**Mathematica Pannonica**
**6**/1 (1995), 145 – 153

# SYMPLECTIC PLÜCKER TRANSFORMATIONS

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**Dedicated to Professor Hans Vogler on the occasion of his 60th birthday**

*Received* October 1994

*MSC 1991*: 51 A 50

*Keywords:* Symplectic space, Plücker transformation, line geometry.

**Abstract:** Plücker transformations of symplectic spaces with dimensions other than three are induced by orthogonality-preserving collineations. For three-dimensional symplectic spaces all Plücker transformations can be obtained — up to orthogonality preserving collineations — by replacing some but not necessarily all non-isotropic lines by their absolute polar lines.

In this paper we discuss bijections of the set $\mathcal{L}$ of lines of a symplectic space, i.e. a (not necessarily finite-dimensional) projective space with orthogonality based upon an absolute symplectic[1] quasipolarity. Following [1], two lines are called related, if they are concurrent and orthogonal, or if they are identical. A bijection of $\mathcal{L}$ that preserves this relation in both directions is called a (*symplectic*[2]) *Plücker transformation*. We shall show that any bijection $\mathcal{L} \to \mathcal{L}$ taking related lines to related lines is already a Plücker transformation. Moreover, a complete description of all Plücker transformations (cf. the abstract above) will be given.

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[1] Instead of 'symplectic' some authors are using the term 'null'.

[2] We shall omit the word 'symplectic', since we do not discuss other types of Plücker transformations in this paper. Cf., however, [1], [2], [4], [5], [8, p. 80ff], [9], [10] for results on other Plücker transformations.

# 1. Symplectic spaces

Let $(\mathcal{P}, \mathcal{L})$ be a projective space, $3 \leq \dim(\mathcal{P}, \mathcal{L}) \leq \infty$. Assume that $\pi$ is a symplectic quasipolarity [11], [12]. Thus $\pi$ assigns to each point $X$ of $\mathcal{P}$ a hyperplane $X^\pi$ with $X \in X^\pi$; furthermore $Y \in X^\pi$ implies $X \in Y^\pi$ for all $X, Y \in \mathcal{P}$. Cf. also [6] for an axiomatic description of projective spaces endowed with a quasipolarity.

We define a mapping from the lattice of subspaces of $(\mathcal{P}, \mathcal{L})$ into itself by setting

(1) $\mathcal{T} \mapsto \bigcap(X^\pi \mid X \in \mathcal{T})$ for all subspaces $\mathcal{T} \neq \emptyset$ and $\emptyset \mapsto \mathcal{P}$.

This mapping is again written as $\pi$ and is also called a *quasipolarity.* If $(\mathcal{P}, \mathcal{L})$ is finite-dimensional, then it is well known that $\pi$ is an anti-automorphism of the lattice of subspaces of $(\mathcal{P}, \mathcal{L})$. In case of infinite dimension the mapping (1) still has the properties

$$(\mathcal{T}_1 \vee \mathcal{T}_2)^\pi = \mathcal{T}_1^\pi \cap \mathcal{T}_2^\pi, \quad (\mathcal{T}_1 \cap \mathcal{T}_2)^\pi \supset \mathcal{T}_1^\pi \vee \mathcal{T}_2^{=}\pi, \quad \mathcal{T} \subset \mathcal{T}^{\pi\pi}$$
$$\mathcal{T}_1 \subset \mathcal{T}_2 \Rightarrow \mathcal{T}_1^\pi \supset \mathcal{T}_2^\pi$$

for all subspaces $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T} \subset \mathcal{P}$. Note that in the last formula strict inclusions are not necessarily preserved, if $\mathcal{T}_1$ and $\mathcal{T}_2$ both have infinite dimension[3]. Moreover, it is an easy induction to show for all finite-dimensional subspaces $\mathcal{T} \subset \mathcal{P}$ that $\mathcal{T}^{\pi\pi} = \mathcal{T}$ and that every completment of $\mathcal{T}^\pi$ has the same finite dimension as $\mathcal{T}$.

