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THE ISOTROPIC 3–SPHERE

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Abstract: Let S^3 be the 3-sphere which is canonically imbedded in the euclidean 4-space E^4. First we introduce a degenerate metric g of rank 2 on S^3 and consider the metric properties of (S^3, g), the so-called isotropic 3-sphere $S^{3(1)}$. Next we define a connection ∇ on $S^{3(1)}$, the so-called standard connection, which is symmetric and compatible with the degenerate metric g, and study some properties of the curvature of ∇. The last chapter is devoted to the curve theory. We develop the fundamentals of this theory up to the derivation equations. There are some analogies to the well-known isotropic 3-space $I_3^{(1)}$.

1. The degenerate metric

We consider the 3-sphere S^3 with its canonical imbedding in the euclidean 4-space E^4. Our aim is to establish a degenerate metric g of rank 2 on S^3 and to give an introduction to the geometry of the so-called *isotropic 3-space* $S^{3(1)} = (S^3, g)$. There are some known examples of isotropic 3-manifolds, besides the isotropic submanifolds, e.g. the well-known 1-fold isotropic 3-space $I_3^{(1)}$, which was studied by K. Strubecker, H. Sachs and others, the 2-fold isotropic 3-space $I_3^{(2)}$, the geometry of which was developed by H. Brauner, the 1-fold isotropic manifold $S^2 \times \times \mathbb{R}$, recently introduced by K. Spitzmüller, and the 1-fold isotropic conform space $C_3^{(1)}$ by W. Vogel.

Let E^4 denote the 4-space $\mathbb{R}^4$ with euclidean metric $< >$ and suppose the 3-sphere S^3 to be canonically imbedded in the euclidean space

E^4. For a point $p \in S^3 \subset E^4$ with position vector $P = (x^1, x^2, x^3, x^4)^\top$ we have

$$(1) \qquad < P, P >= 1.$$

Then the vectors $X_1 = (-x^3, -x^4, x^1, x^2)^\top, X_2 = (-x^4, x^3, -x^2, x^1)^\top$, $X_3 = (-x^2, x^1, x^4, -x^3)^\top$ are, together with P, mutually orthonormal in E^4, i.e.

$$(2) \qquad < X_i, X_j >= \delta_{ij}, < P, X_i >= 0; \quad i, j \in \{1, 2, 3\}.$$

X_1, X_2, X_3 span the tangent space T_pS^3 when we identify the space T_pS^3 with the corresponding 3-subspace of $\mathbb{R}^4$. They form a basis of the Lie algebra $\mathcal{G}$ of the Lie group $G = S^3$.

Now we define a degenerate metric g_p in T_pS^3, $p \in S^3$, analogously as in the isotropic 3-space $I_3^{(1)}$, namely

$$(3) \qquad \begin{aligned} g_p(X_a, X_b) &= \delta_{ab} \qquad && a, b \in \{1, 2\}, \\ g_p(X_i, X_3) &= 0 && i \in \{1, 2, 3\}. \end{aligned}$$

X_3 fixes the *isotropic direction* in the tangent space T_pS^3. Then for two vectors $X, Y \in T_pS^3$, $\quad X = \xi^i X_i, Y = \eta^i X_i$, the *scalar-product* is given by

$$(4) \qquad (X, Y) = g_p(X, Y) = \delta_{ab}\xi^a\eta^b.$$

Here and in the following we adopt Einstein's summation convention for the indices $a, b, \ldots$ and $i, j, \ldots$. For the *length (norm)* of a vector X we have

$$(5) \qquad |X| = \sqrt{(X, X)} \geq 0.$$

X is called an *isotropic vector* if $(X, Y) = 0 \quad$ for all $\quad Y \in T_pS^3$. Thus X is isotropic iff $|X| = 0$. In this case $X = \xi^3 X_3; \xi^1 = \xi^2 = 0$, and we call

$$(6) \qquad ||X|| = \xi^3$$

the *colength (conorm)* of the isotropic vector X. This is the analogue to the notion „Spanne" in the isotropic 3-space $I_3^{(1)}$. For two non-isotropic vectors X, Y we define the *angle* φ between X, Y in the usual manner as

$$(7) \qquad \cos\varphi = \frac{(X, Y)}{|X||Y|}.$$

In the case $(X, Y) = |X||Y| \neq 0, \quad \varphi = 0$, we introduce the *coangle* between X, Y analogously as in the isotropic 3-space, and so on. This is possible because we are given three distingueshed vectors X_1, X_2, X_3

