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# BOSE'S VERTEX THEOREM FOR SIMPLY CLOSED POLYGONS

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**Dedicated to Professor Hans Vogler on occasion of his 60th birthday**

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**Abstract:** The notion of curvature for polygons, developed in the context of exterior parallelism for polyhedra (see the author's paper in Math. Pannonica **2**/2 (1991), 95–106), is studied. It is shown that in the case of constant side lengths, a simply closed polygon in the plane admits at least four sides with locally extremal curvature unless it is circrumscribed to a circle. These extrema are called curvature vertices of the polygon.
This kind of four-vertex-theorem is a special case of the polygonal version of a more general vertex-theorem going back to R.C. Bose (Math. Z. **35** (1932), 16–24) in the case of smooth ovals. Here the number of vertices is estimated from below by counting osculations of the curve with circles, contained in the interior of the curve. A polygonal version of Bose's theorem is formulated and proved in this paper. The main tool for the proof is the part of the sysmmetry set which is contained in the interior of the polygon.

## 1. Introduction

For sufficiently smooth ovals $\alpha$ in the Euclidean plane R.C. Bose [2] developed a method to determine a lower bound for the number of their vertices by counting the number of intersections of $\alpha$ with circles contained in the closure of the interior of $\alpha$ as follows: If $\alpha$ admits $n$ distinct circles $c_i$, $i = 1, \dots, n$, of this type with $\nu_i$ ($\nu_i \geq 2$) points of osculation, then $\alpha$ has at least $2 + \sum_{i=1}^{n}(\nu_i - 2)$ vertices (points with locally extremal curvature). An order geometric version of this theorem has been established by O. Haupt [4]. That Bose's theorem can be extended to simply closed curves which are sufficiently smooth, has been shown in the framework of classical differential geometry in [10], using a very simple method of proof.

By establishing a suitable notion of curvature for polygons a four-vertex-theorem has been proved for closed convex polygons in the plane in [12] by elementary geometric means. The classical four-vertex-theorem may be obtained from this version by approximation. It should be pointed out here, that also other notions of curvature for polygons have been studied and that four-vertex theorems have been shown for these notions (see S. Bilinski [1] and A. Schatteman [6]).

The extension of the four-vertex-theorem to simply closed polygons fails in the general case. Counterexamples have been constructed in the master's thesis of A. Walz [9] for both notions of curvature, that in [1] and that in [12]. They all have the common property that their side lengths vary very strongly. On the contrary, the approximation of simply closed smooth curves by inscribed or circumscribed polygons leads to an approximation of the evolutes of the smooth curves by the corresponding polygonal evolutes. In the case of inscribed polygons their side lengths have to be assumed almost constant for this purpose, if the notion of curvature given in [12] is considered. This indicates that simply closed polygons of constant side lengths will be good candidates to satisfy a four-vertex-theorem. Indeed, here it will be shown that even the stronger version of this theorem given by Bose is valid for these polygons.

## 2. Curvatures and symmetry sets

In this section we shall establish the tools for the proof of our main theorem. The notion of curvature follows the lines of [11] resp. [12].

Let $\mathcal{P}$ denote a polygon in the Euclidean plane $E^2$, given by its consecutive vertices $p_i$, $i = 1, \ldots, k$, and its oriented sides $\overrightarrow{p_i p_{i+1}}$ resp. unoriented sides $s_i := \overline{p_i p_{i+1}}$, where in the closed case the subscripts are considered $\bmod k$. The length of the side $s_i$ is given by $\Delta_i := \|p_i - p_{i+1}\|$. The straight line generated by $p_i$ and $p_{i+1}$ is called $l_i := p_i \vee p_{i+1}$, and the interior oriented angle between $l_{i-1}$ and $l_i$ at $p_i$ is denoted by $\gamma_i := \angle(\overrightarrow{p_i p_{i+1}}, \overrightarrow{p_i p_{i-1}})$. $\gamma_i$ is defined to be between 0 and $\pi$, if the pair $(\overrightarrow{p_ip_{i+1}}, \overrightarrow{p_ip_{i-1}})$ is linearly independent and positively oriented; $\gamma_i$ is defined to be between $\pi$ and $2\pi$, if this pair is negatively oriented. If no confusion is possible $\mathcal{P}$ also will denote the subset of $E^2$ defined by $\mathcal{P}$. In the case that $\mathcal{P}$ is simply closed, it bounds a compact domain $D(\mathcal{P})$, the interior of $\mathcal{P}$. Furthermore we assume in this case, that the ordering of its vertices defines the counter-clockwise orientation of $\mathcal{P}$, such that for interior angles $\gamma_i$ with values between 0 and $\pi$ $D(\mathcal{P})$ is locally convex at $p_i$.

