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VELOCITY AND CORIOLIS QUADRICS OF ROBOT-MANIPULATORS

Marie **Kargerová**

Department of Technical Mathematics, Czech Technical University, Horská 3, 128 03 Praha 2, Czech Republic

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Abstract: The connection between the velocity and Coriolis acceleration of 3R robot-manipulators is described. It is shown that the velocity and Coriolis quadrics have the same axes but of different length. Classification of Coriolis quadratical surfaces is given.

1. Introduction

This paper is a continuation of [3],where the basic properties of the velocity and and acceleration fields of 3R robot-manipulators have been described. At the beginning we shall briefly summarize denotations and basic properties of velocity and acceleration operators from [3].

The geometry of robot-manipulators with p-degrees of freedom is determined by the product of p-revolutions or translations given by axes $X_1, \dots, X_p$. We suppose that axes $X_1, \dots, X_p$ are determined by their Plücker coordinates $X_i = (\vec{x}_i; \vec{y}_i)$, $i = 1, \dots, p$.

Remark. For simplicity reasons we consider only rotational axes, for prismatic joints we have to change all formulas accordingly.

The motion of the end-effector of such a robot-manipulator is expressed by the matrix

$$g(\varphi_1, \varphi_2, \dots, \varphi_p) = r_1(\varphi_1)\; r_2(\varphi_2) \dots r_p(\varphi_p),$$

where $r_i(\varphi_i)$ is the matrix of the revolution around X_i. If $\varphi_i = \varphi_i(t)$ are functions of time we obtain a one parametric motion of the end-effector,

$$g(t) = r_1(\varphi_1(t))\ r_2(\varphi_2(t)) \dots r_p(\varphi_p(t)).$$

Trajectory of a point $\bar{A}$ of the end-effector space is $A(t) = g(t)\bar{A}$. Let us denote $\Omega(\Theta)$ the velocity (acceleration) operator of $g(t)$, respectively. We have

$$\Omega = g' g^{-1},\ \Theta = \Omega' + \Omega^2,\ \Omega = \sum_{i=1}^{p} Y_i v_i,$$

where Y_i is the instantaneous position of $i-$th axis and v_i is the angular velocity of $r_i(\varphi_i(t)); v_i = \frac{d\varphi_i}{dt}$. For the derivative of Ω we have

$$\Omega' = \sum_{i-1}^{p} Y_i'\ v_i + \sum_{i=1}^{p} Y_i \frac{dv_i}{dt}.$$

We can split the acceleration operator into three parts:

a) Ω^2 is the centrifugal acceleration;

b) $\sum\limits_{i-1}^{p} Y_i \varepsilon_i$ is the Euler acceleration, where $\varepsilon = \frac{dv_i}{dt}$ is the angular acceleration of $r_i(\varphi_i(t))$;

c) $\sum\limits_{i=1}^{p} Y_i' v_i = \sum\limits_{i<j=1}^{p} Y_i \times Y_j\ v_i v_j$ is the Coriolis acceleration where $Y_i \times \times Y_j$ denotes the cross product of Plücker coordinates of Y_i with Y_j.

2. Velocity and Coriolis quadrics

In the following we shall concentrate on velocity and acceleration properties of 3R robot-manipulators. We shall show that both velocity and Coriolis acceleration operators are connected with quadratical surfaces. Let us have a 3R robot-manipulator determined by axes X_1, X_2, X_3. Let us consider an instantaneous position Y_1, Y_2, Y_3 of these axes. Then the velocity operator Ω for this configuration is given by

$$\Omega = \omega_1 Y_1 + \omega_2 Y_2 + \omega_3 Y_3\ , \tag{1}$$

Coriolis acceleration C is given by the formula

$$C = Y_1 \times Y_2 \omega_1 \omega_2 + Y_1 \times Y_3 \omega_1 \omega_3 + Y_2 \times Y_3 \omega_2 \omega_3, \tag{2}$$

