle. The problem now consists of finding the set of all possible positions of triangle $\mathcal{R}$ for which vertex R_i lies within the side $Q_{i+1}Q_{i-1}$, for $i = 1, 2, 3$, as shown in Fig. 3a. By carrying $\mathcal{R}$ back into its fixed configuration, while attaching $\mathcal{Q}$ rigidly to it, we determine the set of possible configurations of the MT for the given values of actuator coordinates.

Fig. 3: Triangles $\mathcal{Q}$ and $\mathcal{R}$ a) Planar;
b) Spherical

In Fig. 3a we note that each vertex R_i is common to three angles labelled with numbers 1, 2 and 3. We will denote these angles by a subscripted capital letter. The subscript indicates one of the three angles common to that vertex, while the capital letter corresponds to the lower-case label of the opposite side of the triangle $R_1R_2R_3$. We thus have at vertices R_1, R_2 and R_3 the angles D_i, E_i and F_i, for $i = 1, 2, 3$.

Considering triangle $Q_1R_3R_2$, the law of sines for triangles yields

(9) $$\overline{Q_1R_2} = a_1 \sin(F_1)$$

where $a_1 = d/\sin(Q_1)$. Similarly, for triangle $Q_3R_2R_1$ we have

(10) $$\overline{Q_3R_2} = a_2 \sin(D_3)$$

where $a_2 = f/\sin(Q_3)$. Adding sidewise eq.(9) to eq.(10) gives

(11) $$a_1 \sin(F_1) + a_2 \sin(D_3) = b$$

where $b = \overline{Q_1Q_3}$.

From triangle $Q_2R_1R_3$, we have

(12) $$D_1 = \pi - F_3 - Q_2.$$

But

(13) $$F_3 = \pi - F_1 - F_2.$$

Substitution of F_3 from eq.(13) into eq.(12) yields

(14) $$D_1 = F_1 + F_2 - Q_2.$$

Again, we have

$$D_3 = \pi - D_1 - D_2. \tag{15}$$

Substitution of D_1 from eq.(14) into eq.(15) yields, in turn,

$$D_3 = G - F_1 \tag{16}$$

where $G = \pi - D_2 - F_2 + Q_2$. Substituting the expression for $\sin(D_3)$ from eq.(16) into eq.(11), we obtain

$$b_1 \sin(F_1) + b_2 \cos(F_1) = b \tag{17}$$

where $b_1 = a_1 - a_2 \cos(G)$ and $b_2 = a_2 \sin(G)$. In eq.(17), we substitute now the equivalent expressions for cosines and sines, i.e., $\cos(F_1) = (1 - x^2)/(1 + x^2)$ and $\sin(F_1) = 2x/(1 + x^2)$, where $x \equiv \tan(F_1/2)$. Therefore, eq.(17), upon simplification, leads to

$$c_1 x^2 + c_2 x + c_3 = 0 \tag{18}$$

where $c_1 = -b_2 - b$, $c_2 = 2b_1$ and $c_3 = b_2 - b$. Solving eq.(18) for x gives

$$x = \frac{-b_1 \pm \sqrt{b_1^2 + b_2^2 - b^2}}{-(b_2 + b)}. \tag{19}$$

The above expression leads to the result below:

Theorem 1. *Given two triangles $\mathcal{R}$ and $\mathcal{Q}$, we can inscribe $\mathcal{R}$ in $\mathcal{Q}$ in at most two poses such that vertex R_i is located on the edges $Q_{i+1}Q_{i-1}$ of triangle $\mathcal{Q}$, for $i = 1, 2, 3$.*

Example 3.1.1. Consider the following sides assigned to the triangles $\mathcal{P}$ and $\mathcal{Q}$:

$$Q_1Q_2 = 0.4\,\mathrm{m}, \qquad Q_2Q_3 = 0.5\,\mathrm{m}, \quad Q_3Q_1 = 0.6\,\mathrm{m}$$

$$P_1P_2 = 0.29065\,\mathrm{m}, \quad P_2P_3 = 0.5\,\mathrm{m}, \quad P_3P_1 = 0.47875\,\mathrm{m}$$