For the definition of the curvature of $\mathcal{P}$ we consider the so-called *symmetric normals* $n_i$ of $\mathcal{P}$ which are defined at the vertices $p_i$ different from the ends of $\mathcal{P}$ as the straight lines bisecting the interior angles $\gamma_i$. In analogy to the differentiable case we call the intersection $f_i := n_i \vee n_{i+1}$ the *focal point* of $\mathcal{P}$ at $s_i$. If $n_i$ and $n_{i+1}$ are parallel, we define $f_i$ as the corresponding point at infinity in the sense of projective geometry. The *evolute* of $\mathcal{P}$ is given by the "polygon" defined by the $f_i$ in the given order. This polygon is allowed pass through infinity and to have degenerated sides. If $f_i$ is not at infinity then the radius of curvature $\rho_i$ of $\mathcal{P}$ at $s_i$ is given by

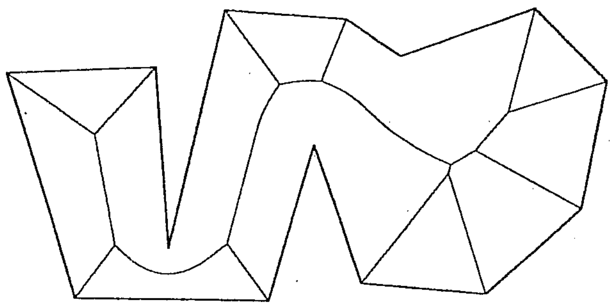
$$\rho_i := d(f_i, l_i) = d(f_i, l_{i-1}) = d(f_i, l_{i+1}).$$

Hence the circle of radius $\rho_i$ around $f_i$ is simultaneously tangential to $l_i, l_{i-1}$, and $l_{i+1}$. It is called the *circle of curvature* if $\mathcal{P}$ at $s_i$. Finally, we define the *curvature* $\kappa_i$ of $\mathcal{P}$ at $s_i$ by

$$\kappa_i := \begin{cases} 0 & f_i \text{ at infinity} \\ \rho_i^{-1} & f_i \text{ finite, and } \overrightarrow{p_if_i} \text{ points to the interior of } \mathcal{P} \\ -\rho_i^{-1} & f_i \text{ finite, and } \overrightarrow{p_if_i} \text{ points to the exterior of } \mathcal{P}. \end{cases}$$

In order to distinguish it from the vertices of $\mathcal{P}$ we call a side of $\mathcal{P}$ with locally extremal curvature a *curvature-vertex* of $\mathcal{P}$. Here "locally" refers to neighboring sides.

Symmetry sets of curves have been introduced by J.W. Bruce, P.J. Giblin and C.G. Gibson [3] as collections of centers for a local reflectional symmetry of the curve. The proof of the version of Bose's vertex theorem in [10] is based on the study of the part of this set which is contained in the interior of the curve. Symmetry sets have been studied in another context by H. Stachel and H. Abdelmoez in [7] and [8] for collections of simply closed polygons bounding some domain in the plane. There also a program for the computation of these sets is established. A very effective algorithm has been developed for this purpose by S. Meyer [5]. These sets also are called medial axes, skeletons or cyclographic sets in other work. Because the curve can be reconstructed as the envelope of a suitable family of circles with centers in that part of the symmetry set which will be of interest for our subsequent considerations, we shall call this part the cyclographic generator.



The figure shows a simply closed polygon (thick lines) with its cyclographic generator (thin lines). It has been computed with the algorithm in [5].

