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PROPER SHAPE INVARIANTS: SMOOTHNESS AND CALMNESS

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Abstract: We study properly $(\mathcal{B},\mathcal{C})$-smooth and properly $\mathcal{C}$-calm spaces, where $\mathcal{B}$ and $\mathcal{C}$ denote classes of topological spaces. Both proper smoothness and proper calmness are invariants of a recently invented author's proper shape theory and are described by the use of proper multi-valued functions. The dual notions are also examined.

1. Introduction

The notions and results in this paper belong to the part of topology that could be described as proper shape theory. As shape theory is an improved homotopy theory designed to handle more successfully complicated spaces so is proper shape theory a modification of proper homotopy theory made with the same goal to provide us with a new insight into global properties even of those spaces for which the classical proper homotopy gives doubtful information.

In [7] the author has described proper shape category of all topological spaces using Sanjurjo's method of multi-valued functions from [12]. Our approach was formally very similar to the one taken by Ball and Sher [2]. Instead of proper fundamental nets we considered proper multi-nets. The other steps were identical. We defined a notion of

a proper homotopy for proper multi-nets and took for the morphisms of the proper shape category $\mathcal{S}h_p$ proper homotopy classes of proper multi-nets.

In the present paper we shall introduce and investigate proper shape invariants called smoothness and calmness. It is useful to consider these notions in terms of arbitrary classes $\mathcal{B}$ and $\mathcal{C}$ of topological spaces. In other words, we shall define $M_p^{\mathcal{B},\mathcal{C}}$-smooth and $M_p^{\mathcal{C}}$-calm spaces and explore their properties. In our notation the letter "M" suggests the use of multi-valued functions while "p" replaces "proper" or "properly".

Let us describe the content of the paper in greater detail. In §2 we recall notions and results from [7] that are necessary in further developments. The next §3 studies $M_p^{\mathcal{B},\mathcal{C}}$-smooth spaces. The idea is that we require that small enough proper multi-valued functions from members of a class of spaces $\mathcal{B}$ into a given space X which are properly homotopic over members of another class $\mathcal{C}$ are already properly homotopic through sufficiently small proper multi-valued functions. This concept is related to the notion of n-types of Whitehead and it could be regarded as a substitute for it in the proper shape theory. We prove that this is an invariant in the category $\mathcal{S}h_p$, explore the role of classes $\mathcal{B}$ and $\mathcal{C}$, and study what kind of maps will preserve and inversely preserve $M_p^{\mathcal{B},\mathcal{C}}$-smooth spaces. The classes of proper $\mathcal{B}$-surjections and proper $\mathcal{B}$-injections from [8] are of key importance.

In the following §4 we consider $M_p^{\mathcal{B}}$-calm spaces. The calm spaces have proved useful in shape theory and geometric topology and are dual in many respects to the movable spaces of Borsuk [4] just as smooth spaces are dual to tame spaces which becomes clear when comparing this paper with [9]. For the first time we have now this concepts in the proper shape theory of arbitrary topological spaces.

Since the method of investigating properties of spaces by looking at maps from some objects into a space has an obvious dual approach where we utilize maps from a space into those objects, we also consider in §§5 and 6 so called $N_p^{\mathcal{B},\mathcal{C}}$-smooth, $P_p^{\mathcal{B},\mathcal{C}}$-smooth and $N_p^{\mathcal{B}}$-calm classes of spaces, where the change from the letter "M" to the letters "N" and "P" should reflect duality between these notions. As the reader will see this duality is striking.

Finally, in §7 we consider dependence of these notions on classes $\mathcal{B}$ and $\mathcal{C}$ under the assumption that they are connected with each other by morphisms from [8].

2. Preliminaries on proper shape theory

In this section we shall introduce notions and results from [7] that are required for our theory.

Let X and Y be topological spaces. By a *multi-valued function* or an *M-function* $F\colon X \to Y$ we mean a rule which associates a non-empty subset $F(x)$ of Y to every point x of the space X. An M-function $F\colon X \to Y$ is *próper* provided for every compact subset C of Y its small counterimage $F'(C) = \{x \in X |\, F(x) \subset C\}$ is a compact subset of X. On the other hand, F is *pròper* provided for every compact subset C of Y its big counterimage $F''(C) = \{x \in X |\, F(x) \cap C \neq \emptyset\}$ is a compact subset of X. We shall use the term *proper* to name either próper or pròper. However, in a given situation, once we make a selection between two different kinds of properness it is understood that it will be retained throughout. Instead of proper multi-valued function we shall use the shorter name *M_p-function*

Observe that for single-valued functions the two notions of properness coincide. Classes of próper and pròper M-functions are completely unrelated [7]. It follows that each of our notions and results on M_p-functions actually has two versions.