Choose three points, R_1, R_2 and R_3, located by three actuator coordinates specified as $\rho_1 = 0.2\,\mathrm{m}$, $\rho_2 = 0.14161\,\mathrm{m}$ and $\rho_3 = 0.03064\,\mathrm{m}$. These values produce the lengths d, e and f given below:

$$d = 0.33166\,\mathrm{m}, \qquad e = 0.26458\,\mathrm{m}, \qquad f = 0.2\,\mathrm{m}$$

The two roots of eq.(19) are: $x_1 = 1.0788$, $x_2 = 0.4447$ i.e., $(F_1)_1 = 94.34°$, $(F_1)_2 = 48°$. Equations (9–16) are used to compute the other parameters, which leads to two poses of the triangle.

3.2. Spherical manipulator

Consider a unit sphere with center at O and a spherical triangle $P_1P_2P_3$, referred to as $\mathcal{P}$, on its surface. Moreover, a second spherical triangle, labelled $Q_1Q_2Q_3$, likewise referred to as $\mathcal{Q}$, is defined. Furthermore, the side P_2P_3 of $\mathcal{P}$, arbitrarily regarded as the FT, intersects the arc Q_2Q_3 of $\mathcal{Q}$, regarded as the MT, at point R_1. We denote by R_2 and R_3 the other intersection points, that are defined correspondingly. Moreover R_i, for $i = 1, 2, 3$, cannot lie outside its corresponding vertices. Thus, feasible or admissible motions maintain R_i within edges $Q_{i+1}Q_{i-1}$ and $P_{i+1}P_{i-1}$, for $i = 1, 2, 3$. Thus, the motion of triangle $\mathcal{Q}$ can be described through the arc lengths μ_i of Fig. 1b, or *actuator coordinates*, for $i = 1, 2, 3$. Likewise, the Cartesian coordinates of the moving triangle $\mathcal{Q}$ are the set of variables defining its orientation. Note that the Cartesian coordinates of the three vertices of $\mathcal{Q}$ can be determined once its orientation is given.

Similar to the direct kinematics of the planar DT manipulator, the same problem, as pertaining to the spherical manipulator, may be formulated as: *Given the actuator coordinates μ_i, for $i = 1, 2, 3$, find the Cartesian coordinates of the vertices of triangle $\mathcal{Q}$.*

Again, we solve this problem by *kinematic inversion*, i.e., by fixing the MT $\mathcal{Q}$ and letting the FT $\mathcal{P}$ accommodate itself to the constraints imposed. To this end, we define points R_i at given arc lengths μ_i, for $i = 1, 2, 3$, on the edges of $\mathcal{P}$, thereby defining a triangle $R_1R_2R_3$, henceforth termed triangle $\mathcal{R}$, that is fixed to $\mathcal{P}$. Next, we let d, e and f be the sides of this triangle. The problem now consists of finding the set of all possible orientations of triangle $\mathcal{R}$ for which vertex R_i lies within the side $Q_{i+1}Q_{i-1}$, for $i = 1, 2, 3$, as shown in Fig. 3b. By carrying $\mathcal{R}$ back into its fixed configuration, while attaching $\mathcal{Q}$ rigidly to it, we determine the set of possible configurations of the MT for the given values of actuator coordinates. In Fig. 3b we note that each vertex R_i is common to the three spherical angles labelled with numbers 1, 2 and 3. Similar to the planar mechanism, we label these angles D_i, E_i and F_i, for $i = 1, 2, 3$.

We introduce now the definitions below:

$$s \equiv \frac{d+e+f}{2}, \quad k \equiv \sqrt{\frac{\sin(s-d)\ \sin(s-e)\ \sin(s-f)}{\sin(s)}}. \tag{20}$$