Its precise definition is given as follows: Let $\mathcal{P}$ be a simply closed polygon for all subsequent considerations. A *cyclographic center* $z$ of $\mathcal{P}$ is characterized by the property that there is some radius $r > 0$ such that the closed disk $D_r(z)$ of radius $r$ around the center $z$ is contained in $D(\mathcal{P})$ and its bounding circle $S_r(z)$ has at least two points in common with $\mathcal{P}$. Clearly, the set $S_r(z) \cap \mathcal{P}$ is finite. The points of this set are called the *base points* of the cyclographic center $z$ and $r$ is called its *cyclographic radius*. If $S_r(z)$ is tangential to some side of $\mathcal{P}$ at the base point $p$, then $p$ will be called an *s-base point*. The remaining case, that this intersection is non-tangential, only can happen at vertices with interior angle greater than $\pi$. Then $p$ will be called a *v-base point*. Note that a vertex of $\mathcal{P}$ can be base point for different cyclographic centers of $\mathcal{P}$ and may be of different types with respect to these centers. The closure of the set of cyclographic centers of $\mathcal{P}$ is obtained, if we add to this set the vertices of $\mathcal{P}$, where $D(\mathcal{P})$ is locally convex. Hence we we define the *cyclographic generator* $G(\mathcal{P})$ of $\mathcal{P}$ by

$$G(\mathcal{P}) := \{z \in E^2 \mid z \text{ is a cyclographic center of } \mathcal{P}\} \cup \{p_i \in \mathcal{P} \mid \gamma_i \in (0, \pi)\}.$$

Obviously $G(\mathcal{P})$ is a finite graph in $E^2$ composed from straight line segments and segments of parabolas. For the subsequent considerations we have to describe this structure more precisely. A cyclographic center $z$ with at least three base points is a vertex of this graph. Considering two neighboring base points $p$ and $q$ on $S_r(z)$, $r$ denoting the cyclographic radius of $z$, the intersection of $G(\mathcal{P})$ with the sector of

$D_r(z)$ bounded by $\overline{zp}$, $\overline{zq}$ and the arc $B$ on $S_r(z)$ between $p$ and $q$, which does not contain other base points, is given locally at $z$ by

a) a segment of a straight line emanating from $z$, bisecting the angle between $l_i$ and $l_j$, if $p$ and $q$ are s-base points located on $s_i$ resp. $s_j$, and $s_i$ resp. $s_j$ coincides locally with the one-sided tangential ray to $B$ emanating from $p$ resp. $q$,

b) a segment of a parabola emanating from $z$, defined as the locus of equal distance to the point $p$ and the line $l_j$, if $p$ is an s-base point not satisfying the requirements of part a) or a v-base point, and $q$ is an s-base point satifying the requirements of part a) with respect to $s_j$,

c) a segment of a straight line emanating from $z$, defined as the locus of equal distance to the points $p$ and $q$, if both $p$ and $q$ are s-base points not satisfying the requirements of case a) or if $p$ and $q$ are v-base points.

Up to interchange of the roles of $p$ and $q$, the description given above represents all possible cases. This shows that the valency of $z$ as a vertex of the graph $G(\mathcal{P})$ coincides with the number of base points belonging to $z$.

Arguments of the same type show that if the number of base points of the cyclographic center $z$ is 2, then this remains unchanged for all cyclographic centers in a small neighborhood of $z$. Then $z$ will be considered as a point on a chord of the graph of $G(\mathcal{P})$, if the types of the two base points remain unchanged in a suitable neighborhood of $z$. It it easy to see by the arguments presented above, that otherwise the types of the base points must change at $z$, and that the cyclographic centers of this kind are isolated. In the latter case $z$ will be considered as a vertex of the graph $G(\mathcal{P})$ of valency 2. The only case which is not covered by the preceding considerations of the points of $G(\mathcal{P})$ is that $z \in G(\mathcal{P}) \cap \mathcal{P}$, i.e. $z$ is a vertex of $\mathcal{P}$. But these points are end points of the chords of the graph $G(\mathcal{P})$, given by segments of the symmetric normals at the vertices $p_i$ of $\mathcal{P}$ having interior angle $\gamma_i \in (0, \pi)$. Hence they have to be considered as end points of the graph $G(\mathcal{P})$.