$(\mathcal{P}, \mathcal{L}, \pi)$ is a *symplectic space* with *absolute quasipolarity* $\pi$ [7, p. 384ff], [11]. In terms of an underlying vector space $\mathbf{V}$ of $(\mathcal{P}, \mathcal{L})$ the symplectic quasipolarity $\pi$ can be described by a non-degenerate alternating bilinear form of $\mathbf{V} \times \mathbf{V}$ into the (necessarily commutative) ground field of $\mathbf{V}$. If $(\mathcal{P}, \mathcal{L})$ is finite-dimensional, then it is well known that $\dim(\mathcal{P}, \mathcal{L})$ is odd.

We are introducing two binary relations on $\mathcal{L}$: Given $a, b \in \mathcal{L}$ then define $a$ and $b$ to be *orthogonal* ($\perp$), if $a \cap b^\pi \neq \emptyset$. The lines $a$ and $b$ are called *related* ($\sim$), if $a \perp b$ and $a \cap b \neq \emptyset$, or if $a = b$. Given orthogonal lines $a, b$ there exists a point $R \in a \cap b^\pi$. Therefore

$$R^\pi \supset (a \cap b^\pi)^\pi \supset a^\pi \vee b^{\pi\pi} = a^\pi \vee b.$$

[3] There are, e.g., hyperplanes $\mathcal{H} \subset \mathcal{P}$ with $\mathcal{H} \neq X^\pi$ for all $X \in \mathcal{P}$. For all such hyperplanes $\mathcal{H}^\pi = \mathcal{P}^\pi = \emptyset$, although $\mathcal{H} \neq \mathcal{P}$.

The line $b$ has a point in common with $a^\pi$, since $R^\pi$ is a hyperplane and $a^\pi$ is a co-line. Consequently, $\perp$ and $\sim$ are symmetric relations. Each line $a \in \mathcal{L}$ either is contained in $a^\pi$ or is a complement of $a^\pi$, since $a \cap a^\pi$ being a single point would imply that $X^\pi = a \vee a^\pi$ for all points $X \in a \setminus a^\pi$, in contradiction to $\pi \mid \mathcal{P}$ being injective. A line $a \in \mathcal{L}$ is *isotropic* (self-orthogonal) if and only if $a$ is *totally isotropic*, i.e., $a \subset a^\pi$. We shall write $\mathcal{J}$ for the set of all isotropic lines.

If $Q$ is a point, then $\mathcal{L}[Q]$ stands for the star of lines with centre $Q$ and $\mathcal{J}[Q] := \mathcal{L}[Q] \cap \mathcal{J}$ for the set of all isotropic lines through $Q$. In the following Lemma 1 we state two simple properties of isotropic lines that are well known in case of finite dimension [3, p. 181ff], [7, p. 384ff] but hold as well for infinite dimension:

**Lemma 1.** *If $Q \in \mathcal{P}$, then all isotropic lines through $Q$ are given by*

$$\mathcal{J}[Q] = \{x \in \mathcal{L} \mid Q \in x \subset Q^\pi\}.$$

*Let $a \in \mathcal{L} \setminus \mathcal{J}$ be non-isotropic. The set of isotropic lines intersecting the line $a$ equals the set of all lines intersecting both $a$ and $a^\pi$.*

**Proof.** Let a line $x$ with $Q \in x \subset Q^\pi$ be given. This implies $Q^\pi \supset \supset x^\pi$ so that $x$ and $x^\pi$ are in the same hyperplane $Q^\pi$. Since $x^\pi$ is a co-line, $x$ and $x^\pi$ cannot be skew, i.e. $x \in \mathcal{J}[Q]$. On the other hand, from $x \in \mathcal{J}[Q]$ follows immediately that $x \subset x^\pi \subset Q^\pi$. Next let $a \in \mathcal{L} \setminus \mathcal{J}$. If $b \in \mathcal{J}$ intersects $a$ at a point $Q$, say, then $Q \in b \subset b^\pi$ implies $b^{\pi\pi} = b \subset Q^\pi$, whereas $Q \in a$ tells us $a^\pi \subset Q^\pi$. Thus, as before, $b$ and $a^\pi$ are not skew. Conversely, given points $Q \in a$ and $R \in a^\pi$ then $R \in A^\pi \subset Q^\pi$ and $Q \in a \subset R^\pi$, whence $Q \vee R \subset Q^\pi \cap R^\pi = (Q \vee R)^\pi$. ◊