From spherical trigonometry, we have

$$D_2 = 2\arctan\left(\frac{k}{\sin(s-d)}\right) \tag{21}$$

$$E_2 = 2\arctan(\frac{k}{\sin(s-e)}) \tag{22}$$

$$F_2 = 2\arctan(\frac{k}{\sin(s-f)}). \tag{23}$$

Consider now the spherical triangle $Q_1R_3R_2$. Using the law of cosines for spherical triangles, we have

$$\cos Q_1 = -\cos F_1 \cos E_3 + \sin F_1 \sin E_3 \cos d. \tag{24}$$

Similarly, for the spherical triangles $Q_2R_1R_3$ and $Q_3R_2R_1$ we have

$$\cos Q_2 = -\cos D_1 \cos F_3 + \sin D_1 \sin F_3 \cos e \tag{25}$$

$$\cos Q_3 = -\cos E_1 \cos D_3 + \sin E_1 \sin D_3 \cos f. \tag{26}$$

However,

$$D_3 = \pi - D_1 + D_2 \tag{27}$$

$$E_3 = \pi - E_1 + E_2 \tag{28}$$

$$F_3 = \pi - F_1 + F_2. \tag{29}$$

Substitution of the expressions for $\cos E_3$ and $\sin E_3$ from eq.(28) into eq.(24), we obtain

$$a_{11}c_1c_2 + a_{12}c_1s_2 + a_{13}s_1s_2 + a_{14}s_1c_2 + a_{15} = 0 \tag{30}$$

where

$$a_{11} = \cos E_2, \quad a_{12} = -\sin E_2, \quad a_{13} = \cos d \cos E_2$$
$$a_{14} = \cos d \sin E_2, \quad a_{15} = -\cos Q_1, \quad c_1 = \cos F_1$$
$$s_1 = \sin F_1, \quad c_2 = \cos E_1, \quad s_2 = \sin E_1.$$

Similarly, substitution of eq.(29) into eq.(25) yields:

$$a_{21}c_3c_1 + a_{22}c_3s_1 + a_{23}s_3s_1 + a_{24}s_3c_1 + a_{25} = 0 \tag{31}$$

where

$$a_{21} = \cos F_2, \quad a_{22} = -\sin F_2, \quad a_{23} = \cos e \cos F_2$$
$$a_{24} = \cos e \sin F_2, \quad a_{25} = -\cos Q_2, \quad c_3 = \cos D_1$$
$$s_3 = \sin D_1.$$

Likewise, substitution of eq.(27) into eq.(26) yields:

$$a_{31}c_2c_3 + a_{32}c_2s_3 + a_{33}s_2s_3 + a_{34}s_2c_3 + a_{35} = 0 \tag{32}$$

where

$$a_{31} = \cos D_2, \quad a_{32} = -\sin D_2, \quad a_{33} = \cos f \cos D_2$$
$$a_{34} = \cos f \sin D_2, \quad a_{35} = -\cos Q_3.$$

Equations (30–32) must be solved simultaneously to determine the values of angles D_1, E_1 and F_1. In the above equations, we substitute now the equivalent expressions for cosines and sines, i.e., $c_i = (1-x_i^2)/(1+ + x_i^2)$ and $s_i = 2x_i/(1+x_i^2)$, for $i = 1,2,3$, where $x_1 \equiv \tan(F_1/2)$, $x_2 \equiv \tan(E_1/2)$ and $x_3 \equiv \tan(D_1/2)$. Upon simplification, eqs.(30–32) lead to three trivariate polynomial equations in x_1, x_2 and x_3, namely,

$$d_1 x_2^2 + d_2 x_2 + d_3 = 0 \tag{33}$$

$$d_4 x_2^2 + d_5 x_2 + d_6 = 0 \tag{34}$$

$$d_7 x_3^2 + d_8 x_3 + d_9 = 0 \tag{35}$$

where

$$\begin{aligned}
d_1 &= (a_{11}+a_{15})x_1^2 - 2a_{14}x_1 + (a_{15}-a_{11})\\
d_2 &= -2a_{12}x_1^2 + 4a_{13}x_1 + 2a_{12}\\
d_3 &= (a_{15}-a_{11})x_1^2 + 2a_{14}x_1 + (a_{15}-a_{11})\\
d_4 &= (a_{31}+a_{35})x_3^2 - 2a_{34}x_3 + (a_{35}-a_{31})\\
d_5 &= -2a_{32}x_3^2 + 4a_{33}x_3 + 2a_{32}\\
d_6 &= (a_{35}-a_{31})x_3^2 + 2a_{34}x_3 + (a_{35}-a_{31})\\
d_7 &= (a_{21}+a_{25})x_1^2 - 2a_{24}x_1 + (a_{25}-a_{21})\\
d_8 &= -2a_{22}x_1^2 + 4a_{23}x_1 + 2a_{22}\\
d_9 &= (a_{25}-a_{21})x_1^2 + 2a_{24}x_1 + (a_{25}-a_{21}).
\end{aligned}$$