**Proposition 1.** *The cyclographic generator of $\mathcal{P}$ is a deformation retract of $D(\mathcal{P})$. In particular $G(\mathcal{P})$ is connected (see also [15] for this statement in a more general context).*

**Proof.** The idea for the construction of the retraction $R\colon D(\mathcal{P}) \to$ $\to G(\mathcal{P})$ is obtained as follows: Clearly the points of $G(\mathcal{P})$ have to remain fixed. For $x \in D(\mathcal{P}) \setminus (\mathcal{P} \cup G(\mathcal{P}))$ we find a unique point on $y \in \in \mathcal{P}$ which is of minimal distance to $x$; if there were two such points on $\mathcal{P}$, then $x$ would be a cyclographic center. Considering the circles with centers on the ray emanating from $y$ in direction of $\overrightarrow{yx}$ which are passing through $y$ there will be a unique one which bounds a closed disk totally contained in $D(\mathcal{P})$, which contains $x$, and which has more than one point in common with $\mathcal{P}$. The center $z$ of this circle is a cyclographic center of $\mathcal{P}$; $z$ is located outside the segment $\overline{xy}$. The retraction $R$ is defined to move $x$ to $z$. This can be used to show that $G(\mathcal{P}) \setminus \mathcal{P}$ is a deformation retract of $D(\mathcal{P}) \setminus \mathcal{P}$. To extend the retraction to the open sides and "convex" vertices of $\mathcal{P}$ is trivial. To cover also the vertices of $\mathcal{P}$ with this construction where $D(\mathcal{P})$ has an interior angle greater than $\pi$, we have to modify the given map in a small neighborhood of these points accordingly. But this is an easy topological exercise. Because $G(\mathcal{P})$ is the continuous image of the connected set $D(\mathcal{P})$, it must be connected itself. $\Diamond$

**Proposition 2**. *The graph $G(\mathcal{P})$ is a tree.*

**Proof**. By Prop. 1, $G(\mathcal{P})$ is connected. Let $z$ be a point on $G(\mathcal{P})$ which is not an end point. Then $z$ is a cyclographic center with at least two base points $p$ and $q$. Removing $\overline{zp} \cap \overline{zq}$ from the closed set $D(\mathcal{P})$ disconnects the remaining set. But $G(\mathcal{P}) \setminus \{z\}$ is contained in the remaining set. Hence the removal of $z$ from $G(\mathcal{P})$ disconnects the cyclographic center, which shows that the graph $G(\mathcal{P})$ does not contain any cycle. $\Diamond$

The proof of Bose's theorem in [12] uses the fact that the ends of the cyclographic generator defined there coincide with centers of curvature belonging to points with locally maximal curvature. Since these ends are vertices of $\mathcal{P}$ in our case, this method will not work in the case of polygons. Nevertheless the cyclographic generator can be used for our purposes, if it is modified slightly as follows:

**Reduction**. To obtain the *reduced cyclographic generator* $G_{\text{red}}(\mathcal{P})$ of $\mathcal{P}$ we remove

a) all ends from $G(\mathcal{P})$, i.e. we obtain $G(\mathcal{P}) \setminus \mathcal{P}$;

b) for such an end $p_i \in G(\mathcal{P}) \cap \mathcal{P}$ we remove the open chord $\sigma_i$ of the graph $G(\mathcal{P})$ ending at $p_i$ from this set (this chord is is an open segment on the symmetric normal $n_i$);

c) if the other end point $p_i^+$ of $\sigma_i$ is of valency 2, then we remove $p_i^+$ and the other open chord $\sigma_i^+$ of $G(\mathcal{P})$ ending at $p_i^+$ from the

cyclographic generator of $\mathcal{P}$.
This completes the reduction procedure.

In case c) $p_i^+$, as a cyclographic center of $\mathcal{P}$, has two base points which obviously are s-base points. This is inherited from the cyclographic centers on $\sigma_i$. The type changes at $p_i^+$ if we consider the cyclographic centers on $\sigma_i^+$. There one or both base points will be v-base points. Hence $\sigma_i^+$ is of type b) (segment of a parabola) or c) (straight line segment) mentioned in the description of the combinatorial structure of $G(\mathcal{P})$ given above.