In this paper by a *cover* we mean an open normal cover [1]. Let $\mathrm{Cov}(Y)$ denote the collection of all covers of a topological space Y. With respect to the refinement relation $>$ the set $\mathrm{Cov}(Y)$ is a directed set. Two covers σ and τ of Y are equivalent provided $\sigma > \tau$ and $\tau > \sigma$. In order to simplify our notation we denote a cover and it's equivalence class by the same symbol. Consequently, $\mathrm{Cov}(Y)$ also stands for the associated quotient set.

If σ is a cover of a space Y, let σ^+ be the collection of all covers of Y which refine σ while $\sigma^\star$ denotes the set of all covers τ of Y such that the star $st(\tau)$ of τ refines σ. Similarly, for a natural number n, $\sigma^{\star n}$ denotes the set of all covers τ of Y such that the n-th star $st^n(\tau)$ of τ refines σ.

Let $\mathrm{Inc}(Y)$ denote the collection of all finite subsets c of $\mathrm{Cov}(Y)$ which have a unique (with respect to the refinement relation) maximal element which we denote by $[c]$. The notation $\mathrm{Inc}(Y)$ comes from "indices of covers". The set $\mathrm{Inc}(Y)$ will be used as indexing set for proper multi-nets into Y. We consider $\mathrm{Inc}(Y)$ ordered by the inclusion relation and regard $\mathrm{Cov}(Y)$ as a subset of single-element subsets of

$\mathrm{Cov}(Y)$. Notice that $\mathrm{Inc}(Y)$ is a cofinite directed set.

For our proper shape theory the following notion of size for M-functions will play the most important role. Let $F\colon X \to Y$ be an M_p-function and let $\alpha \in \mathrm{Cov}(X)$ and $\gamma \in \mathrm{Cov}(Y)$. We shall say that F is an $M_p^{\alpha,\gamma}$-*function* provided for every $A \in \alpha$ there is a $C_A \in \gamma$ with $F(A) \subset C_A$. On the other hand, F is γ-*small* or an M_p^{γ}-*function* provided there is an $\alpha \in \mathrm{Cov}(X)$ such that F is an $M_p^{\alpha,\gamma}$-function. For an M_p^{σ}-function $F\colon X \to Y$ we use $S(F, \sigma)$ to denote the family of all $\alpha \in \mathrm{Cov}(X)$ such that F is an $M_p^{\alpha,\sigma}$-function.

Next we introduce the notions which correspond to the equivalence relation of proper homotopy for proper maps. Let F and G be M_p-functions from a space X into a space Y and let γ be a cover of Y. We shall say that F and G are *properly* γ-*homotopic* or M_p^{γ}-*homotopic* and write $F \overset{\gamma}{\sim} G$ provided there is an M_p^{γ}-function H from the product $X \times I$ of X and the unit segment $I = [0, 1]$ into Y such that $F(x) = H(x, 0)$ and $G(x) = H(x, 1)$ for every $x \in X$. We shall say that H is a *proper* γ-*homotopy* or an M_p^{γ}-*homotopy* that *joins* F and G or that it *realizes* the relation $F \overset{\gamma}{\sim} G$.

The following lemma from [7] is crucial because it provides an adequate substitute for the transitivity of the relation of proper homotopy.

2.1 Lemma. *Let F, G, and H be M_p-functions from a space X into a space Y. Let $\sigma \in \mathrm{Cov}(Y)$ and $\tau \in \sigma^*$. If $F \overset{\tau}{\sim} G$ and $G \overset{\tau}{\sim} H$, then $F \overset{\sigma}{\sim} H$.*

The proof of Lemma 2.1 requires an interesting proposition from A. Dold's book [10, p. 358] on covers of the product $X \times I$ of a space X with the unit segment I. We assume that the reader is familiar with this result and the notion of a stacked covering of $X \times I$ over a cover of X. For a cover σ of $X \times I$, we shall use $D(X, \sigma)$ to denote the collection of all covers τ of X such that some stacked covering of $X \times I$ over τ refines σ. As a consequence of the above proposition, this collection is always non-empty.

The following two definitions correspond to Ball and Sher's definitions of proper fundamental net and proper homotopy for proper fundamental nets.

Let X and Y be topological spaces. By a *proper multi-net* or an M_p-net from X into Y we shall mean a collection $\varphi = \{F_c\}_{c \in \mathrm{Inc}(Y)}$ of M_p-functions $F_c\colon X \to Y$ such that for every $\gamma \in \mathrm{Cov}(Y)$ there

is a $c \in \mathrm{Inc}(Y)$ with $F_d \overset{\gamma}{\sim} F_c$ for every $d > c$. We use functional notation $\varphi\colon X \to Y$ to indicate that φ is an M_p-net from X into Y. Let $MN_p(X, Y)$ denote all M_p-nets $\varphi\colon X \to Y$.