We now eliminate x_2 from eqs.(33) and (34), using Bezout's method [9]. The resulting equation thus contains only x_1 and x_3, namely,

$$\det\begin{bmatrix}\Delta_{11} & \Delta_{12}\\ \Delta_{21} & \Delta_{11}\end{bmatrix} = 0 \tag{36}$$

where quantities Δ_{11}, Δ_{12} and Δ_{21} are defined below:

$$\Delta_{11} \equiv \det\begin{bmatrix}d_1 & d_3\\ d_4 & d_6\end{bmatrix},\ \Delta_{12} \equiv \det\begin{bmatrix}d_5 & d_2\\ d_4 & d_1\end{bmatrix},\ \Delta_{21} \equiv \det\begin{bmatrix}d_2 & d_3\\ d_5 & d_6\end{bmatrix}.$$

After expansion and simplification, eq.(36) reduces to

$$A_1 x_3^4 + A_2 x_3^3 + A_3 x_3^2 + A_4 x_3 + A_5 = 0 \tag{37}$$

where

$$A_i = \sum_{p=0}^{4} A_{ip} x_1^p \quad i = 1, \cdots, 5 \tag{38}$$

and the coefficients A_{ip} are constants and depend only on the data. Detailed expressions for A_{ip} are not given here because these expansions would be too large to serve any useful purpose.

Now, x_2 is eliminated from eqs.(33) and (34), while x_3 is likewise eliminated from eqs.(35) and (37), thereby obtaining a single equation in x_1, namely,

$$\det \begin{bmatrix} d_{11} & d_{12} & A_4 d_7 & A_5 d_7 \\ d_{21} & d_{22} & A_4 d_8 + A_5 d_7 & A_5 d_8 \\ d_7 & d_8 & d_9 & 0 \\ 0 & d_7 & d_8 & d_9 \end{bmatrix} = 0 \tag{39}$$

where

$$d_{11} = A_2 d_7 - A_1 d_8 \qquad d_{12} = A_3 d_7 - A_1 d_9$$
$$d_{21} = A_3 d_7 - A_1 d_9 \qquad d_{22} = A_3 d_8 - A_2 d_9 + A_4 d_7.$$

The foregoing determinant is now expanded and simplified, thereby leading to

$$\sum_{i=0}^{16} k_i x_1^i = 0 \tag{40}$$

where k_i depend only on kinematic parameters, and are related by

$$k_i = (-1)^i k_{16-i}, \quad i = 1, \cdots, 7. \tag{41}$$

The detailed expressions for k_i are not given here for the same reasons given above in connection with coefficients A_{ip} of eq.(38). What is important to point out here is that the above equation admits 16 solutions, whether real or complex, among which we are interested only in the real positive solutions. The real negative solutions lead to the same configurations as the positive ones, with the exception that the sides of the triangle $\mathcal{R}$, d, e and f, are replaced by another triangle with the same vertices $R_1R_2R_3$, but different sides, namely, $2\pi - d$, $2\pi - e$ and $2\pi - f$. So, the negative solutions can be discarded. The upper bound for the number of real positive solutions of a polynomial is given by Descartes theorem [4], namely,

The number of real positive solutions of a polynomial is given by the number of change of signs of the coefficients $k_o, k_1, \cdots, k_n$ minus $2m$, where $m \geq 0$.

The maximum of change of sign in the foregoing polynomial is eight. Therefore, the problem leads to a maximum of eight real positive solutions and, as a result, triangle $\mathcal{Q}$ of Fig. 1b admits up to eight different orientations, for the specified values of μ_1, μ_2 and μ_3.