We apply this result to show

**Lemma 2.** *Distinct lines $a, b \in \mathcal{L}$ with $a \cap b \neq \emptyset$ are related if and only if $a \in \mathcal{J}$ or $b \in \mathcal{J}$.*

**Proof.** If one of the given lines is isotropic, then $a \sim b$. Conversely, if $a \sim b$ and $a \notin \mathcal{J}$, say, then $b \in \mathcal{J}$ by Lemma 1. ◊

As an immediate consequence we obtain

**Lemma 3.** *Let $\mathcal{M}$ be a set of mutually related lines. Then at most one line of $\mathcal{M}$ is non-isotropic.*

Given lines $a, b \in \mathcal{L}$ then there is always a finite sequence

$$a \sim a_1 \sim \ldots \sim a_n \sim b.$$

This is trivial when $a = b$. If $a \cap b =: Q$ is a point, then there exists a line $a_1 \in \mathcal{J}[Q]$ so that $a \sim a_1 \sim b$ by Lemma 2. If $a$ and $b$ are skew then there exists a common transversal line of $a$ and $b$, say $c$, whence repeating the previous construction for $a, c$ and then for $c, b$ gives the required sequence. Thus $(\mathcal{L}, \sim)$ *is a Plücker space*[4] [1, p. 199]. A (symplectic) *Plücker transformation* is a bijective mapping $\varphi: \mathcal{L} \to \mathcal{L}$ preserving the relation $\sim$ in both directions. We say that $\varphi$ is *induced* by a mapping $\kappa: \mathcal{P} \to \mathcal{P}$, if

$$(A \vee B)^{\varphi} = A^{\kappa} \vee B^{\kappa} \text{ for all } A, B \in \mathcal{P} \text{ with } A \neq B.$$

The group $\mathrm{P\Gamma Sp}(\mathcal{P}, \pi)$ consists of all collineations $\mathcal{P} \to \mathcal{P}$ commuting with $\pi$ [7, p. 388ff], [8, p. 19]. Obviously, each $\kappa \in \mathrm{P\Gamma Sp}(\mathcal{P}, \pi)$ is inducing a Plücker transformation.

If $\dim(\mathcal{P}, \mathcal{L}) = 3$, then for each duality $\tau$ with $\mathcal{J}^{\tau} = \mathcal{J}$ the restriction $\tau \mid \mathcal{L}: \mathcal{L} \to \mathcal{L}$ is a Plücker transformation. Moreover, in the three-dimensional case there are always Plücker transformations not arising from collineations or dualities: Let $\mathcal{L}_1$ be any subset of $\mathcal{L} \setminus \mathcal{J}$ such that $\mathcal{L}_1^{\pi} = \mathcal{L}_1$. Then define

$$\delta: \mathcal{L} \to \mathcal{L}, \quad \begin{cases} x \mapsto x & \text{if } x \in \mathcal{L} \setminus \mathcal{L}_1, \\ x \mapsto x^{\pi} & \text{if } x \in \mathcal{L}_1. \end{cases} \tag{2}$$

Such a bijection $\delta$ will be called *partial $\pi$-transformation* (with respect to $\mathcal{L}_1$); it is a Plücker transformation of $(\mathcal{L}, \sim)$, since

$$a \sim b \Leftrightarrow a \sim b^{\pi} \Leftrightarrow a^{\pi} \sim b \Leftrightarrow a^{\pi} \sim b^{\pi} \text{ for all } a, b \in \mathcal{L},\ a \neq b.$$