in T_pS^3. Finally we mention the *scalar triple product* of the three vectors X, Y, Z, that is

$$(X,Y,Z) = \det(X,Y,Z). \tag{8}$$

Now let g be the metric C^∞-tensor field on S^3 with $g(p) = g_p$, and $S^{3(1)} = (S^3, g)$. $\mathcal{F}S^3, \mathcal{X}S^3, \mathcal{X}_0S^3$ denote the set of the C^∞-scalar fields, C^∞-vector fields, C^∞-vector fields of isotropic vectors of S^3 respectively. The Lie derivative of the metric g in the isotropic direction is given by

$$\begin{aligned} L_{Z_0}g(X,Y) &= Z_0(g(X,Y)) - g([Z_0,X],Y) - g(X,[Z_0,Y]), \\ & X,Y \in \mathcal{X}S^3, Z_0 \in \mathcal{X}_0S^3. \end{aligned} \tag{9}$$

It is easy to see that in $S^{3(1)}$

$$L_{Z_0}g = 0\ , \quad Z_0 \in \mathcal{X}_0S^3. \tag{10}$$

As is well-known a degenerate metric g of a manifold M is *absolutely reducible* iff (10) holds (Dautcourt [2], 320). That means to each point $p \in S^{3(1)}$ there exists a chart $(U,\varphi), p \in U, \varphi(p) = u = (u^1,u^2,u^3) \in \mathbb{R}^3$, such that the component matrix of g has the form

$$\left(g_{ij}(u^k)\right) = \begin{pmatrix} g_{11}(u^a) & g_{12}(u^a) & 0 \\ g_{21}(u^a) & g_{22}(u^a) & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \begin{matrix} i,j,k \in \{1,2,3\} \\ a \in \{1,2\} \end{matrix}.$$

If for example the position vector P of a point $p \in S^3 \subset \mathbb{R}^4$ is given by (12)

$$P = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} = \begin{pmatrix} \cos u^1 \cos u^2 \cos u^3 \\ \sin u^1 \cos u^2 \cos u^3 \\ \sin u^2 \cos u^3 \\ \sin u^3 \end{pmatrix}, \quad \cos u^2 \neq 0, \cos u^3 \neq 0, \tag{12}$$

the components of the degenerate metric g read

$$\begin{aligned} g_{11} &= \cos^2 u^2 \cos^2 u^3 (1 - \cos^2 u^2 \cos^2 u^3) \\ g_{12} &= g_{21} = -\cos^3 u^2 \cos^3 u^3 \sin u^3 \\ g_{13} &= g_{31} = \sin u^2 \cos^2 u^2 \cos^2 u^3 \\ g_{22} &= \cos^2 u^3 (1 - \cos^2 u^2 \sin^2 u^3) \\ g_{23} &= g_{32} = \cos u^2 \sin u^2 \cos u^3 \sin u^3 \\ g_{33} &= \cos^2 u^2, \end{aligned}$$

where rank $(g_{ij}) = 2$. Then we can introduce a new coordinate system $(\bar{u}^1, \bar{u}^2, \bar{u}^3)$ such that the component matrix $(\bar{g}_{ij})$ has the form (11), namely

$$(14) \qquad \left(\bar{g}_{ij}(\bar{u}^k)\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos^2\bar{u}^1 \sin^2\bar{u}^1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

2. The standard connection

Unfortunately the induced connection of $S^3 \subset E^4$ is not a metric connection relative to g, i.e. it is not compatible with the degenerate metric g. But as is well-known a degenerate metric g admits a symmetric and metric connection ∇ iff g is absolutely reducible (Vogel [7], 107). On the other hand ∇ is, if it exists, not uniquely determined by the degenerate metric g. With the help of the vector fields X_1, X_2, X_3 we define a linear connection ∇ as follows:

(i) ∇ is the induced connection of $S^3 \subset E^4$ relative to X_1, X_2,
(ii) the covariant derivative relative to X_3 vanishes,
(iii) ∇ is symmetric.