**Proposition 3.** *The reduced cyclographic generator $G_{\text{red}}(\mathcal{P})$ is a tree.*

**Proof.** This is obvious, because according to Prop. 2 and the reduction procedure we have removed from the tree $G(\mathcal{P})$ only semi-open arcs, containing ends of $G(\mathcal{P})$ and possibly passing through vertices of valency 2. $\Diamond$

## 3. Curvature vertices of polygons

**Lemma 1.** *Let $\mathcal{P}$ be a simply closed polygon with constant side lengths, $z$ an end point of $G_{\text{red}}(\mathcal{P})$, such that for some $i = 1, \dots, k$ the segments $\overline{zp_i}$ and $\overline{zp_{i+1}}$ are chords of $G(\mathcal{P})$ ending on $\mathcal{P}$. Then $\kappa_{i-1} \leq \kappa_i$ and $\kappa_{i+1} \leq \kappa_i$. The equality is satisfied if and only if $\overline{zp_{i-1}}$ resp. $\overline{zp_{i+1}}$ are chords of $G(\mathcal{P})$.*

**Proof.** Let $r$ denote the cyclographic radius of $\mathcal{P}$ at $z$. According to our assumptions $S_r(z)$ coincides with the circle of curvature of $\mathcal{P}$ at $s_i$, and $z$ is of valency $\geq 3$ in $G(\mathcal{P})$ having s-base points on $s_{i-1}, s_i$, and $s_{i+1}$. Hence we have $\rho_i = r > 0$. Furthermore $\Delta_{i-2} = \Delta_{i-1} = \Delta_i$ implies that the symmetric normal $n_{i-1}$ does not meet $\overline{zp_i}$ in its interior, because $s_{i-2}$ has empty intersection with the interior of $S_r(z)$. Hence we get for the radii of curvature $\rho_{i-1} \geq \rho_i$ in the case that $\kappa_{i-1}$ is positive. This implies $\kappa_{i-1} \leq \kappa_i$. Furthermore the equality is valid, if and only if $n_{i-1}$ meets $\overline{zp_i}$ at $z$ which is equivalent to $s_{i-2} \cap S_r(z) \neq \emptyset$. The other inequality is obtained similarly. $\Diamond$

**Remark 1.** The same method of proof applies if $\mathcal{P}$ is assumed to be convex instead of assuming constant side lengths. In this case the convexity of $\mathcal{P}$ implies that $l_{i-1}$ has empty intersection with the interior of $S_r(z)$, from which we again conclude $\rho_{i-1} \geq \rho_i$.

**Lemma 2.** *Let $\mathcal{P}$ be a simply closed polygon with constant side lengths, $z$ an end point of $G_{\text{red}}(\mathcal{P})$, which is obtained by removing a piece $\sigma_i$ of a* *symmetric normal for some $i = 1, \ldots, k$, removing the bounding vertex $p_i^+$ of valency 2, and the other semi-open chord $\sigma_i^+$ of the graph $G(\mathcal{P})$ ending at $p_i^+$ from $G(\mathcal{P})$ according to our reduction procedure. Let $z^+$ + be the vertex next to $z$ on $G_{\text{red}}(\mathcal{P})$ and $\mathcal{Q}$ be the minimal polygonal subarc of $\mathcal{P}$ containing two neighboring base points $p$ and $q$ of $z^+$ and $p_i$. Then the curvature of $\mathcal{P}$ attains a local maximum (with respect to $\mathcal{P}$) at a side of $\mathcal{Q}$ which is not at one of the ends of this polygonal arc, if the corresponding base point is an* s-*base point.*

