

LOCALISATION OF A COMMUTATIVITY CONDITION FOR S -UNITAL RINGS

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Abstract: We consider here s -unital rings R satisfying the following condition: For each subset F of R having at most four elements, there exist non-negative integers $m = m(F)$, $n = n(F)$, $r = r(F)$, $s = s(F)$, $t = t(F)$ and $t' = t'(F)$ such that $m > 0$ or $n > 0$, $n + t + t' \neq r + 1$ or $m + s > 1$ if $m = n > 0$, and $x^{t'}[x^n, y]x^t = \pm x^r[x, y^m]y^s$ for all $x, y \in F$. Under appropriate additional conditions we prove some commutativity theorems for R . Especially, in the "global case", where $F = R$, instead of $F \subseteq R$, $|F| \leq 4$, i.e. where m, n, r, s, t and t' are fixed, we improve some earlier results obtained by several authors, among them H. A. S. Abujabal and the present author.

1. Introduction

We investigate here the commutativity of a ring R satisfying the following property

(P_4) For each subset F of R having at most four elements, $|F| \leq 4$, there exist non-negative integers $m = m(F)$, $n = n(F)$, $r = r(F)$, $s = s(F)$, $t = t(F)$ and $t' = t'(F)$ such that

$$(1) \quad x^{t'}[x^n, y]x^t = \pm x^r[x, y^m]y^s \text{ for all } x, y \in F.$$

Our general assumption on the above integers will be:

$$(2) \quad m(F) > 0 \text{ or } n(F) > 0 \text{ for each } F \subseteq R, |F| \leq 4,$$

and moreover

$$(3) \quad \begin{aligned} &\text{if } m(F) = n(F), \text{ then } n(F) + t(F) + t'(F) \neq r(F) + 1 \\ &\text{or } m(F) + s(F) > 1 \text{ for each } F \subseteq R, |F| \leq 4. \end{aligned}$$

The assumption (2) is natural, since for $m(F) = 0$ and $n(F) = 0$ the condition (1) is trivially satisfied. Concerning the condition (3) we remark that if $m(F) = n(F)$, $n(F) + t(F) + t'(F) = r(F) + 1$ and $m(F) + s(F) = 1$, i.e. $s(F) = 0$, then for every ring R with the additional condition

$$(4) \quad [x, [x, y]] = 0 \text{ for all } x, y \in R$$

the condition (1₊) (resp. (1₋)) is surely fulfilled (if R is of characteristic 2). Namely, for such a ring, in view of (4), (1) is equivalent (see Lemma 2) with

$$(5) \quad nx^{n+t+t'-1}[x, y] = \pm mx^r[x, y]y^{m+s-1} \text{ for all } x, y \in F.$$

Since a non-commutative ring can satisfy (4), the condition (3) is thus also reasonable. Thereby, we have denoted by (i₊), resp. by (i₋) the condition obtained from a condition (i) which contains the sign \pm by setting +, resp. - instead of \pm .

A similar condition denoted by (Q₄) with

$$(1') \quad x^{t'}[x^n, y]x^t = \pm y^{s'}[x, y^m]y^s \text{ for all } x, y \in F$$

instead of (1), we consider in another paper [13].

If in (P₄), $r(F) = 0$ for each $F \subseteq R$, $|F| \leq 4$, resp. in (Q₄), $s'(F) = 0$ for all $F \subseteq R$, $|F| \leq 4$, then (P₄), resp. (Q₄) is a special case of (Q₄), resp. of (P₄). Otherwise, these two conditions are not easily comparable.

We remark that instead of (1) we could consider

$$(1'') \quad x^{t'}[x^n, y]x^t = \pm y^s[x, y^m]x^r \text{ for all } x, y \in F.$$

But going from R to the opposite ring R' , (1'') becomes

$$(1''') \quad x^t[x^n, y]x^{t'} = \pm x^r[x, y^m]y^s \text{ for all } x, y \in F',$$

where $F' = F$ is considered as a subset of R' . Thus, we see that it suffices to consider only one of the conditions (1) and (1''): the results

concerning one can be obtained as corollaries from the results concerning the other of the conditions.

For $F = R$ instead of $F \subseteq R$, $|F| \leq 4$, the condition (P_4) reduces to
 (P) There exist non-negative integers m, n, r, s, t and t' such that

$$(6) \quad x^{t'} [x^n, y] x^t = \pm x^r [x, y^m] y^s \text{ for all } x, y \in R,$$

and

$$(7) \quad m > 0 \text{ or } n > 0, \text{ and if } m = n, \text{ then } n+t+t' \neq r+1 \text{ or } m+s > 1.$$

We call (P) the globalisation of (P_4) , and (P_4) a localisation of (P), and in this connection we will talk about the local and the global case of a statement or a condition.

Occasionally we will make other additional conditions on R , or on integers $m(F), n(F), r(F), s(F), t(F)$ and $t'(F)$. For the local case we will additionally assume (4) or

(I-A) For each $x \in R$, $x \in A$ or there exists $f(X) \in X^2\mathbb{Z}[X]$ such that $x - f(x) \in A$, for a suitable non-void subset A of R .