Two M_p-nets $\varphi = \{F_c\}$ and $\psi = \{G_c\}$ between topological spaces X and Y are *M_p-homotopic* and we write $\varphi \sim \psi$ provided for every $\gamma \in \mathrm{Cov}(Y)$ there is a $c \in \mathrm{Inc}(Y)$ such that $F_d \overset{\gamma}{\sim} G_d$ for every $d > c$. On the other hand, we write $\varphi \overset{\gamma}{\sim} \psi$ and call φ and ψ *M_p^{γ}-homotopic* provided there is a $c \in \mathrm{Inc}(Y)$ such that $F_d \overset{\gamma}{\sim} G_d$ for every $d > c$.

It follows from Lemma 2.1 that the relation of M_p-homotopy is an equivalence relation on the set $MN_p(X, Y)$. The M_p-homotopy class of an M_p-net φ is denoted by $[\varphi]$ and the set of all M_p-homotopy classes by $\mathcal{S}h_p(X, Y)$.

Our goal now is to define a composition for M_p-homotopy classes of M_p-nets. Let $\varphi = \{F_c\}\colon X \to Y$ be a M_p-net. Let $\varphi\colon \mathrm{Inc}(Y) \to$ $\to \mathrm{Inc}(Y)$ be an increasing function such that for every $c \in \mathrm{Inc}(Y)$ the relation $d, e > \varphi(c)$ implies the relation $F_d \overset{[c]}{\sim} F_e$. Here we make an assumption that an increasing function φ from a partially ordered set P into itself always satisfies the condition that $\varphi(p) > p$ for every $p \in P$. Let $\mathcal{C} = \{(c, d, e) | \, c \in \mathrm{Inc}(Y), d, e > \varphi(c)\}$. Then $\mathcal{C}$ is a subset of $\mathrm{Inc}(Y) \times \mathrm{Inc}(Y) \times \mathrm{Inc}(Y)$ that becomes a cofinite directed set when we define that $(c, d, e) > (c', d', e')$ if and only if $c > c'$, $d > d'$, and $e > e'$. We shall use the same notation φ for an increasing function $\varphi\colon \mathcal{C} \to \mathrm{Cov}(X \times I)$ such that F_d and F_e are joined by a proper $(\varphi(c, d, e), [c])$-homotopy whenever $(c, d, e) \in \mathcal{C}$. Let $\bar{\varphi}\colon \mathcal{C} \to \mathrm{Inc}(X)$ be an increasing function such that $[\bar{\varphi}(c, d, e)] \in D(X, \varphi(c, d, e))$ for every $(c, d, e) \in \mathcal{C}$. In [7] it was proved that there is an increasing function $\varphi^*\colon \mathrm{Inc}(Y) \to \mathrm{Inc}(X)$ such that (1) $\varphi^*(c) > \bar{\varphi}(c, \varphi(c), \varphi(c))$ for every $c \in \mathrm{Inc}(Y)$, and (2) φ^* is cofinal in $\bar{\varphi}$, i. e., for every $(c, d, e) \in$ $\in \mathcal{C}$ there is an $m \in \mathrm{Inc}(Y)$ with $\varphi^*(m) > \bar{\varphi}(c, d, e)$. With the help of functions φ and φ^* we shall define the composition of M_p-homotopy classes of M_p-nets as follows.

Let $\varphi = \{F_c\}\colon X \to Y$ and $\psi = \{G_s\}\colon Y \to Z$ be M_p-nets. Let $\chi = \{H_s\}$, where $H_s = G_{\psi(s)} \circ F_{\varphi(\psi^*(s))}$ for every $s \in \mathrm{Inc}(Z)$. Observe that each H_s is a M_p-function because the composition of two M_p-functions is an M_p-function. In [7] it was proved that the collection χ is an M_p-net from X into Z. We now define the composition of M_p-homotopy classes of M_p-nets by the rule $[\{G_s\}] \circ [\{F_c\}] =$

$= [\{G_{\psi(s)} \circ F_{\varphi(\psi^*(s))}\}]$. This composition of M_p-homotopy classes of M_p-nets is well-defined and associative.

For a space X, let $\iota^X = \{I_a\}\colon X \to X$ be the identity M_p-net defined by $I_a = \mathrm{id}_X$ for every $a \in \mathrm{Inc}(X)$. It is easy to show that for every M_p-net $\varphi\colon X \to Y$, the following relations hold: $[\varphi] \circ [\iota^X] = [\varphi] = [\iota^Y] \circ [\varphi]$.