The identity on $\mathcal{L}$ and the restriction of $\pi$ to $\mathcal{L}$ are partial $\pi$-transformations, as follows from setting $\mathcal{L}_1 := \emptyset$ and $\mathcal{L}_1 := \mathcal{L} \setminus \mathcal{J}$, respectively. For every other choice of $\mathcal{L}_1$(e.g., $\mathcal{L}_1 := \{a, a^{\pi}\}$) it is easily seen that there exist two non-isotropic concurrent lines $x \in \mathcal{L} \setminus \mathcal{L}_1$, $y \in \mathcal{L}_1$. Then $x^{\delta} = x$ and $y^{\delta} = y^{\pi}$ are skew lines. Such a Plücker transformation cannot arise from a collineation or duality.

## 2. The three-dimensional case

**Theorem 1.** *Let $(\mathcal{P}, \mathcal{L}, \pi)$ be a 3-dimensional symplectic space and let $\beta: \mathcal{L} \to \mathcal{L}$ be a bijection such that*

$$a \sim b \text{ implies } a^{\beta} \sim b^{\beta} \text{ for all } a, b \in \mathcal{L}.$$

[4] Alternatively, $\mathcal{L}$ may be seen as the set of vertices of a graph with two vertices joined by an edge if and only if the corresponding lines are distinct and related. We refrain, however, from using terminology of graph theory.

*Then there exists a partial $\pi$-transformation $\delta\colon \mathcal{L} \to \mathcal{L}$ such that $\delta\beta$ is induced by a collineation $\kappa \in \mathrm{P\Gamma Sp}(\mathcal{P}, \pi)$.*

Th. 1 is a consequence of the subsequent Propositions 1.1–1.4 in which $\beta$ and $(\mathcal{P}, \mathcal{L}, \pi)$ are given as above.

**Proposition 1.1.** *There exists an injective mapping $\kappa\colon \mathcal{P} \to \mathcal{P}$ such that*

$$\mathcal{J}[Q]^\beta = \mathcal{J}[Q^\kappa] \text{ for all } Q \in \mathcal{P}. \tag{3}$$

*Moreover, $\beta$ is a Plücker transformation, since*

$$\mathcal{J}^\beta = \mathcal{J}. \tag{4}$$

**Proof.** By the invariance of $\sim$ under $\beta$, the elements of $\mathcal{J}[Q]^\beta$ are mutually related. We infer from Lemma 3 that $\mathcal{J}[Q]^\beta$ contains at most one non-isotropic line. Thus $\mathcal{J}[Q]^\beta \cap \mathcal{J}$ has at least two distinct elements, whence it is a subset of a pencil of isotropic lines, say $\mathcal{J}[Q']$ with $Q' \in \mathcal{P}$.

We show $\mathcal{J}[Q'] \subset \mathcal{J}[Q]^\beta$: Assume, to the contrary, that there exists a line $x \notin \mathcal{J}[Q]$ with $x^\beta \in \mathcal{J}[Q']$. Recall that at most one line of $\mathcal{J}[Q]^\beta$ is non-isotropic. Therefore there is a point $X' \in x^\beta$ that is not incident with any line of $\mathcal{J}[Q]^\beta$. Thus we can draw a line $b' = b^\beta$ through $X'$ that is not related to any line of $\mathcal{J}[Q]^\beta$. Hence $b$ is not related to any line of $\mathcal{J}[Q]$. By $\dim(\mathcal{P}, \mathcal{L}) = 3$, $b$ and the plane $Q^\pi$ have a common point lying on some line $c \in \mathcal{J}[Q]$, so that $c \sim b$, a contradiction.