**Proof.** Let $r$ be the cyclographic radius of $z$. Then according to our construction $p_{i-1}$ and $p_{i+1}$ must be v-base points of the cyclographic center $z$, because $\mathcal{P}$ has constant side lengths. Since $z$ is a vertex of the graph $G(\mathcal{P})$, there must be another base point of $z$. Otherwise $S_r(z)$ could be enlarged by moving $z$ on the line at equal distance to $p_{i-1}$ and $p_{i+1}$ and increasing $r$ suitably without having points of $\mathcal{P}$ in its interior, which would imply that $z$ is in the interior of a chord of the graph $G(\mathcal{P})$. Considering from of these additional base points one which is nearest to $p_{i-1}$ or $p_{i+1}$, we conclude from the assumption that $z$ is an end point of $G_{\text{red}}(\mathcal{P})$ and that the side lengths of $\mathcal{P}$ are constant, that this point must be $p_{i-3}$ or $p_{i+3}$. Extending this argument to possibly remaining base points we get, that all base points of $z$ can be described in the form $p_{i+2j-1}$ for $j$ between some suitable integers $m_1 \leq 0$ and $m_2 \geq 1$.

Let $p$ be the base point of $z^+$ next to $p_{i+2m_1-1}$. If $p$ is an s-base point, then $p$ is not contained in the segments $s_m$ for all $m = i + 2m_1 - 1, \ldots, i + 2m_2 - 2$. If $p$ is a v-base point, then $p = p_m$ for some $m \leq i + 2m_1 - 1$. Since $m_2 - m_1 \geq 2$ we have $p_{i+2m_1-1}, p_{i+2m_1}, p_{i+2m_1+1}, p_{i+2m_1+2}, p_{i+2m_1+3} \in S_r(z)$. Furthermore $\Delta_{i+2m_1-1} = \Delta_{i+2m_1} = \Delta_{i+2m_1+1} = \Delta_{i+2m_1+2}$ implies $z \in n_{i+2m_1} \cap n_{i+2m_1+2}$, and therefore $f_{i+2m_1} \in \overline{zp_{i+2m_1}}$ or $f_{i+2m_1+1} \in \overline{zp_{i+2m_1+2}}$ from which we conclude $\kappa_{i+2m_1} > 0$ or $\kappa_{i+2m_1+1} > 0$. Furthermore the following conditions are equivalent: $f_{i+2m_1} = Z \Leftrightarrow \kappa_{i+2m_1} = \kappa_{i+2m_1+1} > 0 \Leftrightarrow f_{i+2m_1+1} = z$.

Case 1: If $f_{i+2m_1} \in \overline{zp_{i+2m_1}}$ and $f_{i+2m_1} \neq z$, then this implies $\kappa_{i+2m_1} > 0$ and $\kappa_{i+2m_1} > \kappa_{i+2m_1+1}$. If $f_{i+2m_1-1} \notin \overline{zp_{i+2m_1}} \setminus \{z\}$, then we have $\kappa_{i+2m_1} > \kappa_{i+2m_1-1}$ again, which shows that the curvature of $\mathcal{P}$ attains a local maximum at $s_{i+2m_1}$. If $f_{i+2m_1-1} \in \overline{zp_{i+2m_1}} \setminus \{z\}$, we have to consider $f_{i+2m_1-2}$. In this case $\kappa_{i+2m_1-1}$ is guaranteed to be positive. By assumption on $z^+$ and the minimality of $i + 2m_1 - 1$ $p_{i+2m_1-3}$ must remain outside $S_r(z)$. Hence $\Delta_{i+2m_1-3} = \Delta_{i+2m_1-2} =$

$= \Delta_{i+2m_1-1}$ implies that $n_{i+2m_1-2}$ does not intersect $\overline{zp_{i+2m_1-1}}$ and therefore has no point in common with $\overline{f_{i+2m_1-1}p_{i+2m_1-1}}$. This shows that $\kappa_{i+2m_1-1} > \kappa_{i+2m_1-2}$. Hence we conclude that the curvature of $\mathcal{P}$ attains a local maximum at $s_{i+2m_1}$ or $s_{i+2m_1-1}$.