We can summarize the above with the following main result from [7].

2.2 Theorem. *The topological spaces as objects together with the M_p-homotopy classes of M_p-nets as morphisms and the composition of M_p-homotopy classes form the proper shape category $\mathcal{S}h_p$.*

The above constructions may be done without any reference to próper and pròper M-functions. In this way we shall get the shape category $\mathcal{S}h$. On the other hand, in both cases, we may require that all M-functions belong to a class of M-functions which is closed with respect to pastings from the proof of Lemma 2.1 in [7] and compositions. In particular, we may assume that they are either upper semi-continuous or lower semi-continuous.

3. $M_p^{\mathcal{B},\mathcal{C}}$-smooth spaces

In this section we shall explore the following interesting notion which in the case of compacta reduces to the author's $(\mathcal{B}, \mathcal{C})$-smoothness from [5] and [6].

Let $\mathcal{D}$ be a class of spaces, let F and G be M_p-functions from a space X into a space Y, and let σ be a cover of Y. We shall say that F and G are *properly σ-homotopic over $\mathcal{D}$* and write $F \overset{\tau}{\sim}_{\mathcal{D}} G$ provided there is a cover τ of X such that for every M_p^τ-function H from a member of $\mathcal{D}$ into X the compositions $F \circ H$ and $G \circ H$ are M_p^σ-homotopic.

Let $\mathcal{B}$ and $\mathcal{C}$ be classes of topological spaces. A space X is *$M_p^{\mathcal{B},\mathcal{C}}$-smooth* provided for every cover σ of X there is a cover τ of X with the property that for M_p^τ-functions F and G from a member of $\mathcal{B}$ into X the relation $F \overset{\tau}{\sim}_{\mathcal{C}} G$ implies the relation $F \overset{\sigma}{\sim} G$. A class of spaces is $M_p^{\mathcal{B},\mathcal{C}}$-smooth provided each member of it is $M_p^{\mathcal{B},\mathcal{C}}$-smooth.

We shall first show that the property of being $M_p^{\mathcal{B},\mathcal{C}}$-smooth is a proper shape invariant, i. e., that if X and Y are equivalent objects of the category $\mathcal{S}h_p$ and X is $M_p^{\mathcal{B},\mathcal{C}}$-smooth then Y is also $M_p^{\mathcal{B},\mathcal{C}}$-smooth.

In fact, a much better result is true. The $M_p^{\mathcal{B},\mathcal{C}}$-smooth spaces are preserved under the following weak form of domination.

A class of spaces $\mathcal{B}$ is *M_p-dominated* by a class of spaces $\mathcal{A}$ provided for every $B \in \mathcal{B}$ and every $\beta \in \mathrm{Cov}(B)$ there is an $A \in \mathcal{A}$ and an M_p^β-function $G\colon A \to B$ such that for every $\alpha \in \mathrm{Cov}(A)$ we can find an M_p^α-function $F\colon B \to A$ with $G \circ F \overset{\beta}{\sim} \mathrm{id}_B$.

3.1 Theorem *A space X is $M_p^{\mathcal{B},\mathcal{C}}$-smooth if and only if it is M_p-dominated by a class of $M_p^{\mathcal{B},\mathcal{C}}$-smooth spaces.*

Proof. Since every space M_p-dominates itself, it remains to prove the "if" part. Let a cover σ of X be given. Let $\eta \in \sigma^*$. By assumption, there is an $M_p^{\mathcal{B},\mathcal{C}}$-smooth space Y and an M_p^η-function $D\colon Y \to X$ such that for every $\varepsilon \in \mathrm{Cov}(Y)$ there is an M_p^ε-function $U\colon X \to Y$ with $G \circ F \overset{\eta}{\sim} \mathrm{id}_X$.

Let $\delta \in S(D, \eta)$. Since Y is $M_p^{\mathcal{B},\mathcal{C}}$-smooth, there is an $\varepsilon \in \mathrm{Cov}(Y)$ such that for every M_p^ε-functions K and L from a member of $\mathcal{B}$ into Y the relation $K \overset{\varepsilon}{\sim}_{\mathcal{C}} L$ implies the relation $K \overset{\delta}{\sim} L$.