In both cases we will need one of the conditions:

$$(8) \quad m[x, y] = m[x, y]f(y) \Rightarrow [x, y] = [x, y]kf(y),$$

$$(8') \quad n[x, y] = n[x, y]f'(y) \Rightarrow [x, y] = [x, y]k'f'(y),$$

$$(8'') \quad (n \mp 1)[x, y] = [x, y]f''(y) \Rightarrow [x, y] = [x, y]k''f''(y),$$

$$(8''') \quad 2[x, y] = [x, y]f'''(y) \Rightarrow [x, y] = [x, y]k'''f'''(y)$$

for all $x, y \in F$ in the local case, and for all $x, y \in R$ in the global case. Thereby, k, k', k'' and k''' are appropriate integers (or polynomials in $X\mathbb{Z}[X]$ taken at $X = y$), and

$$(9) \quad f(X) = -(X+1)^{m+s-1} + X^{m+s-1} + 1,$$

$$(9') \quad f'(X) = -(X+1)^{n+t+t'-1} + X^{n+t+t'-1} + 1,$$

$$(9'') \quad f''(X) = -n(X+1)^{n+t+t'-1} + nX^{n+t+t'-1} \pm (X+1)^r \mp X^r + (n \mp 1),$$

and

$$(9''') \quad f'''(X) = -(X+1)^{t+t'} + X^{t+t'} - (X+1)^r + X^r + 2$$

are polynomials in $X\mathbb{Z}[X]$ for suitable values of m, n, r, s, t and t' .

Obviously, (8), resp. (8') implies

$$(8)^* \quad m[x, y] = 0 \Rightarrow [x, y] = 0 \text{ for all } x, y \in F, [x, y]y^2 = 0,$$

resp.

$$(8')^* \quad n[x, y] = 0 \Rightarrow [x, y] = 0 \text{ for all } x, y \in R, [x, y]y^2 = 0.$$

Concerning $(8'')$, resp. $(8''')$ we have only

$$\begin{aligned}(n \mp 1)[x, y] &= [x, y]f''(y) \Rightarrow (n \mp 1)^2[x, y] = 0, \\ [x, y] &= [x, y]k''f''(y) \Rightarrow [x, y] = 0,\end{aligned}$$

resp.

$$\begin{aligned}2[x, y] &= [x, y]f''(y) \Rightarrow 2^2[x, y] = 0, \\ [x, y] &= [x, y]k''f''(y) \Rightarrow [x, y] = 0\end{aligned}$$

for all $x, y \in F$. $[x, y]y^2 = 0$. Therefore, the conditions

$$(8'')^* \quad (n \mp 1)[x, y] = 0 \Rightarrow [x, y] = 0 \text{ for all } x, y \in F, [x, y]y^2 = 0$$

resp.

$$(8''')^* \quad 2^2[x, y] = 0 \Rightarrow [x, y] = 0 \text{ for all } x, y \in F, [x, y]y^2 = 0,$$

which we will also consider here, does not follow from $(8'')$, resp. $(8''')$.

We remark that in the local case, the condition (8) , $(8)^*$, resp. $(8')$, $(8')^*$, resp. $(8'')$, $(8'')$, resp. $(8''')$, $(8''')$ follow from

$$Q(q) \quad q[x, y] = 0 \Rightarrow [x, y] = 0 \text{ for all } x, y \in R$$

$$\text{with } q = m, \text{ resp. } q = n, \text{ resp. } q = n \mp 1, \text{ resp. } q = 2.$$

But the conditions (8) , $(8')$ are surely satisfied if $[x, y]f(y)$, resp. $[x, y]f'(y)$ is torsion-free or m , resp. n is relatively prime to the additive order of $[x, y]f(y)$, resp. of $[x, y]f'(y)$. Similarly, the conditions $(8'')$, $(8''')$ are satisfied provided $[x, y]f''(y)$, resp. $[x, y]f'''(y)$ is a torsion element, and $n \mp 1$, resp. 2 is relatively prime to the additive order of the element $[x, y]f''(y)$, resp. $[x, y]f'''(y)$. Finally, $(8)^*$, $(8')^*$, $(8'')$, and $(8''')$ are satisfied if and only if $[x, y]$ is torsion-free or the additive order of $[x, y]$ is relatively prime to $m, n, n \mp 1$ and 2 , respectively.

In the global case some commutativity results were obtained by H.A.S. Abujabal and the present author in [2], and more generally in [4]. For a very special form of $(1''_+)$ with $s = t = 1$ (in our notations), but for $m > 1$ and $r > 0$ depending on x and y were combined by Ashraf and Quadri with (I-A) for some commutative subset A of $N(R)$. They proved that each ring R satisfying these two conditions must be

commutative ([4], Th. 1). Later H.A.S. Abujabal, M. Ashraf and M. Obaid enlarged this result to the case where $t' > 0$ also depending on x and y ([4], Th. 1). They also proved the corresponding result for (1_+) instead of $(1''_+)$ (assuming that R is s -unital ([1], Th. 2). Recently, H. Komatsu, T. Nishinaka and H. Tominaga [11] proved somewhat more ($f(y)$ with $f(X) \in X^2\mathbb{Z}[X]$ and $f(1) = \pm 1$ instead of y^m , and $A = N$ instead of $A \subseteq N$ and A commutative) ([11], Th. 3.2)) and without condition (I-A) for $t' \geq 0$ and $r \geq 0$ fixed) ([11], Th. 3.3; see also [12], Th. 1).