That means

$$(15) \qquad \begin{gathered} \nabla_{X_1}X_1 = 0, \;\; \nabla_{X_1}X_2 = X_3, \;\; \nabla_{X_2}X_1 = -X_3, \;\; \nabla_{X_2}X_2 = 0, \\ \nabla_{X_1}X_3 = 0, \;\; \nabla_{X_2}X_3 = 0, \;\; \nabla_{X_3}X_3 = 0, \\ \nabla_{X_3}X_1 = 2X_2, \;\; \nabla_{X_3}X_2 = -2X_1. \end{gathered}$$

Then for $X, Y \in \mathcal{X}S^3; X = \xi^i X_i, Y = \eta^i X_i$, we have

$$(16) \qquad \nabla_X Y = \left(\xi^j X_j \eta^i\right) X_i + \xi^j \eta^i \nabla_{X_j} X_i \quad ; \quad i, j \in \{1, 2, 3\}.$$

∇ is a uniquely determined, symmetric and metric linear connection relative to g. We don't write down the components Λ^i_{jk} of the connection ∇ relative to a coordinate system (u^i) of $S^{3(1)}$ because the formulas are a little complicated. We call ∇ the *standard connection* of $S^{3(1)}$.

Since ∇ is symmetric, the torsion vanishes:

$$(17) \qquad T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = 0.$$

Next we consider the curvature tensor of ∇

$$(18) \qquad R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

and the sectional curvature

$$(19) \qquad K_\sigma = \frac{g(R(X, Y)Y, X)}{g(X, X)g(Y, Y) - (g(X, Y))^2}$$

in the direction of the plane σ spanned by X, Y. For a given vector $X = \xi^i X_i$ let $\bar{X} = \xi^a X_a, a \in \{1, 2\}$, denote the *horizontal projection* of X. Then we find

$$R(X,Y)Z = R(\bar{X},\bar{Y})\bar{Z} = 4(g(\bar{Y},\bar{Z})\bar{X} - g(\bar{X},\bar{Z})\bar{Y}). \tag{20}$$

It is $R(X,Y)Z = 0$ if any of the vectors X, Y, Z is an isotropic vector. Since $g(X,Y) = g(\bar{X},\bar{Y})$ we see that the denominator (and also the numerator) of K_σ vanishes if σ is an *isotropic plane* (contains the isotropic vector X_3), and vice versa, and then K_σ is not defined. In case K_σ is defined we have $K_\sigma = \text{const}$, exactly

$$K_\sigma = 4. \tag{21}$$

The Ricci curvature tensor of ∇

$$\text{Ric}(Y,Z) = \text{trace}(X \to R(X,Y)Z) \tag{22}$$

is a constant multiple of the degenerate metric tensor g, namely

$$\text{Ric} = 4g. \tag{23}$$

3. Curve theory

Let $c : I \to S^{3(1)}, t \mapsto c(t)$ be a curve in $S^{3(1)}$ and X, Y two vector fields along c. In particular $T : t \mapsto T(t)$ denotes the tangent vector field $T = \dot{c}$. The covariant derivative of X along c is defined as usual by

$$\frac{DX}{dt} = \nabla_{\dot{c}}\tilde{X}, \tag{24}$$

where $X = \tilde{X} \circ c$, and ∇ means the standard connection on $S^{3(1)}$. Since ∇ is a metric connection relative to g we have the product formulas

$$\frac{D(g(X,Y))}{dt} = \frac{d(g(X,Y))}{dt} = g\left(\frac{DX}{dt}, Y\right) + g\left(X, \frac{DY}{dt}\right), \tag{25}$$

and especially

$$\frac{D|X|^2}{dt} = \frac{d|X|^2}{dt} = 2g\left(\frac{DX}{dt}, X\right). \tag{26}$$

If X is an isotropic vector then

$$\frac{D||X||}{dt} = \frac{d||X||}{dt} = \left|\left|\frac{DX}{dt}\right|\right|. \tag{27}$$

In the following we make the general assumption that T is not an isotropic vector:

(28) $$|T(t)| \neq 0 \quad , \quad t \in I.$$

Then we can use the curve length of c

(29) $$s = \int_{t_0}^{t} \sqrt{g(\dot{c}(\tau), \dot{c}(\tau))} \quad d\tau$$

as the parameter of the curve. In what follows we consider the mapping $c : I \to S^{3(1)}, s \mapsto c(s)$ and write c' instead of $\dot{c}$. Thus the tangent vector field $T(s) = c'(s)$ is a unit vector field:

(30) $$\boxed{|T(s)| = 1 \, , \quad s \in I.}$$

$B(s) = X_3 \circ c(s)$ describes the isotropic unit vector field along c, hence

(31) $$\boxed{||B(s)|| = 1 \quad s \in I.}$$

At the point $p = c(s)$ the derivative $\frac{DT}{ds}(s)$ may be an isotropic or a non-isotropic vector. As in the isotropic 3-space $I_3^{(1)}$ the theory splits into two cases according as the derivative of the unit tangent vector T along c is an isotropic vector or not.