Example 3.2.1. Consider the spherical triangles $\mathcal{P}$ and $\mathcal{Q}$ given as:

$$Q_1Q_2 = 60^\circ \qquad Q_2Q_3 = 70^\circ \qquad Q_3Q_1 = 50^\circ$$
$$P_1P_2 = 70^\circ \qquad P_2P_3 = 58.6^\circ \qquad P_3P_1 = 81.5^\circ$$

and three points, R_1, R_2 and R_3, located by the three values $\mu_1 = 10^\circ$, $\mu_2 = 49.5^\circ$ and $\mu_3 = 40^\circ$. These values correspond to the angles D_2, E_2 and F_2 given below:

$$D_2 = 43.4745^\circ, \quad E_2 = 37.9120^\circ, \quad F_2 = 106.7287^\circ.$$

Equation (40) is solved for x_1. The solutions are shown in Table 1. For this particular problem, we were able to find two real positive solutions.

N0.	x_1	D_1 (deg.)	E_1 (deg.)	F_1 (deg.)
1	-3.52853659	$180^\circ + (D_1)_{13}$	$180^\circ + (E_1)_{13}$	$180^\circ + (F_1)_{13}$
2	-1.81493883	$180^\circ + (D_1)_{16}$	$180^\circ + (E_1)_{16}$	$180^\circ + (F_1)_{16}$
3	$-0.7122360 - \jmath 0.9461246$	–	–	–
4	$-0.7122360 + \jmath 0.9461246$	–	–	–
5	$-0.4987636 - \jmath 1.6448662I$	–	–	–
6	$-0.4987636 + \jmath 1.6448662$	–	–	–
7	$-0.0110361 - \jmath 1.7618928$	–	–	–
8	$-0.0110361 + \jmath 1.7618928$	–	–	–
9	$0.00355500 - \jmath 0.5675491$	–	–	–
10	$0.0035550 + \jmath 0.5675491$	–	–	–
11	$0.1688234 - \jmath 0.5567607$	–	–	–
12	$0.1688234 + \jmath 0.5567607$	–	–	–
13	0.28340360	31.64584216	76.17273858	42.53021089
14	$0.5078577 - \jmath 0.6746313$	–	–	–
15	$0.5078577 + \jmath 0.6746313$	–	–	–
16	0.55098275	57.70801252	99.32576667	64.91849185

Table 1: The sixteen solutions of Example 3.2.1

3.3. Spatial 6-DOF manipulator

Consider two spatial triangles, $\mathcal{P}$ and $\mathcal{Q}$, with $\mathcal{P}$ connected to $\mathcal{Q}$ via three 6-dof $PRRPRP$ legs, where R stands for revolute and P for prismatic pairs, as shown in Fig. 2b. Moreover, the geometric model of this manipulator is depicted in Fig. 4.

Fig. 4: Geometric model of spatial DT manipulators

Triangle $\mathcal{P}$ consists of three lines given by $\{\, \mathbf{v}_i^* \,\}_1^3$ and their three common perpendiculars given by $\{\, \mathbf{a}_i^* \,\}_1^3$, with $\mathbf{v}_i^*$ defined as

$$\mathbf{v}_i^* \equiv \mathbf{v}_i + \epsilon \mathbf{v}_{i0}, \quad i = 1, 2, 3 \tag{42}$$

where $\mathbf{v}_i$ and $\mathbf{v}_{i0}$ are the direction and the moment vectors of the ith line of $\mathcal{P}$ with respect to the origin, respectively. In the foregoing discussion, $\mathbf{a}_i^*$, the common perpendicular between $\mathbf{v}_{i+1}^*$ and $\mathbf{v}_{i-1}^*$, is defined as

$$\mathbf{a}_i^* \equiv \mathbf{a}_i + \epsilon \mathbf{a}_{i0}, \quad i = 1, 2, 3 \tag{43}$$

where $\mathbf{a}_i$ and $\mathbf{a}_{i0}$ are, respectively, the direction and the moment vectors of the line represented by $\mathbf{a}_i^*$ with respect to the origin.