Case 2: The remaining case is that $\kappa_{i+2m_1} = \kappa_{i+2m_1+1} > 0$ or $f_{i+2m_1} \notin \overline{zp_{i+2m_1}}$. This implies that $f_{i+2m_1+1} \in \overline{zp_{i+2m_1+2}}$, and from this we conclude $\kappa_{i+2m_1} \leq \kappa_{i+2m_1+1}$ and $\kappa_{i+2m_1+1} > 0$. In the case of equality the same argument as in case 1 will show the curvature decreases again on $s_{i+2m_1-1}$ or $s_{i+2m_1-2}$. Hence, if we assume that the curvature of $\mathcal{P}$ attains no local maximum on $s_l$ for some $l = i + 2m_1 - - 1, \ldots i + 2m_2 - 4$, then $f_{i+2m_2-3} \in \overline{zp_{i+2m_2-2}}$ and $\kappa_{i+2m_2-3} > 0$. Now we apply the same argument as in case 1, replacing $p_{i+2m_1+1}$ by $p_{i+2m_2-3}$, $p_{i+2m_1}$ by $p_{i+2m_2-2}$, $p_{i+2m_1-1}$ by $p_{i+2m_2-1}$, $p_{i+2m_1-2}$ by $p_{i+2m_2}$, and $p$ by $q$. Then we get that the curvature of $\mathcal{P}$ attains a local maximum at $s_{i+2m_2-3}$ or $s_{i+2m_2-2}$. This proves our lemma. $\Diamond$

**Special case.** We want to consider the case that $\mathcal{P}$ has constant side lenghts and that $G_{\text{red}}(\mathcal{P})$ consists of one point $z$ only. Then there are two possibilities only:

i) All base points of $z$ are s-base points and there is such a base point on every side of $\mathcal{P}$, i.e. $\mathcal{P}$ is circumscribed tangentially to the fixed circle $S_r(z)$, $r$ denoting the cyclographic radius of $z$. In this case the curvature of $\mathcal{P}$ has the constant value $r^{-1}$.

ii) All base points of $z$ are v-base points, and every second vertex of $\mathcal{P}$ is such a base point. Hence we may assume without loss of generality that $p_i \in S_r(z)$ for all $i = 2j$, $j = 1, \ldots, m$, $2m = k$. Then $z \in n_{2j-1}$ for all $j = 1, \ldots, m$. If we also have $z \in n_{2j}$ for these $j$, then $\mathcal{P}$ has constant curvature. Otherwise some $n_{2j}$ will meet one of the segments $\overline{zp_{2j+1}}$ and $\overline{zp_{2j-1}}$ in its interior and will have no point in common with the other one. Then a refinement of the arguments in the preceding proof will show, that the curvature of $\mathcal{P}$ attains at least two local maxima on $\mathcal{P}$ and at least two local minima. We leave this to the reader because no essentially new argumentation is needed.

**Remark 2.** The type of local maxima (resp. minima) we have found are strict in the following sense: $\kappa_i \geq \kappa_j$ for $j = i+1, i-1$ and there are $m_1 \leq i$, $m_2 \geq i$ such that $\kappa_i = \kappa_j$ for $j = m_1, \ldots, m_2$ and $\kappa_i > > \kappa_j$ for $j = m_1 - 1, m_2 + 1$. This also refers to Lemma 1, if $\mathcal{P}$ is not circumscribed to a fixed circle.

**Theorem 1.** *Let $\mathcal{P}$ be a simply closed polygon with constant side* *lengths. Let $E$ denote the number of ends of it reduced cyclographic generator $G_{\mathrm{red}}(\mathcal{P})$. Then, with exception of the case that the curvature of $\mathcal{P}$ is constant (i.e. that there exists some circle such that all straight lines belonging to sides of $\mathcal{P}$ are tangents to this circle), $\mathcal{P}$ has at least $2E$ curvature vertices. For $E = 1$ $\mathcal{P}$ has at least four curvature vertices.*

**Proof.** According to the considerations made in the special case above, we only have to consider the case when $E \geq 2$. In this case the curvature cannot be constant. Lemma 1 and Lemma 2 imply that there belongs a strict local maximum of curvature (in the sense of Remark 2) to every end point $z$ of $G_{\mathrm{red}}(\mathcal{P})$. This can be found on a polygonal arc of $\mathcal{P}$ which is associated in a unique way with $z$ and cannot be associated in the same way with another end point of $G_{\mathrm{red}}(\mathcal{P})$. This shows that the number of strict local maxima of the curvature of $\mathcal{P}$ is at least $E$. But then we have the same estimate for the number of local minima, which proves our theorem. $\Diamond$