Pick a U as above. Let W be an M_p^η-homotopy joining id_X and $D \circ U$. Let $\tau \in \mathrm{Cov}(X)$ belong to $D(W, \eta)$ and $S(F, \varepsilon)$. Then τ is the required cover of X. To verify this, consider a member B of $\mathcal{B}$ and M_p^τ-functions $F, G\colon B \to X$ with $F \overset{\tau}{\sim}_{\mathcal{C}} G$. Let K and L be $U \circ F$ and $U \circ G$. Then K and L are M_p^ε-functions from B into Y with $K \overset{\varepsilon}{\sim}_{\mathcal{C}} L$. It follows that $K \overset{\delta}{\sim} L$ so that after composing with D we obtain $F \overset{\eta}{\sim} D \circ U \circ F = D \circ K \overset{\eta}{\sim} D \circ L = D \circ U \circ G \overset{\eta}{\sim} G$. Hence, $F \overset{\sigma}{\sim} G$. $\Diamond$

The M_p-domination is weaker than the quasi $\mathcal{S}h_p$-domination and thus also weaker than $\mathcal{S}h_p$-domination [8]. Recall that a class of spaces $\mathcal{A}$ is *$\mathcal{S}h_p$-dominated* by a class of spaces $\mathcal{B}$ provided for every $X \in \in \mathcal{A}$ there is a $Y \in \mathcal{B}$ and M_p-nets $\varphi\colon X \to Y$ and $\psi\colon Y \to X$ with the composition $\psi \circ \varphi$ M_p-homotopic to the identity M_p-net ι^X on X. On the other hand, $\mathcal{A}$ is *quasi $\mathcal{S}h_p$-dominated* by $\mathcal{B}$ provided for every $X \in \mathcal{A}$ and every $\sigma \in \mathrm{Cov}(X)$ there is a $Y \in \mathcal{B}$ and M_p-nets $\varphi\colon X \to \to Y$ and $\psi\colon Y \to X$ with the composition $\psi \circ \varphi$ M_p^σ-homotopic ι^X. The notion of quasi $\mathcal{S}h_p$-domination is similar to the notion of quasi-domination in [3].

3.2 Corollary. *A space is $M_p^{\mathcal{B},\mathcal{C}}$-smooth if and only if it is either $\mathcal{S}h_p$-dominated or quasi $\mathcal{S}h_p$-dominated by a class of $M_p^{\mathcal{B},\mathcal{C}}$-smooth spaces.*

Another example of M_p-domination provides the notion of being properly $\mathcal{B}$-like. Recall that a space X is *properly $\mathcal{B}$-like*, where $\mathcal{B}$ is a class of spaces, provided for every $\sigma \in \mathrm{Cov}(X)$ there is a member Y of $\mathcal{B}$ and a proper map $f: X \to Y$ such that the inverse $f^{-1}: Y \to X$ is an M_p^σ-function. In [8] we showed that if a space X is properly $\mathcal{B}$-like, then X is M_p-dominated by $\mathcal{B}$. Hence, we get the following conclusion.

3.3 Corollary. *A space X is $M_p^{\mathcal{B},\mathcal{C}}$-smooth if and only if it is properly $\mathcal{D}$-like, where $\mathcal{D}$ is a class of $M_p^{\mathcal{B},\mathcal{C}}$-smooth spaces.*

In the following two theorems we explore in which way does the definition of $M_p^{\mathcal{B},\mathcal{C}}$-smooth spaces depend on classes $\mathcal{B}$ and $\mathcal{C}$. The first result uses the following notion from [9].

Let $\mathcal{B}$ and $\mathcal{C}$ be classes of spaces. A space X is *$M_p^{\mathcal{B},\mathcal{C}}$-tame* provided for every $\sigma \in \mathrm{Cov}(X)$ there is a $\tau \in \mathrm{Cov}(X)$ such that for every $B \in \mathcal{B}$ and every M_p^τ-function $F: B \to X$ there is a $C \in \mathcal{C}$ and an M_p^σ-function $H: C \to X$ with the property that for every $\alpha \in \mathrm{Cov}(C)$ there is an M_p^α-function $G: B \to C$ with $F \overset{\sigma}{\sim} H \circ G$. A class of spaces is $M_p^{\mathcal{B},\mathcal{C}}$-tame provided each member of it is $M_p^{\mathcal{B},\mathcal{C}}$-tame.

3.4 Theorem. *Let $\mathcal{A}$ and $\mathcal{C}$ be classes of topological spaces and let $\mathcal{B}$ be a class of $M_p^{\mathcal{A},\mathcal{C}}$-tame spaces. Then every $M_p^{\mathcal{B},\mathcal{A}}$-smooth space X is also $M_p^{\mathcal{B},\mathcal{C}}$-smooth.*

Proof. Let a cover σ of X be given. Let $\mu \in \sigma^*$. Pick a $\mu \in \mathrm{Cov}(X)$ such that for M_p^μ-functions F and G from a member of $\mathcal{B}$ into X the relation $F \overset{\mu}{\sim}_{\mathcal{A}} G$ implies the relation $F \overset{\sigma}{\sim} G$. Let $\tau \in \mu^*$.