Next $\mathcal{J}[Q]^\beta \subset \mathcal{J}[Q']$ will be established: Suppose there is a line $a \in \mathcal{J}[Q]$ such that $a^\beta \notin \mathcal{J}[Q']$. Then $a^\beta \sim \mathcal{J}[Q']$ forces that $a^\beta$ is a non-isotropic line either through the point $Q'$ or in the plane $Q'^\pi$. Let $d \in \mathcal{L}[Q]$ be non-isotropic, whence $\mathcal{J}[Q]^\beta \cup \{d^\beta\}$ is a set of mutually related lines containing the non-isotropic line $a^\beta$. Since $d^\beta \notin \mathcal{J}[Q]^\beta$ and $\mathcal{J}[Q'] \subset \mathcal{J}[Q]^\beta$, the line $d^\beta \neq a^\beta$ also has to be non-isotropic in contradiction to Lemma 3.

To sum up, there is a mapping $\kappa$ satisfying formula (3). The injectivity of $\kappa$ follows from the bijectivity of $\beta$ together with (3).

Finally, we prove (4): $\mathcal{J}^\beta \subset \mathcal{J}$ is a consequence of (3). Conversely, assume that $e \in \mathcal{L} \setminus \mathcal{J}$. Choose a point $R \in e$. Then $\mathcal{J}[R]^\beta \cup \{e^\beta\} = \mathcal{J}[R^\kappa] \cup \{e^\beta\}$ is a set of mutually related lines. Therefore $e^\beta$ is a non-isotropic line either through $R^\kappa$ or in $R^{\kappa\pi}$. Lemma 2 and $\mathcal{J}^\beta = \mathcal{J}$ imply that $\beta$ is a Plücker transformation. ◊

**Proposition 1.2.** *Let $a \in \mathcal{L}$. Then*

$$a^{\beta\pi} = a^{\pi\beta}, \tag{5}$$

$$Q^{\kappa} \in a^{\beta} \cup a^{\pi\beta} \text{ for all } Q \in a. \tag{6}$$

*If* $a \in \mathcal{L} \setminus \mathcal{J}$, *then either*

$$Q^{\kappa} \in a^{\beta} \text{ for all } Q \in a \tag{7}$$

*or*

$$Q^{\kappa} \in a^{\pi\beta} \text{ for all } Q \in a. \tag{8}$$

**Proof.** If $a \in \mathcal{J}$, then $a^{\beta} \in \mathcal{J}$, whence (5) follows from $a = a^{\pi}$ and $a^{\beta} = a^{\beta\pi}$. If $a \in \mathcal{L} \setminus \mathcal{J}$, then, by Lemma 1,

$$\mathcal{C} := \{x \in \mathcal{L} \mid x \neq a, x \sim a\}$$

is a hyperbolic linear congruence of lines with axes $a$ and $a^{\pi}$; moreover $\mathcal{C} \subset \mathcal{J}$. We infer from $\beta$ being a Plücker transformation and (4), that $\mathcal{C}^{\beta} \subset J$ is also a hyperbolic linear congruence with $a^{\beta}, a^{\pi\beta}$ being its axes. Obviously, only $a^{\beta}$ and $a^{\pi\beta}$ are meeting all lines of $\mathcal{C}^{\beta}$. On the other hand, by Lemma 1, the axes of $\mathcal{C}^{\beta}$ are $a^{\beta}$ and $a^{\beta\pi}$. This completes the proof of (5).

If $a \in \mathcal{J}$, then (6) holds true, since $Q^{\kappa} \in a^{\beta} = a^{\pi\beta}$. If $a \in \mathcal{L} \setminus \mathcal{J}$ and $Q^{\kappa} \notin a$, then $a^{\beta} \sim \mathcal{J}[Q]^{\beta} = \mathcal{J}[Q^{k}]$, whence $Q^{\kappa\pi} \supset a^{\beta}$ and therefore $Q^{\kappa} \in a^{\beta\pi} = a^{\pi\beta}$, as required to establish (6).