Our aim is to prove here two commutativity theorems for the local case, and three theorems for the global case. The second local theorem is partially connected with the results of Ashraf and Quadri, resp. of Abujabal, Ashraf and Obaid mentioned above. The second global theorem we proved here improves all results of [2] and [3].

2. Results

In the local case we combine first the condition (P_4) with (4) and prove

Theorem 1. *Let R be an s -unital (a left, or right s -unital) ring satisfying (P_4) and (4). Then R is commutative provided for each $F \subseteq R$, $|F| \leq 4$, any one of the following conditions is fulfilled:*

- 1) $m(F) > 0$, $n(F) > 0$; $m(F) > 1$ or $s(F) > 0$; F satisfies (8) if $r(F) > 0$ and (8) or (8') if $r(F) = 0$ and $(m(F), n(F)) \neq 1$;
- 2) $n(F) = 0$, $m(F) > 0$; F satisfies (8) (and $r(F) = 0$ or $s(F) = 0$);
- 3) $m(F) = 0$, $n(F) > 0$; F satisfies (8') (and $t(F) = 0$ or $t'(F) = 0$);
- 4) $m(F) = 1$, $n(F) > 1$, $s(F) = 0$; and F satisfies (8'');
- 5) $m(F) = n(F) = 1$, $s(F) = 0$.

Theorem 2. *Let R be an s -unital (a left, resp. right s -unital) ring satisfying (P_4) and (I-N). Then R is commutative provided for each $F \subseteq R$, $|F| \leq 4$, one of the following conditions is fulfilled:*

- i) $m(F) > 0$, $n(F) > 0$; $m(F) > 1$ or $s(F) > 1$; ($t(F) = 0$ or $t(F) > 0$ and $s(F) = 0$, resp. $t'(F) = 0$ or $t'(F) > 0$ and $r(F) = 0$); F satisfies (8)* for $r(F) > 0$, except for $n(F) = 1$, $t(F) = 0$ or $t'(F) = 0$, and (8)* or (8')* for $r(F) = 0$ and $(m(F), n(F)) \neq 1$;
- ii) $n(F) = 0$, $m(F) > 0$; ($s(F) = 0$, resp. $r(F) = 0$); and F satisfies (8)*;
- iii) $m(F) = 0$, $n(F) > 0$; ($t(F) = 0$, resp. $t'(F) = 0$); and F satisfies (8')*;

- iv) $m(F) = 1, n(F) > 1, s(F) \leq 1; (t(F) = 0 \text{ or } t(F) > 0 \text{ and } s(F) = 0, \text{ resp. } t'(F) = 0 \text{ or } t'(F) > 0 \text{ and } r(F) = 0);$ and F satisfies $(8'')$ * for $s(F) = 0$;
- v) $m(F) = n(F) = 1, s(F) \leq 1; (t(F) = 0 \text{ or } t(F) > 0 \text{ and } s(F) = 0, \text{ resp. } t'(F) = 0 \text{ or } t'(F) > 0 \text{ and } r(F) = 0);$ and F satisfies (1_-) and $(8''')$ *.

Th. 1 we use in several occasions. Th. 2 is connected with the above mentioned results ([1], Th. 1) and ([4], Th. 1). The localisation in our theorem is not properly complete, but the equation (1) is more general, and moreover, in (I-A) we assume $A = N$, and not that A is a commutative subset N .

In the global case, for a semi-prime ring R we can drop the condition (I-N) in Th. 2 and weak all of the other conditions in this theorem. Precisely, the following theorem holds true:

Theorem 3. *Let R be a semi-prime ring satisfying (P). Then R is commutative provided one of the following conditions is satisfied:*

- a) $m > 0, n > 0; m > 1 \text{ or } s > 0;$
- b) $n = 0, m > 0; \text{ and } r = 0 \text{ or } s = 0 \text{ for } m \text{ even};$
- c) $m = 0, n > 0; \text{ and } t = 0 \text{ or } t' = 0 \text{ for } n \text{ even};$
- d) $m = 1, n > 0, s = 0; t > 0 \text{ or } t = 0 \text{ and } n \text{ even, or } n \text{ and } t' - r \text{ odd.}$

Similarly, in the global case, it is possible to drop the condition (4) in Th. 1, for any ring R , under an appropriate sharpening of other conditions in this theorem:

Theorem 4. *Let R be a left, resp. right s -unital (an s -unital) ring satisfying condition (P). Then R is commutative provided one of the following conditions is fulfilled:*

- A) $m > 0, n > 0; m > 1 \text{ or } s > 1; t = 0 \text{ or } s = 0, \text{ resp. } t' = 0 \text{ or } r = 0$ (only if R is left, resp. right s -unital); R satisfies (8) for $r > 0$, except for $n = 1$ and $t = 0$ or $t' = 0$, and (8) or (8') for $r = 0$ and $(m, n) \neq 1$;
- B) $n = 0, m > 0; r = 0 \text{ or } s = 0$ (for m even); and R satisfies (8);
- C) $m = 0, n > 0; t' = 0 \text{ or } t = 0$ (for n even); and R satisfies (8');
- D) $m = 1, n > 0, s \leq 1; t = 0 \text{ or } s = 0, \text{ resp. } t' = 0 \text{ or } r = 0$ (only if R is left, resp. right s -unital); and R satisfies (8''') for $s = 0$.

