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ON THE FELLER STRONG LAW OF LARGE NUMBERS FOR FIELDS OF B-VALUED RANDOM VARIABLES

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Abstract: The purpose of this note is to provide Feller type strong law of large numbers for sums of i.i.d. B-valued random variables with multidimensional indices.

1. Introduction

Let $\{X_{\bar{n}}, \bar{n} \in \mathbb{N}^r\}$ be a field of independent and identically distributed (i.i.d.) random variables taking values in a separable Banach space $(B, \| \ \|)$. $\mathbb{N}^r$ denotes the positive integer r-dimensional lattice points, r is positive integer. Assume that points of $\mathbb{N}^r$ are denoted by $\bar{n} = (m_1, ..., m_r)$, $\bar{n} = (n_1, ..., n_r)$ etc. and ordered by coordinatewise partial ordering. For $\bar{n} = (n_1, ..., n_r) \in \mathbb{N}^r$, we define $S_{\bar{n}} = \sum\limits_{\bar{k} \leq \bar{n}} X_{\bar{n}}$ and $|\bar{n}| = \prod\limits_{i=1}^{r} n_i$. Throughout this paper, $\bar{n} \to \infty$ means $|\bar{n}| \to \infty$. Further, let $\{a_{\bar{n}}, \bar{n} \in \mathbb{N}^r\}$ be an increasing directed (directed upwards) family of positive numbers, i.e $a_{\bar{m}} \leq a_{\bar{n}}$ whenever $\bar{m} \leq \bar{n}$ and $a_{\bar{n}} \to \infty$ as $\bar{n} \to \infty$.

In order to bring into focus the main aim of this paper we start with a description of Fazekas' result [2]. Let B be separable Banach space, $1 \leq p < 2$ and $\{X_{\bar{n}}, \bar{n} \in \mathbb{N}^r\}$ be i.i.d. B-valued random variables. Assume that $E\|X_1\|^p(\log^+ \|X_1\|)^{d-1} < \infty$. If $S_{\bar{n}}/|\bar{n}|^{1/p} \xrightarrow{P} 0$ as $\bar{n} \to \to \infty$ then $S_{\bar{n}}/|\bar{n}|^{1/p} \to 0$ a.s. as $\bar{n} \to \infty$. Our main aim is to establish the strong law of large numbers with an arbitrary normalizing family and no moment restriction assuming on a random field $\{X_{\bar{n}}, \bar{n} \in \mathbb{N}^r\}$, which is also a generalization of some results obtained for a field of real random variables (cf. Gut [3], Klesov [4]). We exploit the concept of Mikosh and Norvaisa [8], and assume similar properties for normalizing family of positive numbers $\{a_{\bar{n}}, \bar{n} \in \mathbb{N}^r\}$.

2. Auxiliary lemmas

In this section we collect some auxiliary results needed later on. Let $\{a_{\bar{n}}, \bar{n} \in \mathbb{N}^r\}$ be a field of positive numbers such that $\lim_{\bar{n}\to\infty} a_{\bar{n}} = \infty$ and let there exist a sequence D_k, $k \geq 1$ of finite subsets of $\mathbb{N}^r$ such that $D_k \uparrow \mathbb{N}^r$, and satisfies the following conditions:

(A) Set $I_k = D_k - D_{k-1}$, $k \geq 1$. If $\bar{n} \in I_k$, then $(\bar{n}) \subseteq D_k$;

(B) There are constants $d > 1$, $C_1, C_2 > 0$ such that for every $k, \bar{n} \in I_k$, the relation $C_1 d^k \leq a_{\bar{n}} \leq C_2 d^k$ holds;

(C) For every k there exist disjoint rectangles E_{kl} and an appropriate index set R_k such that $I_k = \bigcup_{l \in R_k} E_{kl}$;

(D) $\nu_0 \overline{\lim_{k\to\infty}} \max_{\bar{n}\in I_k} d^{-k} \sum_{i=1}^{k} d^i \big|\{t \in R_i : E_{it} \cap (\bar{n}) \neq \emptyset\}\big| < \infty.$

Conditions (A), (B), (C) and (D) come from Mikosh and Norvaisa [8] and field of numbers satisfying them is said to have the *weak star property*. For examples of the weak star property see Mikosh [7], Mikosh, Norvaisa [8] and the *star property* see Li, Wang, Rao [5–6].