Consider M_p^τ-functions $F, G: B \to X$ such that $B \in \mathcal{B}$ and $F \overset{\tau}{\sim}_{\mathcal{C}} G$. Then there is a $\beta \in \mathrm{Cov}(B)$ with the property that for every M_p^β-function K from a member of $\mathcal{C}$ into B the compositions $F \circ K$ and $G \circ K$ are M_p^τ-homotopic. Let a $\gamma \in \beta^+$ belong to both $S(F, \tau)$ and $S(G, \tau)$. Since B is $M_p^{\mathcal{A},\mathcal{C}}$-tame, there is a $\delta \in \mathrm{Cov}(B)$ such that for every $A \in \mathcal{A}$ and every M_p^δ-function $L: A \to B$ there is a $C \in \mathcal{C}$ and an M_p^γ-function $K: C \to B$ so that for every $\alpha \in \mathrm{Cov}(C)$ there is an M_p^α-function $J: A \to C$ with $K \overset{\gamma}{\sim} K \circ J$.

Let $A \in \mathcal{A}$ and let $L: A \to B$ be an M_p^δ-function. Pick a C and a K as above. Our choices imply $F \circ K \overset{\tau}{\sim} G \circ K$. Let N be a M_p^τ-homotopy which realizes this relation. Let $\alpha \in D(N, \tau)$. Choose a J as above. Then we obtain $F \circ L \overset{\tau}{\sim} F \circ K \circ J \overset{\tau}{\sim} G \circ K \circ J \overset{\tau}{\sim} G \circ L$. It follows that $F \overset{\mu}{\sim}_{\mathcal{A}} G$ and therefore $F \overset{\sigma}{\sim} G$. $\Diamond$

3.5 Theorem. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ be classes of spaces such that $\mathcal{B}$ and $\mathcal{D}$ are M_p-dominated by $\mathcal{A}$ and $\mathcal{C}$, respectively. If a space X is $M_p^{\mathcal{A},\mathcal{D}}$-smooth, then it is also $M_p^{\mathcal{B},\mathcal{C}}$-smooth.*

Proof. Let a cover σ of X be given. Let $\pi \in \sigma^*$. Since X is $M_p^{\mathcal{A},\mathcal{D}}$-smooth, there is a $\varrho \in \pi^+$ such that for M_p^{ϱ}-functions K and L from a member of $\mathcal{A}$ into X the relation $K \overset{\varrho}{\sim}_{\mathcal{D}} L$ implies the relation $K \overset{\pi}{\sim} L$. Let $\tau \in \varrho^*$. Then τ is the required cover of X. Indeed, consider a member B of $\mathcal{B}$ and M_p^{τ}-functions $F, G\colon B \to X$ such that $F \overset{\tau}{\sim}_{\mathcal{C}} G$. Let $\beta \in \mathrm{Cov}(B)$ belong to both $S(F,\tau)$ and $S(G,\tau)$ and be such that for every M_p^{β}-function R from a member of $\mathcal{C}$ into B we have $F \circ R \overset{\tau}{\sim} G \circ R$. Since the class $\mathcal{B}$ is M_p-dominated by the class $\mathcal{A}$, there is an $A \in \mathcal{A}$ and an M_p^{β}-function $J\colon A \to B$ such that for every $\alpha \in \mathrm{Cov}(A)$ there is an M_p^{α}-function $E\colon B \to A$ with $J \circ E \overset{\beta}{\sim} \mathrm{id}_B$.

Let K and L be the compositions $F \circ J$ and $G \circ J$, respectively. Then K and L are M_p^{ϱ}-functions from A into X. We claim that $K \overset{\varrho}{\sim}_{\mathcal{D}} L$.

In order to verify this, let $\alpha \in S(J,\beta)$. Suppose that $D \in \mathcal{D}$ and $T\colon D \to A$ is an M_p^{α}-function. Let $\delta \in S(T,\alpha)$. We utilize now the assumption that the class $\mathcal{D}$ is M_p-dominated by the class $\mathcal{C}$ to select a $C \in \mathcal{C}$ and an M_p^{δ}-function $W\colon C \to D$ with the property that for every cover γ of C there is an M_p^{γ}-function $V\colon D \to C$ with $W \circ V \overset{\delta}{\sim} \mathrm{id}_D$. The composition $J \circ T \circ W$ is an M_p^{β}-function from C into B. It follows that there is an M_p^{τ}-homotopy $P\colon C \times I \to X$ joining $F \circ J \circ T \circ W$ and $G \circ J \circ T \circ W$. Let $\gamma \in D(P,\tau)$. Choose a V as above. Then we have $K \circ T = F \circ J \circ T \overset{\tau}{\sim} F \circ J \circ T \circ W \circ V \overset{\tau}{\sim} G \circ J \circ T \circ W \circ V \overset{\tau}{\sim} G \circ J \circ T = L \circ T$. Hence, $K \circ T \overset{\varrho}{\sim} L \circ T$ and the claim has been verified.