Similarly, triangle $\mathcal{Q}$ consists of three lines given by $\{\, \mathbf{u}_i^* \,\}_1^3$ and their three common perpendiculars given by $\{\, \mathbf{b}_i^* \,\}_1^3$, with $\mathbf{u}_i^*$ defined as

$$\mathbf{u}_i^* \equiv \mathbf{u}_i + \epsilon \mathbf{u}_{i0}, \quad i = 1, 2, 3 \tag{44}$$

where $\mathbf{u}_i$ and $\mathbf{u}_{i0}$ are the direction and the moment vectors of the ith line of $\mathcal{Q}$ with respect to the origin, respectively. In the foregoing discussion, $\mathbf{b}_i^*$, the common perpendicular between $\mathbf{u}_{i+1}^*$ and $\mathbf{u}_{i-1}^*$, is defined as

$$\mathbf{b}_i^* \equiv \mathbf{b}_i + \epsilon \mathbf{b}_{i0}, \quad i = 1, 2, 3 \tag{45}$$

where $\mathbf{b}_i$ and $\mathbf{b}_{i0}$ are, respectively, the direction and the moment vectors of the line represented by $\mathbf{b}_i^*$ with respect to the origin.

Triangle $\mathcal{P}$ is designated the FT, while $\mathcal{Q}$ is the MT. Moreover, the MT can move freely on the FT such that $\mathbf{r}_i^*$, for $i = 1, 2, 3$, does not lie outside its corresponding line-segments. Thus, for feasible or admissible motions, $\mathbf{r}_i^*$ must intersect $\mathbf{a}_i^*$ and $\mathbf{b}_i^*$ within their line-segments. The motion of triangle $\mathcal{Q}$ can thus be described through changes in the edge-length parameters ρ_i, which locate $\mathbf{r}_i^*$ along a side of $\mathcal{P}$, measured from P_{i+1}, and changes in the twist angle between $\mathbf{v}_{i+1}^*$ and $\mathbf{r}_i^*$, μ_i, for $i =$ $= 1, 2, 3$. In other words, this motion can be described through changes in the dual angles $\hat{\mu}_i \equiv \mu_i + \epsilon\rho_i$, for $i = 1, 2, 3$. In this discussion, $\mathbf{r}_i^*$ is the dual representation of a line whose direction and moment vectors are specified by $\mathbf{r}_i$ and $\mathbf{r}_{i0}$, respectively, i.e.,

$$\mathbf{r}_i^* \equiv \mathbf{r}_i + \epsilon\mathbf{r}_{i0} \tag{46}$$

The changes in $\hat{\mu}_i$, for $i = 1, 2, 3$, are assumed to be produced by actuators, and hence, they are termed the *actuator coordinates*. The three lines $\{\, \mathbf{b}_i^* \,\}_1^3$ of the moving triangle, in turn, are the set of variables used to define the pose of the triangle. Note that any three lines define a spatial triangle.

The direct kinematic problem of the manipulator described above is the subject of this subsection. This problem may be formulated as: *Given the actuator coordinates $\hat{\mu}_i$, for $i = 1, 2, 3$, find the three lines of triangle $\mathcal{Q}$, namely, $\mathbf{b}_i^*$, for $i = 1, 2, 3$.* Thus, given $\{\, \hat{\mu}_i \,\}_1^3$, we define a spatial triangle whose sides are the three axes $\{\, \mathbf{r}_i^* \,\}_1^3$. The DK problem thus consists of finding all triangles $\mathcal{Q}$ whose three common perpendiculars, namely $\{\, \mathbf{b}_i^* \,\}_1^3$, intersect these three axes at right angles.

Note that $\mathbf{a}_i^*$ can be transformed into $\mathbf{b}_i^*$ via a screw motion represented by a unit dual quaternion $\hat{r}^*{}_i$, namely,

$$\mathbf{b}_i^* = \hat{r}^*{}_i\mathbf{a}_i^*, \quad i = 1, 2, 3 \tag{47}$$

where $\hat{r}^*{}_i$ is defined as

$$\hat{r}^*{}_i \equiv \cos\hat{\psi}_i + \mathbf{r}_i^* \sin\hat{\psi}_i, \quad i = 1, 2, 3 \tag{48}$$

in which $\hat{\psi}_i$ is the dual angle defined as

$$\hat{\psi}_i \equiv \psi_i + \epsilon r_i, \quad i = 1, 2, 3 \tag{49}$$

where ψ_i and r_i are the twist angle and the distance between lines $\mathbf{a}_i^*$ and $\mathbf{b}_i^*$, respectively. Substitution of the value of $\hat{r}^*{}_i$ from eq.(48) into eq.(47), upon simplification, leads to

$$\mathbf{b}_i^* = \cos\hat{\psi}_i\mathbf{a}_i^* + \mathbf{r}_i^* \sin\hat{\psi}_i\mathbf{a}_i^*, \quad i = 1, 2, 3. \tag{50}$$