The conditions (8), resp. (8) or (8') in Th. 4 A), which are trivially satisfied if $m = 1$, can be eliminated if we assume that R satisfies (P) for $m = m_j, n = n_j, r = r_j, s = s_j, t = t_j$ and $t' = t'_j$, where j

runs over a finite index set J such that the greatest common divisor $(m_j; j \in J)$ is equal to 1:

Theorem 5. *Let R be a left, resp. right s -unital (an s -unital) ring satisfying (P) for $m = m_j$, $n = n_j$, $r = r_j$, $s = s_j$, $t = t_j$ and $t' = t'_j$, where j runs over a finite index set J , such that $(m_j; j \in J) = 1$. If moreover, for each $j \in J$,*

$$\begin{aligned} m_j > 0, n_j > 0; m_j > 1 \text{ or } s_j > 0; \\ t_j = 0 \text{ or } s_j = 0, t'_j = 0 \text{ or } r_j = 0 \\ \text{(only if } R \text{ is left, resp. right } s\text{-unital),} \end{aligned}$$

then R is commutative.

Since, as just mentioned, $Q(m(F)) \Rightarrow (8) \Rightarrow (8)^*$, and $Q(n(F)) \Rightarrow (8') \Rightarrow (8')^*$, we have especially the following results:

Corollary 1. *Let R be an s -unital (a left, resp. right s -unital) ring satisfying $(P)_4$ and (4). Then R is commutative if for each $F \subseteq R$, $|F| \leq 4$, one of the following conditions is fulfilled:*

- 1') $m(F) > 0$, $n(F) > 0$; $m(F) > 1$ or $s(F) > 0$; R satisfies $Q(m(F))$ if $r(F) > 0$, and $Q(m(F))$ or $Q(n(F))$ if $r(F) = 0$ and $(m(F), n(F)) \neq 1$;
- 2') $n(F) = 0$, $m(F) > 0$; R satisfies $Q(m(F))$ (and $r(F) = 0$ or $s(F) = 0$);
- 3') $m(F) = 0$, $n(F) > 0$; R satisfies $Q(n(F))$ (and $t(F) = 0$ or $t'(F) = 0$);
- 4') $m(F) = n(F) = 1$, and $s(F) = 0$.

Corollary 2. *Let R be an s -unital (a left, resp. right s -unital) ring satisfying (P_4) and (I-N). Then R is commutative if for each $F \subseteq R$, $|F| \leq 4$, one of the following conditions is fulfilled:*

- i') $m(F) > 0$, $n(F) > 0$; $m(F) > 1$ or $s(F) > 1$; ($t(F) = 0$ or $T(F) > 0$ and $s(F) = 0$, resp. $t'(F) = 0$ or $t'(F) > 0$ and $r(F) = 0$); F satisfies $Q(m(F))$ for $r(F) > 0$, except for $n(F) = 1$, $t(F) = 0$ or $t'(F) = 0$, and $Q(m(F))$ or $Q(n(F))$ if $r(F) = 0$ and $(m(F), n(F)) \neq 1$;
- ii') $n(F) = 0$, $m(F) > 0$; ($s(F) = 0$, resp. $r(F) = 0$); and F satisfies $Q(m(F))$;
- iii') $m(F) = 0$, $n(F) > 0$; ($t(F) = 0$, resp. $t'(F) = 0$); and F satisfies $Q(n(F))$;
- iv') $m(F) = 1$, $n(F) > 0$, $s(F) = 1$; ($t(F) = 0$, resp. $t'(F) = 0$, or $t'(F) > 0$ and $r(F) = 0$).

Corollary 3. *Let R be a left, resp. right s -unital (an s -unital) ring satisfying condition (P). Then R is commutative provided one of the following conditions is fulfilled:*

- A') $m > 0, n > 0; m > 1$ or $s > 1; t = 0$ or $s = 0$, resp. $t' = 0$ or $r = 0$ (only if R is left, resp. right s -unital); and R satisfies $Q(m)$ for $r > 0$, except for $n = 1$ and $t = 0$ or $t' = 0$, and $Q(m)$ or $Q(n)$ for $r = 0$ and $(m, n) \neq 1$;
- B') $n = 0, m > 0; s = 0$ or $r = 0$ (for m even); and R satisfies $Q(m)$;
- C') $m = 0, n > 0; t = 0$ or $t' = 0$ (for n even); and R satisfies $Q(n)$;
- D') $m = 1, n > 0, s = 1; t = 0$, resp. $t' = 0$ or $r = 0$ (only if R is left, resp. right s -unital).

Cor. 1 follows immediately from Th. 1, Cor. 2 from Th. 2, and Cor. 3 from Th. 4. Cor. 2 is related to the mentioned results of [1] and [4], and Cor. 3 contains almost all main results of [2] and also of [3].

3. Preparations for the proofs

In all of our theorems, except for Th. 3, we suppose that R is an s -unital (a left or/resp. right s -unital) ring. It is well known that in an s -unital (a left, resp. right s -unital) ring R , for arbitrary elements x, y there exists an element e such that $ex = xe = x$ and $ey = ye = y$ ($ex = x$ and $ey = y$, resp. $xe = x$ and $ye = y$). We call such an element e a local (left, resp. right) unity for x and y . Also, it is known ([14], Lemma) that if R is a left, resp. right s -unital ring, and for every two elements x and y in R there exists a positive integer $k = k(x, y)$ and an element $e = e(x, y)$ such that $x^k e = x^k$ and $y^k e = y^k$, resp. $ex^k = x^k$ and $ey^k = y^k$, then R is s -unital.