Now our assumption implies existence of an M_p^{π}-homotopy $Q\colon A \times I \to X$ joining K and L. Let $\alpha \in D(Q,\pi)$. Pick an E as above. Then we obtain that $F \overset{\pi}{\sim} F \circ J \circ E = K \circ E \overset{\pi}{\sim} L \circ E = G \circ J \circ E \overset{\pi}{\sim} G$. Hence, $F \overset{\sigma}{\sim} G$. $\Diamond$

3.6 Corollary. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ be classes of spaces such that $\mathcal{B}$ and $\mathcal{D}$ are (quasi) Sh_p-dominated by $\mathcal{A}$ and $\mathcal{C}$, respectively. If a space X is $M_p^{\mathcal{A},\mathcal{D}}$-smooth, then it is also $M_p^{\mathcal{B},\mathcal{C}}$-smooth.*

The following weak form of the notion of being properly $\mathcal{B}$-like is more in line with our point of view because it is based on M_p-functions. It offers us the possibility to improve Cor. 3.3 in Th. 3.7.

Let $\mathcal{C}$ be a class of spaces. A space X is *$M_p^{\mathcal{C}}$-like* provided for

every $\sigma \in \mathrm{Cov}(X)$ there is a member Y of $\mathcal{C}$ and a cover α of Y such that for every $\beta \in \mathrm{Cov}(Y)$ there is an M_p^β-function $F\colon X \to Y$ such that F^{-1} is an $M_p^{\alpha,\sigma}$-function.

3.7 Theorem *A space X is $M_p^{\mathcal{B},\mathcal{C}}$-smooth if and only if it is $M_p^{\mathcal{D}}$-like, where $\mathcal{D}$ is a class of $M_p^{\mathcal{B},\mathcal{C}}$-smooth spaces.*

Proof. Let a cover σ of X be given. Let $\mu \in \sigma^*$. Since X is $M_p^{\mathcal{D}}$-like, there is a $Y \in \mathcal{D}$ and a cover α of Y such that for every $\beta \in \in \mathrm{Cov}(Y)$ there is an M_p^β-function $R\colon X \to Y$ such that R^{-1} is an $M_p^{\alpha,\mu}$-function. Since Y is $M_p^{\mathcal{B},\mathcal{C}}$-smooth, there is a $\beta \in \mathrm{Cov}(Y)$ such that for M_p^β-functions K and L from a member of $\mathcal{B}$ into Y the relation $K \overset{\beta}{\sim}_{\mathcal{C}} L$ implies the relation $K \overset{\alpha}{\sim} L$. Choose an R as above and let $\tau \in S(R, \beta)$. The cover τ is the one we were looking for. In fact, let $B \in \mathcal{B}$ and assume that $F, G\colon B \to X$ are M_p^π-functions with $F \overset{\tau}{\sim}_{\mathcal{C}} G$. Let K and L be $R \circ F$ and $R \circ G$. Then K and L are M_p^β-functions from B into Y and $K \overset{\beta}{\sim}_{\mathcal{C}} L$. As in the proof of Theorem 3.4 in [9], it follows that $F \overset{\mu}{\sim} R^{-1} \circ R \circ F \overset{\mu}{\sim} R^{-1} \circ R \circ G \overset{\mu}{\sim} G$. Hence, $F \overset{\sigma}{\sim} G$. $\Diamond$

In the rest of this section we shall address the question of identifying those proper maps which will preserve or inversely preserve $M_p^{\mathcal{B},\mathcal{C}}$-smooth spaces. The answer provide proper maps studied in [8] whose definitions we now recall.

Let $\mathcal{B}$ be a class of spaces. A proper map $f\colon X \to Y$ is called an *$M_p^{\mathcal{B}}$-injection* provided for every $\sigma \in \mathrm{Cov}(X)$ there is a $\tau \in \mathrm{Cov}(X)$ and a $\xi \in \mathrm{Cov}(Y)$ such that for M_p^τ-functions F and G from a member B of $\mathcal{B}$ into X the relation $f \circ F \overset{\xi}{\sim} f \circ G$ implies the relation $F \overset{\sigma}{\sim} G$. A proper map $f\colon X \to Y$ is *M_p^l-placid* provided for every $\sigma \in \mathrm{Cov}(X)$ there is an M_p^σ-function $J\colon Y \to X$ such that $J \circ f \overset{\sigma}{\sim} \mathrm{id}_X$.