First we consider

case 1 $$\left| \frac{DT}{ds}(s) \right| = 0 \, , \quad s \in I.$$

Thus $\frac{DT}{ds} = kB$. The factor k is the colength of $\frac{DT}{ds}$,

(32) $$k = \left\| \frac{DT}{ds} \right\|,$$

and is called the *curvature* of the curve c in the case 1. As we have seen the covariant derivative of the isotropic vector field B along c vanishes. All together we obtain the derivation equations

(33) $$\boxed{\begin{aligned} \frac{DT}{ds} &= kB \\ \frac{DB}{ds} &= 0 \end{aligned}} \, .$$

If we write the tangent vector T in the form

(34) $$T = t^1 X_1 \circ c + t^2 X_2 \circ c + t^3 X_3 \circ c,$$

it follows

(35) $$t^1 = \cos\varphi(s), t^2 = \sin\varphi(s), t^3(s),$$

where

(36) $$\boxed{\begin{array}{l} t^{3'} = k \\ \varphi' = -2t^3 \end{array}}.$$

The derivative of the third component t^3 of T with respect to s is equal to the curvature k.

To find the curve $P(s)$ in $S^3 \subset E^4$ in the case 1 we have to integrate the differential equation $P' = T$, T given by (34). In detail

(37) $$P'(s) = \begin{pmatrix} \frac{dx^1}{ds} \\ \frac{dx^2}{ds} \\ \frac{dx^3}{ds} \\ \frac{dx^4}{ds} \end{pmatrix} = t^1 \begin{pmatrix} -x^3 \\ -x^4 \\ x^1 \\ x^2 \end{pmatrix} + t^2 \begin{pmatrix} -x^4 \\ x^3 \\ -x^2 \\ x^1 \end{pmatrix} + t^3 \begin{pmatrix} -x^2 \\ x^1 \\ x^4 \\ -x^3 \end{pmatrix},$$

where t^1, t^2, t^3 are the functions (35), and $\varphi(s)$, $t^3(s)$ satisfy (36). $P(s)$ can be described as follows. There exists an orthonormal frame in E^4 with coordinates (y^1, y^2, y^3, y^4) such that

(38) $$P(s) = \begin{pmatrix} y^1(s) \\ y^2(s) \\ y^3(s) \\ y^4(s) \end{pmatrix} = \begin{pmatrix} r_0 \cos\alpha_1(s) \\ r_0 \sin\alpha_1(s) \\ \bar{r}_0 \cos\alpha_2(s) \\ \bar{r}_0 \sin\alpha_2(s) \end{pmatrix},$$

$r_0 = \text{const}, \bar{r}_0 = \text{const}, r_0^2 + \bar{r}_0^2 = 1$. The projections of the curve into the (y^1, y^2)-plane and the (y^3, y^4)-plane lie on circles. The radius r_0 is allowed to be arbitrary. The angular velocities $\alpha' = \frac{d\alpha}{ds}$ depend on the curvature k of the curve and are given by

(39) $$\alpha' = t^3 \pm 1 \quad , \quad t^{3'} = k.$$

In the case of $k = 0$, $\frac{DT}{ds} = 0$, the angular velocities are constant $\alpha' = t_0^3 \pm 1$. There are closed and non-closed curves with vanishing curvature k (geodesics) according as t_0^3 is a rational or an irrational number.

Now we suppose

case 2 $$\left|\tfrac{DT}{ds}(s)\right| \neq 0 \quad , \quad s \in I.$$

We define the *principal normal vector* $H(s)$ of the curve by

(40) $$H = \pm \left|\frac{DT}{ds}\right|^{-1} \frac{DT}{ds},$$

and the *binormal vector* $B(s)$ by

$$B = X_3 \circ c. \tag{41}$$

As can be seen the triple product of the vectors T, H, B is equal to ± 1. We choose the sign resp. orientation of H so that

$$(T, H, B) = 1. \tag{42}$$

The vectors T, H, B form the moving frame of the curve. The *derivation equations* can be found in the usual manner and read

$$\boxed{\begin{array}{lccc} \frac{DT}{ds} = & & \kappa H & \\ \frac{DH}{ds} = & -\kappa T & & +\tau B \\ \frac{DB}{ds} = & & 0 & \end{array}} . \tag{43}$$