$Y_i \times Y_j$ is the cross product of Plücker coordinates which is defined as follows. Denote $Y_i = (\vec{x}_i; \vec{y}_i), Y_j = (\vec{x}_j; \vec{y}_j)$. Then

$$(\vec{x}_i; \vec{y}_i) \times (\vec{x}_j; \vec{y}_j) = (\vec{x}_i \times \vec{x}_j; \vec{x}_i \times \vec{y}_j + \vec{y}_i \times \vec{x}_j),$$

see [2]. We can see that velocity operator is always a linear combination of Y_1, Y_2, Y_3. This shows that we have to work in the 6-dimensional vector space of screws. It is the vector space of all pairs $(\vec{x}; \vec{y})$ of ordinary vectors of the Euclidean space E_3. V_6 contains Plücker coordinates of all straight lines of E_3. Their image is called Klein's quadratical hypersurface K.

Let us assume that the direction vectors $\vec{x}_1, \vec{x}_2, \vec{x}_3$ of Y_1, Y_2, Y_3 are independent. All velocity operators Ω for the given configuration Y_1, Y_2, Y_3 generate a 3-dimensional subspace V_3 of V_6. The intersection of K with V_3 is a ruled hyperboloid Q_v. (We know that it contains three straight lines with independent directions.) Q_v is connected with the velocity operator and it is uniquely determined by the instantaneous configuration Y_1, Y_2, Y_3 of axes X_1, X_2, X_3 of the robot-manipulator. We shall call it *velocity quadrics.*

We have similar situation with Coriolis acceleration. According to (2) the Coriolis acceleration operator C for the given configuration Y_1, Y_2, Y_3 lies in the 3-dimensional space W_3 generated by screws $Y_1 \times \times Y_2, Y_1 \times Y_3, Y_2 \times Y_3$.

Remark. The difference between V_3 and W_3 is that W_3 need not contain any straight lines and it is also not true that any screw of W_3 is a Coriolis acceleration. Let us denote Q_c the quadratical surface obtained as the intersection of W_3 with K, we shall call it *Coriolis quadrics.*

3. Properties of velocity and Coriolis quadrics

Lemma. *The Coriolis quadrics Q_c is independent of the choice of screws Y_1, Y_2, Y_3 in the subspace W_3.*

Proof. W_3 is generated by independent screws Y_1, Y_2, Y_3. Let us choose another basis $\bar{Y}_1, \bar{Y}_2, \bar{Y}_3$ of W_3 by

$$\begin{aligned}\bar{Y}_1 &= a_{11}Y_1 + a_{12}Y_2 + a_{13}Y_3\\ \bar{Y}_2 &= a_{21}Y_1 + a_{22}Y_2 + a_{23}Y_3\\ \bar{Y}_3 &= a_{31}Y_1 + a_{32}Y_2 + a_{33}Y_3,\end{aligned}$$

where the determinant $D = |a_{ij}| \neq 0$. The Coriolis quadrics corresponding to $\bar{Y}_1, \bar{Y}_2, \bar{Y}_3$ is determined by screws $\bar{Y}_1 \times \bar{Y}_2, \bar{Y}_1 \times \bar{Y}_3, \bar{Y}_2 \times \bar{Y}_3$.

Computation yields

$$\bar{Y}_1 \times \bar{Y}_2 = b_{11}Y_1 \times Y_2 + b_{12}Y_1 \times Y_3 + b_{13}Y_2 \times Y_3$$
$$\bar{Y}_1 \times \bar{Y}_3 = b_{21}Y_1 \times Y_2 + b_{22}Y_1 \times Y_3 + b_{23}Y_2 \times Y_3$$
$$\bar{Y}_2 \times \bar{Y}_3 = b_{31}Y_1 \times Y_2 + b_{32}Y_1 \times Y_3 + b_{33}Y_2 \times Y_3,$$

where for instance $b_{11} = a_{11}a_{22} - a_{12}a_{21}$ and similarly for the others. Computation shows that for the determinant $D_1 =\mid b_{ij} \mid$ we have $D_1 = = D^2$ and therefore $D_1 \neq 0$. $\Diamond$

From Lemma we see that the conection between Q_v and Q_c is independent of the choice of screws which determine them. This means that we can choose screws corresponding to axes of Q_v. We can prove the following theorem.