We state first some known results we will use in this paper. The following two lemmas are well known and easy to prove.

Lemma 1. *Let R be a ring with unity element 1, and $x, y \in R$. If $x^k y = (x + 1)^{k'} y = 0$ or $yx^k = y(x + 1)^{k'} = 0$ for some non-negative integers k and k' , then $y = 0$.*

Lemma 2. *Let x and y be given elements of an arbitrary ring R . If $[x, [x, y]] = 0$, then $[x^k, y] = kx^{k-1}[x, y]$ for all integers $k \geq 1$.*

The next lemma, for $f(X) \in X\mathbb{Z}[X]$ a fixed polynomial, is a very special case of a result due to Streb ([14], Hauptsatz 3; see also [2], Th. S), and is also generally valid by a result due to Bell ([6], Th. 1). In the present form it was proved in a simple manner by this author

([13], Lemma 3).

Lemma 3. *Let R be a ring with the following property:*

- (10) *for arbitrary elements x, y in R there exists a polynomial $f(X) \in X\mathbb{Z}[X]$ such that $[x, y] = [x, y]f(y)$.*

Then R is commutative.

The first of the next two results is due to Kezlan and Bell, and the second to Herstein.

Theorem KB ([5], Th. 1; [10], Theorem). *Let f be a polynomial in n non-commuting indeterminates X_1, \dots, X_n with (relatively prime) coefficients. Then the following are equivalent:*

- 1) *for any ring R satisfying the polynomial identity*

$$f(x_1, \dots, x_n) = 0 \text{ for all } x_1, \dots, x_n \in R$$

the commutator ideal $C = C(R)$ of R is a nil ideal;

- 2) *for every prime p , the matrix ring $R = M_2(GF(p))$ fails to satisfy the above identity;*
 3) *every semi-prime ring R satisfying the above identity is commutative.*

Theorem H ([8], Theorem). *Let R be a ring. If for each $x \in R$ there exists a polynomial $f(X) \in X\mathbb{Z}[X]$ such that $x - f(x) \in Z(R)$, then R is commutative.*

4. Proofs

The next three lemmas concern a ring R which satisfies the condition (P_4) . The first of them shows that the ring R in all our theorems, except for Th. 3, is in fact an s -unital ring. This will enable us to prove these theorems for a ring with unity element 1 (see [9], Prop. 1).

Lemma 4. *Let R be a left, resp. right s -unital ring satisfying (P_4) . Then R is s -unital, provided for each $F \subset R$, $|F| \leq 4$, any one of the following conditions is fulfilled:*

- 1'' *$m(F) > 0$, $n(F) > 0$; $t(F) = 0$ and $m(F) > 1$ or $s(F) > 0$, or $t(F) > 0$ and $s(F) > 0$, resp. $t'(F) = 0$ and $m(F) > 1$ or $r(F) > 0$ or $s(F) > 0$, or $t'(F) > 0$ and $r(F) = 0$;*
 2'' *$n(F) = 0$, $m(F) > 1$; and $s(F) = 0$, resp. $r(F) = 0$;*
 3'' *$m(F) = 0$, $n(F) > 1$; and $t(F) = 0$, resp. $t'(F) = 0$.*

Proof. Let x and y be arbitrary elements of R , and e a left, resp. right local unity for x and y . Set $F = \{x, y, x + 1, y + 1\}$.

Case 1''): Let e be a left local unity for x and y . If $t(F) = 0$, then

$$e^t [e^n, x] = \pm e^r [e, x^m] x^s, \text{ hence } x = x e^n \pm x^{m+s} \mp x^m e x^s,$$

and thus, $x \in xR$ if $m(F) > 1$ or $s(F) > 0$. If $t(F) > 0$ and $s(F) = 0$, then

$$z^t [z^n, e] z^t = \pm z^r [z, e^m], \text{ i.e. } z^{r+1} e^m = z^{r+1} \text{ for } z \in \{x, y\}.$$

Let now e be a right local unity for x and y . If $t'(F) = 0$, then

$$[e^n, x] e^t = \pm e^r [e, x^m] x^s, \text{ hence } x = e^n x \pm e^r x^{m+s} \mp e^{r+1} x^{m+s},$$

and thus, $x \in Rx$ if $m(F) > 1$ or $r(F) > 0$ or $s(F) > 0$. If $t'(F) > 0$, and $r(F) = 0$, then

$$x^t [x^n, e] x^t = \pm [x, e^m] e^s, \text{ hence } x = e^m x.$$

Case 2''): Let e be left, resp. right local unity for x, y . Then

$$\text{for } s(F) = 0, z^r [z, e^m] = 0, \text{ i.e. } z^{r+1} = z^{r+1} e^m,$$

resp.

$$\text{for } r(F) = 0, [e, z^m] z^s = 0, \text{ i.e. } z^{m+s} = e z^{m+s},$$

for $z \in \{x, y\}$.