Lemma 2.1 (Mikosh, Norvaisa [8]). *Let* $\{X_{\bar{n}}, \bar{n} \in \mathbb{N}^r\}$ *be a field of independent, symmetric* B*-valued r.v.'s. Assume that* (A)–(D) *hold. Then the condition*

$$\sum_{k} \sum_{l \in R_k} P\big(\|S_{E_{kl}}\| > \varepsilon d^k\big) < \infty, \quad \forall \varepsilon > 0 \tag{2.1}$$

is equivalent to the strong law of large numbers

$$S_{\bar{n}}/a_{\bar{n}} \to 0, \quad a.s.\ as\ \bar{n} \to \infty \tag{2.2}$$

Lemma 2.2. *Let* $\{X_{\bar{n}}, \bar{n} \in \mathbb{N}^r\}$ *be a field of independent, symmetric* B*-valued r.v.'s. Assume that* (A)–(C) *hold and*

(2.3) $$\|X_{\bar{k}}\| \le a_{\bar{k}} \quad a.s. \quad (\bar{k} \in \mathbb{N}^r),$$

(2.4) $$S_{\bar{n}}/a_{\bar{n}} \to 0 \quad in\ probability.$$

Then for all $p > 0$, $E\big\|S_{E_{kl}}/d^k\big\|^p \to 0$ *as* $k \to \infty$ *uniformly in* $l \in R_k$.
Proof. Lemma V-1-1 of Neveu [9] implies that, the family $\{x_{\bar{n}}, \bar{n} \in \in \mathbb{N}^r\}$ converges to x if the net $\{x_{\bar{n}}, \bar{n} \in K\}$ converges to x for any infinite linearly ordered subset $(K, \le)$ of $\mathbb{N}^r$, thus by Lemma 3.1 of de Acosta [1] $E\|S_{\bar{n}}/a_{\bar{n}}\|^p \to 0$ as $\bar{n} \to \infty$. Then it is easy to see that for an arbitrary $l \in R_k$, $\lim_{k\to\infty} E\big\|S_{E_{kl}}\big\|^p/d^{kp} \to 0$. Since for every k, R_k are finite, therefore $\lim_{k\to\infty} \max_{l\in R_k} E\big\|S_{E_{kl}}\big\|^p/d^{kp} \to 0$. ◊

3. Results

Let $M_j = \mathrm{card}\{\bar{n} \in \mathbb{N}^r : a_{\bar{n}} \le j\}$ and $m_j = M_j - M_{j-1}$ for every integer $j \ge 1$.
Theorem 3.1. *Let* $\{X_{\bar{n}}, \bar{n} \in \mathbb{N}^r\}$ *be a field of i.i.d. Banach space valued random variables and* $\{a_{\bar{n}}, \bar{n} \in \mathbb{N}^r\}$ *be an increasing directed field of positive numbers. Suppose that there exist* $j_0 \ge 1$ *and positive numbers* C_3, C_4 *such that*

(3.1) $$\forall_{j>j_0} M_j \le C_3 M_{j-1}, \quad \sum_{i\ge j} i^{-3} M_i \le C_4 j^{-2} M_j.$$

Then

(3.2) $$\sum_{\bar{n}} P\big(|X_{\bar{n}}| \ge a_{\bar{n}}\big) < \infty,$$

(3.3) $$S_{\bar{n}}/a_{\bar{n}} \to 0 \quad in\ probability$$

are equivalent to

(3.4) $$S_{\bar{n}}/a_{\bar{n}} \to 0 \quad a.s.$$

Proof. It is enough to prove (3.2) and (3.3)⇒(3.4). We assume, without loss of generality, that $\{X_{\bar{n}}, \bar{n} \in \mathbb{N}^r\}$ are symmetric. Let us put

$$Y_{\bar{j}} = X_{\bar{j}}\big(\|X_j\| > a_{\bar{j}}\big) \quad \text{and} \quad T_{\bar{n}} = \sum_{\bar{j}\le\bar{n}} Y_{\bar{j}}.$$