Theorem. *Let us suppose that Q_v has the equation*

$$v_1x^2 + v_2y^2 + v_3z^2 + v_1v_2v_3 = 0 \tag{3}$$

in the canonical basis. Then Q_c has the equation

$$(v_2+v_3)x^2+(v_3+v_1)y^2+(v_2+v_1)z^2+(v_2+v_3)(v_3+v_1)(v_2+v_1) = 0 \tag{4}$$

in the same basis.

Proof. If Q_v is given by (3) in a Cartesian system of coordinates $\{O, \vec{e}_1, \vec{e}_2, \vec{v}_3\}$, then the corresponding subspace V_3 is determined by screws $Y_i = (\vec{e}_i; v_i\vec{e}_i)$, $i = 1, 2, 3$. Then

$$Y_1 \times Y_2 = (\vec{e}_1; v_1\vec{e}_1) \times (\vec{e}_2; v_2\vec{e}_2) = (\vec{e}_3; (v_1 + v_2)\vec{e}_3)$$

$$Y_1 \times Y_3 = (\vec{e}_1; v_1\vec{e}_1) \times (\vec{e}_3; v_3\vec{e}_3) = (\vec{e}_2; (v_1 + v_3)\vec{e}_2)$$

$$Y_2 \times Y_3 = (\vec{e}_2; v_2\vec{e}_2) \times (\vec{e}_3; v_3\vec{e}_3) = (\vec{e}_1; (v_2 + v_3)\vec{e}_1).$$

We see that Q_c has the canonical form in the same system of Cartesian coordinates. $\Diamond$

Corollary. *Q_c and Q_v have the same axes. Length a, b, c of axes of these surfaces are not in general the same. Relation between those quantities is following:*

$$Q_v: \quad a = \sqrt{\mid v_2v_3 \mid} \quad b = \sqrt{\mid v_1v_3 \mid} \quad c = \sqrt{\mid v_1v_2 \mid}$$
$$Q_c: \quad a = \sqrt{\mid (v_3 + v_1)(v_2 + v_1) \mid} \quad b = \sqrt{\mid (v_2 + v_3)(v_1 + v_2) \mid}$$

$$c = \sqrt{\mid (v_2 + v_3)(v_3 + v_1) \mid}.$$

4. Classification of velocity and Coriolis quadrics

Let us consider Q_v and Q_c given by (3) and (4). We shall classify them according to different values of v_1, v_2, v_3.

1) $v_1 v_2 v_3 \neq 0$. Q_v is a one sheet hyperboloid.

a) $(v_1+v_2)(v_2+v_3)(v_1+v_3) \neq 0, v_1 \geq v_2 > 0, v_3 < 0$. Q_c is one sheet hyperboloid if $v_1 < -v_3 < v_2$ or $v_1 < v_2 < -v_3$. Q_c is empty if $-v_3 < v_1 < v_2$.

b) $(v_1 + v_2)(v_1 + v_3) \neq 0, v_2 + v_3 = 0$. Q_c has the equation

$$(v_1 - v_2)y^2 + (v_1 + v_2)z^2 = 0.$$

According to the sign of $v_1 + v_2$ and $v_1 - v_2$ we obtain two planes which can be different, coinciding or imaginary.

c) $v_1 + v_2 \neq 0, v_1 + v_3 = v_2 + v_3 = 0$, Q_c is one plane.

2) $v_1 v_2 \neq 0, v_3 = 0$. We may suppose $v_1 > 0, v_2 < 0$ and Q_v consists of two pencils of straight lines, two axes intersect. Q_c has the equation

$$v_2 x^2 + v_1 y^2 + (v_1 + v_2)z^2 + (v_1 + v_2)v_1 v_2 = 0.$$

If $v_1 + v_2 \neq 0$, Q_c is one sheet hyperboloid and if $v_1 + v_2 = 0$ we obtain two pencils of straight lines.

3) The case $v_2 = v_3 = 0$ was excluded.

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