They have the same shape as in the isotropic 3-space $I_3^{(1)}$ and are in some sense analogous to the formulas of Frenet in the euclidean space. κ is the *isotropic curvature and* τ *the isotropic torsion* of the curve. The formulas for κ, τ are

$$\kappa = \left(T, \frac{DT}{ds}, B\right), \quad \tau = \left|\frac{DT}{ds}\right|^{-2} \left(T, \frac{DT}{ds}, \frac{D^2T}{ds^2}\right). \tag{44}$$

If we write the tangent vector T in the form

$$T = t^1 X_1 \circ c + t^2 X_2 \circ c + t^3 X_3 \circ c \tag{45}$$

with

$$t^1 = \cos\varphi(s), t^2 = \sin\varphi(s), t^3(s), \tag{46}$$

the curvature κ and the torsion τ can be calculated by

$$\boxed{\kappa = \varphi' + 2t^3, \tau = \frac{1}{\kappa} t^{3''} - \frac{\kappa'}{\kappa^2} t^{3'} + \kappa t^3 + 1} . \tag{47}$$

Conversely if κ and τ are given the differential equations (47) for the functions $\varphi(s), t^3(s)$ can be solved explicitely by integrals.

To find the curve $P(s)$ in $S^3 \subset E^4$ we have to integrate the differential equation $P' = T, T$ given by (45) , (46). Thus $P(s)$ can be represented in the form

$$P(s) = \begin{pmatrix} x^1(s) \\ x^2(s) \\ x^3(s) \\ x^4(s) \end{pmatrix} = \begin{pmatrix} r(s)\cos\alpha_1(s) \\ r(s)\sin\alpha_1(s) \\ \bar{r}(s)\cos\alpha_2(s) \\ \bar{r}(s)\sin\alpha_2(s) \end{pmatrix}, \tag{48}$$

with $r^2 + \bar{r}^2 = 1$. $r(s)$ is a solution of a differential equation of third order, involving the curvature κ but not the torsion τ of the curve. The angular velocities $\alpha' = \frac{d\alpha}{ds}$ are given by

$$\alpha' = t^3 - \frac{1}{2}\kappa \pm \sqrt{\frac{r''}{r} + \frac{1}{4}\kappa^2 + 1}. \tag{50}$$

If $r =$ const , then $\kappa =$ const. The converse is also true. If $\kappa = \kappa_0 =$ = const, there exists an orthonormal frame in E^4 with coordinates (y^1, y^2, y^3, y^4) such that

$$P(s) = \begin{pmatrix} y^1(s) \\ y^2(s) \\ y^3(s) \\ y^4(s) \end{pmatrix} = \begin{pmatrix} r_0 \cos \alpha_1(s) \\ r_0 \sin \alpha_1(s) \\ \bar{r}_0 \cos \alpha_2(s) \\ \bar{r}_0 \sin \alpha_2(s) \end{pmatrix}, \tag{50}$$

with $r_0^2 + \bar{r}_0^2 = 1$. The projections of the curve into the (y^1, y^2)-plane and the (y^3, y^4)-plane lie on circles. The radius r_0 is allowed to be arbitrary. The angular velocities α' satisfy

$$\alpha' = t^3 - \frac{1}{2}\kappa_0 \pm \sqrt{\frac{1}{4}\kappa_0^2 + 1}. \tag{51}$$

There is a remarkable relation between the curve theories in $S^{3(1)}$ and in $I_3^{(1)}$. In the isotropic 3-space $I_3^{(1)}$ the curvature and the torsion of a curve have some nice geometric interpretations. Moreover they satisfy the equation (47.2) without the additional term $+1$ and with t^3 equals to the derivative of the third coordinate function of the curve (Strubecker [6], (2.69)). If we consider curves in $S^{3(1)}$ and in $I_3^{(1)}$ with the same curvature κ and the same function t^3, the torsions differ only by $+1$. This property yields both a geometric interpretation of the torsion in $S^{3(1)}$ and a method for constructing a curve in $S^{3(1)}$ with given curvature and torsion, via the curve theory in $I_3^{(1)}$. For example a curve with constant torsion τ in $S^{3(1)}$ corresponds to a curve with constant torsion $\tau - 1$ in $I_3^{(1)}$, especially a curve with vanishing torsion in $S^{3(1)}$ to a curve with torsion -1 in $I_3^{(1)}$. In this context we can use the elegant theory of curves with constant torsion in $I_3^{(1)}$ due to Strubecker [6] and Sachs [3].

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