By the virtue of (3.2) and from the Borel–Cantelli lemma follows

$$(3.5) \qquad (S_{\bar{n}} - T_{\bar{n}})/a_{\bar{n}} \to 0 \quad \text{a.s. as } \bar{n} \to \infty.$$

Therefore it is enough to prove $T_{\bar{n}}/a_{\bar{n}} \to 0$ a.s. as $\bar{n} \to \infty$. Let us put

$$V_{kl} = \|T_{E_{kl}}\| - E\|T_{E_{kl}}\|.$$

Thus by Th. 2.1 of de Acosta [1] we have

$$\sum_{k=1}^{\infty} P\big(\max_{l\in R_k} |V_{kl}|/d^k > \varepsilon\big) \le \sum_{k=l}^{\infty} \sum_{l\in R_k} P\big(|V_{kl}|/d^k > \varepsilon\big) \le$$

$$\le \frac{1}{\varepsilon^2} \sum_{k=1}^{\infty} \sum_{l\in R_k} E|V_{kl}|^2/d^{2k} \le \frac{4}{\varepsilon^2} \sum_{k=1}^{\infty} \sum_{l\in R_k} \sum_{\bar{j}\in E_{kl}} E\|Y_{\bar{j}}\|^2/d^{2k} \le$$

$$\le \frac{4C_2^2}{\varepsilon^2} \sum_{k=1}^{\infty} \sum_{l\in R_k} \sum_{\bar{j}\in E_{kl}} E\|Y_{\bar{j}}\|^2/a_{\bar{j}}^2 \le \frac{4C_2^2}{\varepsilon^2} \sum_{\bar{n}\in\mathbb{N}^r} E\|Y_{\bar{n}}\|^2/a_{\bar{n}}^2.$$

On the other hand

$$\sum_{\bar{k}} E\|Y_{\bar{k}}\|^2/a_{\bar{k}}^2 = \sum_{\bar{k}} E\big\{\|X\|^2 I(\|X\| < a_{\bar{k}})\big\}/a_{\bar{k}}^2 \le$$

$$\le C + C\sum_{i\ge 1} i^{-2} m_i E\big\{\|X\|^2 I(\|X\| < i)\big\} \le$$

$$\le C + C\sum_{j\ge 1} E\big\{\|X\|^2 I(j-1 \le \|X\| < 1)\big\} \sum_{i\ge j} i^{-2} m_i$$

for some constants C.

Now let us observe that by assumptions and Abel transform we get

$$\sum_{i\ge j} i^{-2} m_i \le C\sum_{i\ge j} i^{-3} M_i \le C j^{-2} M_j.$$

Hence

$$(3.6) \qquad \begin{aligned} \sum_{\bar{k}} E\|Y_{\bar{k}}\|^2/a_{\bar{k}}^2 &\le C + C\sum_{j\ge 1} j^{-2} M_j E\big\{\|X\|^2 I(j-1 \le \|X\| < j)\big\} \le \\ &\le C + C\sum_{i\ge 1} m_i P(\|X\| \ge i) \le C + C\sum_{\bar{n}} P(\|X\| \ge a_{\bar{n}}) < \infty. \end{aligned}$$

Therefore by the Borel–Cantelli Lemma

$$(3.7) \qquad \max_{l\in R_k} |V_{kl}|/d^k \to 0 \quad \text{a.s.}$$

It follows at once from (3.3) and (3.5) that $T_{\bar{n}}/a_{\bar{n}} \xrightarrow{P} 0$ and by Lemma 2.2

$$E\|T_{E_{kl}}/d^k\| \to 0 \quad \text{uniformly in } l \in R_k. \tag{3.8}$$

Now, let us observe that (3.7) and (3.8) imply

$$T_{E_{kl}}/d^k \to 0 \quad \text{a.s. uniformly in } l \in R_k$$

and applications of Lemma 2.1 complete the proof. ◊

Remark. Let us observe that condition (3.2) is essential. However, it is necessary for (3.4), but (3.3) does not imply (3.2). We will give appropriate example. Let $\{X_k, k \geq 1\}$ be a sequence of i.i.d. random variables with probability density function

$$f(x) = \begin{cases} k(1+\ln|x|)/(x^2\ln^2|x|), & \text{for } |x| \geq 2 \\ \frac{1}{4}(1+\ln 2)/(1+\ln 2), & \text{for } |x| \leq 2 \end{cases}$$

$$\text{where constant } k = \ln^2 2/(1+2\ln 2).$$

It is easy to see that $\sum_{i=1}^{n} X_i/n \to 0$ in probability but condition (3.2) is not satisfied.