Case 3''): For $t(F) = 0$ and a left local unity e resp. for $t'(F) = 0$ and a right local unity e , we have

$$e^t [e^n, x] = 0, \text{ resp. } [e^n, x] e^t = 0,$$

hence,

$$x = x e^n, \text{ resp. } x = e^n x. \quad \diamond$$

The next lemma we need only for the global case, but we state it in the general local case:

Lemma 5. *No matrix ring $M_2(GF(p))$, p prime, can satisfy (P_4) , such that for each $F \subset R = M_2(GF(p))$, $|F| \leq 4$, one of the following conditions is fulfilled:*

- a) $m(F) > 0$, $n(F) > 0$, and $m(F) > 1$ or $s(F) > 0$;
- b) $n(F) = 0$, $m(F) > 0$, and $r(F) = 0$ or $s(F) = 0$ for $m(F)$ even;
- c) $m(F) = 0$, $n(F) = 0$, and $t'(F) = 0$ or $t(F) = 0$ for $n(F)$ even;
- d) $m(F) = 1$, $n(F) > 0$, $s(F) = 0$; and $t(F) > 0$ or $t(F) = 0$ and $n(F)$ even, or $n(F)$ and $t'(F) - r(F)$ odd.

Proof. Let e_{ij} be the matrix in R with entree 1 on the position i, j , and with 0 elsewhere. Set $F = \{e_{11}, e_{22}, e_{12}, e_{12} + e_{21}\}$.

Case a): $x = e_{12} + e_{21}$ and $y = e_{12}$ for $n(F)$ odd, resp. $y = e_{11}$ for $n(F)$ even fail to satisfy (1).

Case b): $x = e_{12} + e_{21}$ and $y = e_{11}$ for $m(F)$ odd, and $y = e_{12}$, $x = e_{22}$, resp. $x = e_{11}$ for $m(F)$ even and $t'(F) = 0$, resp. $t(F) = 0$, fail to satisfy (1).

Case c): Similar to Case b).

Case d): $x = e_{11}$ and $y = e_{12}$ for $t(F) > 0$, and $x = e_{12} + e_{21}$ and $y = e_{11}$ for $t(F) = 0$ and $n(F)$ even or $n(F)$ and $t'(F) - r(F)$ odd, fail to satisfy (1).

The following Lemma will be used in the proof of Th. 2 and that of Th. 4, and thus we need its general local form:

Lemma 6 (cf. [2], Lemma 5). *Let R be a ring with unity 1 which satisfies (P_4) . Then every nilpotent element of R is central, i.e. $N(R) \subseteq Z(R)$ provided for each $F \subseteq R$, $|F| \leq 4$, one of the following conditions is fulfilled:*

- i'') $m(F) > 0$, $n(F) > 0$; $m(F) > 1$ or $s(F) > 1$; and F satisfies $(8)^*$ for $r(F) > 0$, except for $n(F) = 1$, $t'(F) = 0$ or $t(F) = 0$, and $(8)^*$ or $(8')^*$ if $r(F) = 0$ and $(m(F), n(F)) \neq 1$;
- ii'') $n(F) = 0$, $m(F) > 0$ and F satisfies $(8)^*$;
- iii'') $m(F) = 0$, $n(F) > 0$ and F satisfies $(8')^*$;
- iv'') $m(F) = 1$, $n(F) > 1$, $s(F) \leq 1$, and F satisfies $(8'')^*$ for $s(F) = 0$;
- v'') $m(F) = n(F) = 1$, $s(F) \leq 1$, and for $s(F) = 0$, F satisfies (1_-) and $(8''')^*$.

Proof. Let $x \in R$ and $a \in N(R)$ be arbitrary, but fixed elements. We have to prove that $[x, a] = 0$. Since $a \in N(R)$, there exists a minimal positive integer p such that

$$[x, a^k] = 0 \text{ for all integers } k \geq p.$$

If $p = 1$, we have nothing to prove. Suppose that $p > 1$ and set $b = a^{p-1}$. Then

$$(11) \quad b^k[x, b] = [x, b^k] = [x, b]b^k \text{ for all integers } k \geq 1$$

and

$$(11') \quad b[x, b] = -[x, b]b.$$

We will prove that $[x, b] = 0$, which in view of the minimality

of p contradicts to the assumption that $p > 1$. For this purpose set $F = \{x, b, x + 1, b + 1\}$.

Case i''): Setting b for y in (1), in view of (11) we get

$$(12) \quad x^{t'}[x^n, b]x^t = 0.$$

Now, setting $b + 1$ in (1) and using (12), we obtain

$$x^r[x(b + 1)^m](b + 1)^s = 0.$$

Since $b + 1$ is invertible, the last equation, in view of (11) and Lemma 1 yields

$$(13) \quad m[x, b] = 0.$$

If $n = 1$, then setting $x + 1$ for x in (12), we get

$$(x + 1)^{t'}[x, b](x + 1)^t = 0,$$

and if $t = 0$ or $t' = 0$, then by Lemma 1, follows $[x, b] = 0$. Also, for $r(F) > 0$, (8)* and (13) imply $[x, b] = 0$. Let now $r(F) = 0$. Then for $n = 1$, $t > 0$ and $t' > 0$, and also for $n > 1$, (1) with b , resp. x instead of x , resp. y , gives

$$[b, x^m]x^s = 0$$

and thus (1), for $b + 1$, resp. x instead of x , resp. y , becomes

$$(b + 1)^{t'}[(b + 1)^n, x](b + 1)^t = 0.$$

In view of (11) and the invertibility of $b + 1$, this yields

$$(13') \quad n[x, b] = 0.$$

If $(m, n) = 1$, then (13) and (13') imply $[x, b] = 0$. If $(m, n) \neq 1$, then we have (8)* or (8')*, and then (13) or (13') implies $[x, b] = 0$ again.