Moreover, $\mathbf{r}_i^*$ is a transformation of $\mathbf{v}_{i+1}^*$ via a screw motion represented by a unit dual quaternion $\hat{a}^*{}_i$, as shown in Fig. 4, namely,

$$\mathbf{r}_i^* = \hat{a}^*{}_i\mathbf{v}_{i+1}^*, \quad i = 1, 2, 3 \tag{51}$$

where

$$\hat{a}^*{}_i \equiv \cos\hat{\mu}_i + \mathbf{a}_i^* \sin\hat{\mu}_i, \quad i = 1, 2, 3 \tag{52}$$

Substitution of the value of $\mathbf{r}_i^*$ from eq.(51) into eq.(50), upon simplification, leads to

$$\mathbf{b}_i^* = \cos\hat{\psi}_i \mathbf{a}_i^* + \cos\hat{\mu}_i \sin\hat{\psi}_i \mathbf{v}_{i+1}^* \mathbf{a}_i^* + \sin\hat{\mu}_i \sin\hat{\psi}_i \mathbf{v}_{i+1}^* (\mathbf{a}_i^*)^2, \quad i = 1, 2, 3 \tag{53}$$

Equation (53) leads to 18 scalar equations in 24 unknowns, namely, the three lines represented by $\{\, \mathbf{b}_i^* \,\}_1^3$ and the three dual quantities $\{\, \hat{\psi}_i \,\}_1^3$.

Moreover, we recall the angular closure equation from eq.(8), which, for the MT, leads to

$$\hat{b}^*{}_2 \, \hat{b}^*{}_1 \, \hat{b}^*{}_3 = 1 \tag{54}$$

where $\hat{b}^*{}_i$, for $i = 1, 2, 3$, are unit dual quaternions, defined as

$$\hat{b}^*{}_i \equiv \cos\hat{\gamma}_i + \mathbf{b}_i^* \sin\hat{\gamma}_i, \quad i = 1, 2, 3 \tag{55}$$

in which $\hat{\gamma}_i$ is the dual angle defined as

$$\hat{\gamma}_i \equiv \gamma_i + \epsilon b_i, \quad i = 1, 2, 3 \tag{56}$$

with γ_i and b_i defined as the twist angle and the distance between lines $\mathbf{u}_{i+1}^*$ and $\mathbf{u}_{i-1}^*$, respectively. Moreover, pre-multiplying eq.(54) by $k(\hat{b}^*{}_2)$, leads to

$$\hat{b}^*{}_1 \, \hat{b}^*{}_3 = k(\hat{b}^*{}_2). \tag{57}$$

Equation (57) thus leads to eight extra equations to give a total of 26 equations, in total, in 24 unknowns, whose roots are the solutions of the direct kinematic problem at hand. Substituting the values of $\hat{b}^*{}_1$, $\hat{b}^*{}_2$ and $\hat{b}^*{}_3$ from eq.(55) into eq.(57), upon simplification, leads to

$$\begin{aligned} &\cos\hat{\gamma}_1 \cos\hat{\gamma}_3 + \mathbf{b}_3^* \cos\hat{\gamma}_1 \sin\hat{\gamma}_3 + \mathbf{b}_1^* \sin\hat{\gamma}_1 \cos\hat{\gamma}_3 + \\ &\quad + \mathbf{b}_1^* \mathbf{b}_3^* \sin\hat{\gamma}_1 \sin\hat{\gamma}_3 - \cos\hat{\gamma}_2 + \mathbf{b}_2^* \sin\hat{\gamma}_2 = 0. \end{aligned} \tag{58}$$

Moreover, substituting the values of $\mathbf{b}_i^*$, for $i = 1, 2, 3$, from eq.(53) into eq.(58) leads to eight equations in six unknowns, namely, six parameters in three dual quantities $\hat{\psi}_i$, for $i = 1, 2, 3$. Among the eight equations, only six are independent, and the problem admits exact solutions.