Corollary 3.1. *Let* $a_{\bar{n}} = |\bar{n}|^{1/p}$, $1 \leq p < 2$ *then* $M_j = \mathrm{O}(j^p(\log j)^{r-1})$ (cf. Smythe [10]) *and* (3.1) *is satisfied. Convergence of series* (3.2) *is equivalent to* $E\|X\|^p(\log_+ \|X\|^{r-1}) < \infty$.

Thus by Th. 3.1 we can obtain immediately result of Fazekas [2].

The following corollary is not only generalization of Marcinkiewicz SLLN but give us a better and deeper understanding of strong laws for random variables with multidimensional indices.

Corollary 3.2. *Let* $a_{\bar{n}} = n_1^{1/p_1} \cdot \ldots \cdot n_r^{1/p_r}$, $1 \leq p_i < 2$, $1 \leq i \leq r$, $t = \max(p_1, \ldots, p_r)$, $q = \mathrm{card}\{i : p_i = t, 1 \leq i \leq r\}$. *Thus* $M_j = \mathrm{O}(j^t(\log_+ j)^{q-1})$ *and* (3.2) *imply* $E\|X\|^t(\log_+ \|X\|^{q-1}) < \infty$. *Then the following are equivalent:*

(i) $$S_{\bar{n}}/n_1^{1/p_1} \cdot \ldots \cdot n_r^{1/p_r} \to 0 \quad a.s. \text{ as } \bar{n} \to \infty;$$

(ii) $$S_{\bar{n}}/n_1^{1/p_1} \cdot \ldots \cdot n_r^{1/p_r} \xrightarrow{P} 0 \quad \text{as } \bar{n} \to \infty.$$

Furthermore, let us observe that for $r=1$ all increasing to infinity, positive sequences $\{a_{\bar{n}}, \bar{n} \in \mathbb{N}^r\}$ have the weak star property and assumption (3.1) implies the well-known Feller condition $\sum_{k \geq n} a_k^{-2} = \mathrm{O}(n/a_n^2)$.

Th. 3.1 and corollaries are established for separable Banach space $(B, \| \|)$. In what follows we will assume geometric conditions on the space $(B, \| \|)$.

Theorem 3.2. *Let Banach space* $(B, \| \|)$ *be of type 2,* $\{X_{\bar{n}}, \bar{n} \in \mathbb{N}^r\}$, $\{a_{\bar{n}}, \bar{n} \in \mathbb{N}^r\}$ *are such as in* Th. 1 *and* $EX_{\bar{n}} = 0$, $\bar{n} \in \mathbb{N}^r$. *If moreover* (3.1) *and* (3.2) *are satisfied then*

$$S_{\bar{n}}/a_{\bar{n}} \to 0 \quad a.s. \text{ as } \bar{n} \to \infty.$$

Proof. A family $\{x_{\bar{n}}, \bar{n} \in \mathbb{N}^r\}$ is said to be Cauchy if for every $\varepsilon > 0$ we have $d(x_{\bar{n}}, x_{\bar{m}}) < \varepsilon$ whenever $\bar{n}, \bar{m} \geq \bar{k}_\varepsilon$ for suitably chosen $\bar{k}_\varepsilon$ in $\mathbb{N}^r$. Since B is of type 2 one can prove using estimation as in (3.6) that, $\{Y_{\bar{n}}/a_{\bar{n}}, \bar{n} \in \mathbb{N}^r\}$ is a Cauchy family. Hence $\{Y_{\bar{n}}/a_{\bar{n}}, \bar{n} \in \mathbb{N}^r\}$ convergence in L^2, then converges almost surely. Further, by multidimensional version of Kronecker Lemma and arguments as in proof of Th. 3.1, we deduce the assertion of Th. 3.2. ◊

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