Case ii''): If $n(F) = 0$, $m(F) > 0$, then (1) for $y = b + 1$ becomes

$$x^r[x, (b + 1)^m](b + 1)^s = 0,$$

hence by (11) and the invertibility of $b + 1$, after applying Lemma 1, we get $m[x, b] = 0$. This in view of (8)* implies $[x, b] = 0$.

Similarly, we can get $[x, b] = 0$ in *Case iii'')*.

Case iv''): If $s(F) = 1$, then (1) for $y = b$ and the same equation for $y = b + 1$ easily give $x^n[x, b] = 0$, and this using Lemma 1 implies $[x, b] = 0$.

Let now $s(F) = 0$. Setting $b + 1$ for x , and x for y in (1), we get in view of (11) and (11'),

$$(n \mp 1)[b, x] = (-nt' + nt \pm r)b[b, x],$$

hence

$$(n \mp 1)^2[x, b] = 0,$$

and thus, by $(8'')^*$, $[x, b] = 0$.

Case v''): For $s = 1$, we can get $[x, b] = 0$ as in *Case iv'*). Let now $s = 0$. Then for $x = b + 1$, and $y = x$, (1_-) becomes

$$(b + 1)^{t'}[b, x](b + 1)^t = -(b + 1)^r[b, x],$$

hence, in view of (11) and (11') we have

$$2[x, b] = (-t' + t - r)b[x, b], \text{ i.e. } 2^2[x, b] = 0,$$

which by $(8''')^*$ gives $[x, b] = 0$. \diamond

Now we can go to the proofs of our theorems.

By Lemma 4, all rings in our theorems, except for Th. 3, are s -unital, and according to ([9], Prop. 1) we can and will assume that, in all these theorems, R is a ring with unity 1.

Proof of Th. 1. In view of (4) and Lemma 2, the identity (1) is equivalent with (5), and also with

$$(14) \quad x^{t+t'}[x^n, y] = [x^n, y]x^{t+t'} = \pm x^r[x, y^m]y^s = \pm y^s[x, y^m]x^r.$$

From (14) and Lemma 4 we easily see that R is s -unital. Now we can assume that R is a ring with unity 1. We fix x and y in R , and set $F = \{x, y, x + 1, y + 1\}$.

Case 1): Setting in (5) $y + 1$ for y , and combining equality (5) with the obtained one, we get

$$mx^r[x, y](1 - f(y)) = 0,$$

hence, using Lemma 1,

$$(15) \quad m[x, y] = m[x, y]f(y),$$

where $f(X) \in XZ[X]$ is given by (9). Similarly, from (5), if $r(F) = 0$, we get

$$(15') \quad n[x, y] = n[x, y]f'(y),$$

where $f'(X) \in XZ[X]$ is given by (9').

Using (15) and Lemma 1, from (5) we easily obtain

$$(16) \quad n[x, y] = n[x, y]f(y),$$

and similarly, from (5) and (15'), if $r(F) = 0$ we can get

$$(16') \quad m[x, y] = m[x, y]f'(y).$$

If $r(F) > 0$, then by assumption, F satisfies (8), and thus (15)

implies

$$(17) \quad [x, y] = [x, y]kf(y).$$

Let now $r(F)=0$. If $(m, n) = 1$, then from (15) and (16), resp. (15') and (16') follows (17), resp.

$$(17') \quad [x, y] = [x, y]k'f'(y).$$

If $(m, n) \neq 1$, then, by assumption, F satisfies (8) or (8'), and thus (15) implies (17) or (15') implies (17').

Case 2): In this case instead of (5) we have

$$mx^r[x, y]y^{m+s-1} = 0, \text{ resp. } n[x, y]x^{n+t+t'-1} = 0,$$

which by applying Lemma 1 give

$$m[x, y] = 0, \text{ resp. } n[x, y] = 0,$$

i.e. in view of (8)*, resp. (8')*, $[x, y] = 0$.

Case 3): Since $m = 1$, and $s = 0$, in this case (5) becomes

$$n[x, y]x^{n+s+t-1} = \pm[x, y]x^r \text{ for all } x, y \in F.$$

From this equation we easily get

$$(n \mp 1)[x, y] = [x, y]f''(y),$$

where $f''(X) \in XZ[X]$ is given by (9''). But the last equation, in view of (8''), implies

$$(17'') \quad [x, y] = [x, y]k''f''(y).$$

Case 4): Now (5₋) becomes

$$[x, y]x^{t+t'} = -[x, y]x^r \text{ for all } x, y \in F.$$

This implies

$$2[x, y] = [x, y]f'''(y),$$

i.e. by (8'''),

$$(17''') \quad [x, y] = [x, y]k'''f'''(y),$$

where $f'''(X) \in XZ[X]$ is given by (9''').