Example 3.3.1. The fixed triangle is given by three dual vectors $\mathbf{v}_i^*$, for $i = 1, 2, 3$, via their directions and moments as explained in eq.(42), i.e.,

$$\mathbf{v}_1 = [1, 0, 0]^T, \qquad \mathbf{v}_{10} = [0, 0, 0]^T$$
$$\mathbf{v}_2 = [0, 0, 1]^T, \qquad \mathbf{v}_{20} = [1, 0, 0,]^T$$
$$\mathbf{v}_3 = [0, -1, 0]^T, \qquad \mathbf{v}_{30} = [1, 0, 1]^T.$$

The directions and moments of the three common perpendiculars to the foregoing lines, $\{\, \mathbf{a}_i^* \,\}_1^3$, are

$$\mathbf{a}_1 = [-1, 0, 0]^T, \quad \mathbf{a}_{10} = [0, -1, 1]^T$$
$$\mathbf{a}_2 = [0, 0, -1]^T, \quad \mathbf{a}_{20} = [0, -1, 0]^T$$
$$\mathbf{a}_3 = [0, 1, 0]^T, \quad \mathbf{a}_{30} = [0, 0, 0]^T.$$

Moreover, the moving triangle is given by its three sides, namely,

$$\hat{\gamma}_1 = 1.99133 + \epsilon 0.37268, \qquad \hat{\gamma}_2 = 0.876816 + \epsilon 0.737494, \qquad \hat{\gamma}_3 = 1.74577 + \epsilon 0.123211. \tag{61}$$

Finally, six actuator coordinates are given in dual form as

$$\hat{\mu}_1 = -\pi + \epsilon 0.5, \quad \hat{\mu}_2 = -2.15873528 + \epsilon 0.75, \quad \hat{\mu}_3 = -3\pi/4 + \epsilon 0.25. \tag{62}$$

Substitution of the foregoing data into eq.(58), upon simplification, leads to

$$\mathbf{q} = \mathbf{0} \tag{63}$$

where $\mathbf{q}$ is an 8-dimensional vector with only six independent components. The eight components of $\mathbf{q}$ are not given here because of space limitations.

Solving eq.(63) for r_i and ψ_i, for $i = 1, 2, 3$, leads to the six real solutions in Table 2.

N0.	r_1 m	r_2 m	r_3 m	ψ_1 Deg.	ψ_2 Deg.	ψ_3 Deg.
1	0. 0558852	1. 0740471	0. 40721916	−154. 806	8. 32983	21. 746
2	0. 34708068	1. 17104435	−0. 20553218	62. 084	−7. 16017	152. 624
3	0. 46406054	1. 03938365	0. 28097901	135. 155	−39. 8682	101. 556
4	0. 4999920	0. 4160031	0. 3535622	−26. 5655	−89. 999	−90. 0004
5	0. 74620074	−0. 10399796	1. 54681873	−46. 5857	146. 239	9. 86011
6	1. 34650493	1. 13581347	0. 01708774	−10. 8435	−161. 372	−46. 3652

Table 2: The six solutions of Example 3.3.1

Substitution of the data from eqs.(59 – 62) and the foregoing values for $-r_i$ and ψ_i, for $i = 1, 2, 3$, into eq.(53) gives $\mathbf{b}_i^*$, for $i = 1, 2, 3$. For example, for solution No. 4, we obtain three lines of the moving triangle as

$$\mathbf{b}_1^* = [-0.894427, -0.447214, 0]^T + \epsilon[0.223608, -0.447214, 1.11803]^T$$

$$\mathbf{b}_2^* = [0.5547, -0.83205, 0]^T + \epsilon[0.208013, 0.138676, 0.416026]^T$$

$$\mathbf{b}_3^* = [0.707107, 0, 0.707107]^T + \epsilon[0.176777, 0.353553, -0.176777]^T$$

which correspond to the pose of the moving triangle.

Conclusions. Note that the spatial architecture with $\{r_i\}_1^3$ of fixed length is a 3-dof double-triangular manipulator, with three actuators $\{\rho_i\}_1^3$. It is therefore the obvious spatial counterpart of the other two double-triangular manipulators described earlier. Moreover, it is anticipated that it will provide an effective mean to furnish and study mixed three parameters systems. Consider that methods to systematically treat 3-dof spatial manipulator kinematics are not presently available.

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