Now let $a \in \mathcal{L} \setminus \mathcal{J}$. Assume to the contrary that there exist points $Q_0, Q_1 \in a$ such that $Q_0^{\kappa} \in a^{\beta}$ and $Q_1^{\kappa} \in a^{\pi\beta}$. Then $a \in \mathcal{J}$ implies $\mathcal{J}[Q_0] \cap \mathcal{J}[Q_1] = \emptyset$ whereas, by Lemma 1 and (3), $Q_0^{\kappa} \vee Q_1^{\kappa} \in \mathcal{J}[Q_0]^{\beta} \cap$ $\cap\, \mathcal{J}[Q_1]^{\beta}$. This is a contradiction to $\beta$ being injective. ◊

**Proposition 1.3.** *Write* $\mathcal{L}_1$ *for the set of all lines* $a \in \mathcal{L} \setminus \mathcal{J}$ *satisfying* (8). *Then*

$$\delta\colon \mathcal{L} \to \mathcal{L}, \quad \begin{cases} x \mapsto x & \text{if } x \in \mathcal{L} \setminus \mathcal{L}_1, \\ x \mapsto x^{\pi} & \text{if } x \in \mathcal{L}_1, \end{cases} \tag{9}$$

*is a partial* $\pi$*-transformation. The Plücker transformation* $\delta\beta\colon \mathcal{L} \to \mathcal{L}$ *takes intersecting lines to intersecting lines.*

**Proof.** In order to show that $\delta$ is a well-defined partial $\pi$-transformation, we just have to establish that $a \in \mathcal{L}_1$ implies $a^{\pi} \in \mathcal{L}_1$: Given $Q_0 \in a$ and $Q_1 \in a^{\pi}$ then $Q_0 \vee Q_1$ and $(Q_0 \vee Q_1)^{\beta} = Q_0^{\kappa} \vee Q_1^{\kappa}$ are isotropic lines. Therefore

$$Q_0^{\kappa} \vee Q_1^{\kappa} \neq a^{\beta\pi} = a^{\pi\beta} \notin \mathcal{J}$$

so that $Q_1^{\kappa} \notin a^{\pi\beta}$. Now, by (8), $a^{\pi} \in \mathcal{L}_1$. If distinct lines $b$ and $c$ intersect at a point $R$, then $b^{\delta\beta} \cap c^{\delta\beta} = R^{\kappa}$ follows from (7), (8) and (9). ◊

**Proposition 1.4.** *The mapping* $\kappa\colon \mathcal{P} \to \mathcal{P}$ *defined in* (3) *belongs to* $\mathrm{P\Gamma Sp}(\mathcal{P}, \pi)$. *The Plücker transformation* $\delta\beta$ *is induced by this collineation* $\kappa$.

**Proof.** The bijection $\delta\beta$ is taking intersecting lines to intersecting lines. Every star of lines is mapped under $\delta\beta$ either onto a star of lines or onto a ruled plane [4], [10, Th. 1]. The latter possibility does not occur, since $\delta\beta$ is induced by $\kappa$. Because of $\dim(\mathcal{P}, \mathcal{L})$ being finite, the mapping $\kappa$ is a collineation [10, Th. 3]. Finally, $\mathcal{J}^\beta = \mathcal{J}$ implies $\kappa \in \mathrm{P\Gamma Sp}(\mathcal{P}, \pi)$. ◊

## 3. The higher-dimensional case

**Theorem 2.** *Let* $(\mathcal{P}, \mathcal{L}, \pi)$ *be an n-dimensional symplectic space* $(5 \le n \le \infty)$ *and let* $\beta\colon \mathcal{L} \to \mathcal{L}$ *be a bijection such that*

$$a \sim b \text{ implies } a^\beta \sim b^\beta \text{ for all } a, b \in \mathcal{L}.$$

*Then* $\beta$ *is induced by a collineation* $\kappa \in \mathrm{P\Gamma Sp}(\mathcal{P}, \pi)$.

As before, Th. 2 will be split into several Propositions subject to the assumptions stated above.