Thus, the ring R in Th. 1 satisfies the condition of Lemma 3, and so R is commutative. \diamond

Remark. Since obviously, for every prime p , the matrix ring $R = M_2(GF(p))$ fails to satisfy (4), then by Th. KB, for every ring R , (4)

implies

$$(4') \quad [x, y] \in N(R) \quad \text{for all } x, y \in R.$$

For an s -unital (a left, resp. right s -unital) ring R in Th. 1, we can replace (4) by the weaker condition (4'), assuming the conditions i'' – v'') of Lemma 6 (and the conditions $1''$)– $3''$) of Lemma 4) were satisfied. Namely (by Lemma 4, R is unital, and) we can assume that R has a unity 1. Moreover, according to Lemma 6, $N(R) \subseteq Z(R)$, hence by (4), $[x, y] \in Z(R)$ for all $x, y \in R$, and especially we have (4).

Proof of Th. 2. By Lemma 4, the ring R in this theorem is s -unital, and we can assume that R is with unity 1. But, then, according to Lemma 6, $N(R) \subseteq Z(R)$. Since, moreover, R satisfies (I–N), R in fact satisfies the conditions in Th. H, and hence R is commutative. \diamond

Proof of Th. 3. By Lemma 5, no matrix ring $M_2(GF(p))$, p prime, can satisfy (P) in situation of Th. 3. Since R is semi-prime and R satisfies (P), R must be commutative in view of Th. KB. \diamond

Proof of Th. 4. According to Lemma 4, the ring R in this theorem is s -unital, and thus we can assume that R is a ring with unity 1. But, then R satisfies all conditions of Lemma 6, and thus, $N(R) \subseteq Z(R)$. Moreover, in the situation of Th. 4, in view of Lemma 5, no matrix ring $M_2(GF(p))$, p prime, can satisfy (P), hence in view of Th. KB, $C(R) \subseteq N(R)$. Therefore, $C(R) \subseteq Z(R)$, and thus R satisfies all conditions of Th. 1, and so R is commutative. \diamond

Proof of Th. 5. The ring R in this theorem obviously satisfies condition $1'$) of Lemma 4. Therefore, R is s -unital, and so we can assume that R is a ring with unity 1. Moreover, since $m_j > 1$ or $s_j > 0$, similarly as we have got $m[x, b] = 0$ in the proof of Lemma 6, we can get $m_j[x, b] = 0$ for all $j \in J$. But, this with $(m_j : j \in J) = 1$ implies $[x, b] = 0$. Hence $N(R) \subseteq Z(R)$. Also, for each $j \in J$, R satisfies condition $1''$) of Lemma 5, and thus, in view of Th. KB, $C(R) \subseteq N(R)$. Hence, $C(R) \subseteq Z(R)$, and R surely satisfies (4). Now, as in the proof of Th. 1, we can prove that

$$(15j) \quad m_j[x, y] = m_j[x, y]f_j(y) \quad \text{for all } x, y \in J,$$

where $f_j(X) \in X\mathbb{Z}[X]$. This, in view of $(m_j : j \in J) = 1$, yields

$$[x, y] = [x, y] \sum_{j \in J} m_j m'_j f_j(y) \quad \text{for all } x, y \in R,$$

and some integers m'_j ($j \in J$). Therefore, R is commutative according to Lemma 3. \diamond

For the appropriate examples in the local, resp. global case, and also for other related results, we refer to [1] and [4], resp. [2] and [3].

We give here an example showing that a ring R can satisfy a local condition of the form (P_4) , which seems to does not satisfy a global condition of the form (P) .

Example 1. Let $M = M(p)$ be the ring of all infinite lower triangular matrices over a Galois field $GF(p)$, p prime, and k be a positive integer. We denote by $M_k = M_k(p)$ the subring of M generated by the identity matrix I and all matrices $(a_{ij}) \in M$ such that $a_{ii} = 0$ for $i = 1, 2, \dots$ and $a_{ij} = 0$ for $i \geq k$, or $j \geq k$. Let R be the union of all the M_k . Then R is a ring with unity I .

The regular matrices in M_k form a finite group of order $(p-1)p^{\frac{k(k-1)}{2}} =: m_k$ and thus, $a^{m_k} = I$ for all regular matrices $a \in M_k$. A non-regular matrix $b \in M_k$ is nilpotent, and $b^k = 0$ for all such matrices b . For sufficiently large p , we have $m_k \geq k$, and hence

$$(18) \quad x^{m_k} \in Z(R) \text{ for all } x \in M_k.$$

Now, for each finite subset F of R , there exists an integer k such that $F \subseteq M_k$, and thus for

$$(19) \quad m = m(F) = m_k, n = n(F) = \varrho(F)m_k$$

and arbitrary non-negative integers $r = r(F)$, $s = s(F)$, $t = t(F)$ and $t' = t'(F)$ we have

$$x^{t'} [x^n, y] x^t = \pm x^r [x, y_m] y^s \text{ for all } x, y \in F.$$

Thereby, $\varrho = \varrho(F)$ is a non-negative integer. Especially, the ring R satisfies a condition of the form (P_4) . Since R is obviously non-commutative, R fails to satisfy some of the additional conditions in our Th. 1 and Th. 2. Actually, the condition $(8)^*$ is not fulfilled, and thus also the condition (8) cannot be satisfied. Namely, for $b = e_{12} \in M_k$, $k > 1$, we have $b^2 = 0$. On the other hand, for $x = e_{2k} \in M_k$, $k > 2$, we have

$$m_k [x, b] = 0, \text{ but } [x, b] = e_{1k} \neq 0.$$

We remark that in view of (18), $C(R) \subseteq N(R)$ by a well known result due to Herstein [7].

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