**Corollary** (Four-Vertex-Theorem). *Every simply closed polygon with constant side lengths is circumscribed to a circle or has at least four curvature vertices.*

Because according to Prop. 3 $G_{\mathrm{red}}(\mathcal{P})$ is a tree, we have the following formula for the valencies $\mathrm{Val}(z_i)$ of the vertices $z_1, \ldots, z_l$ of this graph (in the non-trivial case), where $E$ again denotes the number of its ends:

$$E = 2 + \sum_{i=1,\mathrm{Val}(z_i)>1}^{l} (\mathrm{Val}(z_i) - 2).$$

Hence we obtain as a lower estimate for the number of curvature vertices according to Theorem 1 the number $2(2 + \sum_{i=1,\mathrm{Val}(z_i)>1}^{l} (\mathrm{Val}(z_i) - 2))$.

Let $S_R(x)$ be a circle which does not contain any point of $\mathcal{P}$ in its interior and has at least two points in common with $\mathcal{P}$. Then $x \in G(\mathcal{P})$. We call the contact of $S_R(x)$ with $\mathcal{P}$ *generic* at the point of intersection $p$, if

a) $p$ is in the interior of some side $s_i$ of $\mathcal{P}$ and the neighboring sides $s_{i-1}$ and $s_{i+1}$ have empty intersection with $\mathcal{P}$ or

b) $p$ is some vertex $p_i$ of $\mathcal{P}$ and the sides $s_{i-2}$ and $s_{i+1}$ have empty intersection with $\mathcal{P}$.

**Theorem 2** (Bose's Theorem). *Let $\mathcal{P}$ be a simply closed polygon with constant side lengths. Let $S_{R_j}(x_j)$, $j = 1, \ldots, m$, be distinct circles which do not contain any point of $\mathcal{P}$ in their interior and have at least*

*$\nu_j \geq 2$ points in common with $\mathcal{P}$. Assume that all intersections of these circles with $\mathcal{P}$ are generic. Then $\mathcal{P}$ has at least $2(2+\sum_{j=1}^{m}(\nu_j - 2))$ curvature vertices.*

**Proof.** According to our assumptions $x_j$ is a vertex of the graph $G(\mathcal{P})$ of valency $\nu_j$. The genericity of the contact of $S_{R_j}(x_j)$ with $\mathcal{P}$ at any of the base points of $x_j$ implies that the reduction procedure could not have removed the chords of $G(\mathcal{P})$ emanating from $x_j$. Hence the valency of $x_j$ as a vertex of the graph $G_{\text{red}}(\mathcal{P})$ also is $\nu_j$. From this we conclude with Theorem 1 and the considerations following to that theorem for the number $V$ of curvature vertices of $\mathcal{P}$, using the notation introduced there,

$$V \geq 2(2+\sum_{i=1,\text{Val}(z_i)>1}^{l}(\text{Val}(z_i)-2)) \geq 2(2+\sum_{j=1}^{m}(\nu_j - 2)). \ \Diamond$$

**Remark 3.** The genericity assumption can be removed, if the parts are counted more carefully, where $x_j$ is the center of curvature for some subarc of $\mathcal{P}$. But then the corresponding formulation of Theorem 2 will become very complicated.

**Remark 4.** In the case of a convex polygon, only Lemma 1 will be needed for a proof of Theorems 1 and 2. Hence in view of Remark 1 we have not to assume that the side lengths of $\mathcal{P}$ are constant. This shows that Theorems 1 and 2 are valid for convex polygons in general, as has been demonstrated in [13].

**Remark 5.** The assumption that the side lengths of $\mathcal{P}$ are constant, relates the circles of curvature of $\mathcal{P}$ in a simple way with the interior angles of $\mathcal{P}$. Hence results can be derived from Theorems 1 and 2 for local extrema of these angles or sums of neighboring angles. Results of this type have been described in the convex case in [14].

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