**Proposition 2.1.** *The bijection* $\beta$ *takes intersecting lines to intersecting lines. There exists an injective mapping* $\kappa\colon \mathcal{P} \to \mathcal{P}$ *inducing* $\beta$. *This* $\kappa$ *is preserving collinearity and non-collinearity of points. Moreover*

$$\mathcal{L}[Q]^\beta = \mathcal{L}[Q^\kappa] \text{ for all } Q \in \mathcal{P}. \tag{10}$$

**Proof.** Suppose that $a, b \in \mathcal{L}$ meet at a point $Q$. If $a \sim b$, then $a^\beta$ and $b^\beta$ are intersecting. Otherwise, by Lemma 2, $a \notin \mathcal{J}$ and $b \notin \mathcal{J}$. Then $\mathcal{J}[Q] \cup \{a\}$ and $\mathcal{J}[Q] \cup \{b\}$ are, respectively, sets of mutually related lines. Each line of $\mathcal{L}$ is related to at least one line in $\mathcal{J}[Q]$, since $Q^\pi$ is a hyperplane covered by $\mathcal{J}[Q]$. If $\mathcal{J}[Q]^\beta$ were a set of coplanar lines, then all lines in $\mathcal{L}$ would meet a fixed plane in contradiction to $n \ge 5$. Thus $\mathcal{J}[Q]^\beta$ is not contained in a plane, whence there exists a point $Q'$ with $\mathcal{J}[Q]^\beta \subset \mathcal{L}[Q']$. Since the elements of $\mathcal{J}[Q]^\beta \cup \{a^\beta\}$ are mutually related, $Q' \in a^\beta$. Repeating this for $b$ yields $Q' \in b^\beta$. Now the assertions on $\kappa$ follow from [10, Th. 1]. ◊

**Proposition 2.2.** *The bijection* $\beta$ *is a Plücker transformation, since*

$$\mathcal{J}^\beta = \mathcal{J}. \tag{11}$$

**Proof.** Given $a \in \mathcal{J}$ then choose a point $Q \in a$. We observe that $a \sim \mathcal{L}[Q]$, whence $a^\beta \sim \mathcal{L}[Q^\kappa]$ by (10). Since $\mathcal{L}[Q^\kappa]$ contains more than one non-isotropic line, $a^\beta \in \mathcal{J}$ follows from Lemma 2.

Given $b \in \mathcal{L} \setminus \mathcal{J}$ then choose a point $R \in b$. Assume to the contrary that $b^\beta \in \mathcal{J}$. Then for each line

$$x \in \mathcal{L}[R] \setminus (\mathcal{J}[R] \cup \{b\})$$

there exists a line $\overline{x} \in \mathcal{J}[R]$ such that $b, x, \overline{x}$ are three distinct lines in one pencil. By the invariance of collinearity and non-collinearity of points under $\kappa$, as is stated in Prop. 2.1, $b^\beta, x^\beta, \overline{x}^\beta$ are again three distinct lines in one pencil. However, $b^\beta$ and $\overline{x}^\beta$ are isotropic, so that

$$x^\beta \in \mathcal{J}[R^\kappa].$$

Hence $\mathcal{L}[R]^\beta \subset \mathcal{J}[R^\kappa]$ which is impossible by (10).

Now (11) and Lemma 2 show that $\beta$ is a Plücker transformation. $\Diamond$

**Proposition 2.3.** *The mapping $\kappa: \mathcal{P} \to \mathcal{P}$, described in Prop. 2.1, is a collineation belonging to* $\mathrm{P\Gamma Sp}(\mathcal{P}, \pi)$.

**Proof.** Since $\beta$ is a Plücker transformation of $(\mathcal{L}, \sim)$, Prop. 2.1 can be applied to $\beta^{-1}$. Therefore $\beta$ and $\beta^{-1}$ are preserving intersection of lines. By [10, Th. 2], the mapping $\kappa$ is a collineation and, by formula (11), $\kappa \in \mathrm{P\Gamma Sp}(\mathcal{P}, \pi)$. $\Diamond$

This completes the proof of Th. 2.

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