Observe that every proper map $f\colon X \to Y$ which has a left proper homotopy inverse (i. e., for which there is a proper map $g\colon Y \to X$ with the composition $g \circ f$ properly homotopic to id_X) is M_p^l-placid. The same is true if the map f has a left $\mathcal{S}h_p$-inverse. Moreover, an M_p^l-placid proper map is an $M_p^{\mathcal{S}}$-injection, where $\mathcal{S}$ denotes the class of all topological spaces.

The following result shows that $M_p^{\mathcal{B}}$-injections inversely preserve $M_p^{\mathcal{B},\mathcal{C}}$-smooth spaces.

3.8 Theorem. *If $f\colon X \to Y$ is an $M_p^{\mathcal{B}}$-injection and Y is $M_p^{\mathcal{B},\mathcal{C}}$-smooth, then X is also $M_p^{\mathcal{B},\mathcal{C}}$-smooth.*

Proof. Let a cover σ of X be given. Since f is an $M_p^{\mathcal{B}}$-injection, there is an $\alpha \in \mathrm{Cov}(X)$ and $\beta \in \mathrm{Cov}(Y)$ such that for M_p^{α}-functions F and G from a member of $\mathcal{B}$ into X the relation $f \circ F \overset{\beta}{\sim} f \circ G$ implies the relation $F \overset{\sigma}{\sim} G$. We utilize now the assumption that Y is $M_p^{\mathcal{B},\mathcal{C}}$-smooth to select a $\gamma \in \mathrm{Cov}(Y)$ with the property that for M_p^{γ}-functions K and L from a member of $\mathcal{B}$ into Y the relation $K \overset{\gamma}{\sim}_{\mathcal{C}} L$ implies the relation $K \overset{\beta}{\sim} L$. Let $\tau \in \mathrm{Cov}(X)$ be a common refinement of α and $f^{-1}(\gamma)$. Then τ is the required cover. In fact, let F and G be M_p^{τ}-functions from a member B of $\mathcal{B}$ into X and assume that $F \overset{\tau}{\sim}_{\mathcal{C}} G$. Let K and L be $f \circ F$ and $f \circ G$. Then K and L are M_p^{β}-functions from B into Y. The last relation implies $K \overset{\gamma}{\sim}_{\mathcal{C}} L$ so that $K \overset{\beta}{\sim} L$. It follows that $F \overset{\sigma}{\sim} G$. $\Diamond$

An example of M_p^l-placid maps provide inclusions $i_{A,X}$ of the M_p-retracts A of a space X. Here, we will say that a closed subset A of a space X is an *M_p-retract* of X provided for every cover σ of A there is an M_p^{σ}-function $R\colon X \to A$ such that $a \in R(a)$ for every $a \in A$. Hence, the following is a consequence of Th. 3.8.

3.9 Corollary. *An M_p-retract of an $M_p^{\mathcal{B},\mathcal{C}}$-smooth space is $M_p^{\mathcal{B},\mathcal{C}}$-smooth.*

For the preservation of $M_p^{\mathcal{B},\mathcal{C}}$-smooth spaces from the domain to the codomain we must assume that the map f is either M_p^{τ}-placid or that it is an $M_p^{\mathcal{B},\mathcal{C}}$-bijection. Let us recall the definitions of these notions from [8].

Let $\mathcal{B}$ be a class of spaces. A proper map $f\colon X \to Y$ is an *$M_p^{\mathcal{B}}$-surjection* provided for every $\sigma \in \mathrm{Cov}(X)$ and every $\tau \in \mathrm{Cov}(Y)$ there is a $\varrho \in \mathrm{Cov}(Y)$ such that for every M_p^{ϱ}-function F from a member of $\mathcal{B}$ into Y there is an M_p^{σ}-function G with $F \overset{\tau}{\sim} f \circ G$. A special case of $M_p^{\mathcal{B}}$-surjections are *M_p^{τ}-placid* maps, i. e. proper maps $f\colon X \to Y$ such that for every $\sigma \in \mathrm{Cov}(X)$ and every $\tau \in \mathrm{Cov}(Y)$ there is an M_p^{σ}-function $J\colon Y \to X$ with $f \circ J \overset{\tau}{\sim} \mathrm{id}_Y$. In fact, every M_p^{τ}-placid map is an $M_p^{\mathcal{S}}$-surjection, where $\mathcal{S}$ denotes the class of all topological spaces.

Observe that a proper map $f\colon X \to Y$ which has a right proper homotopy inverse (i. e., for which there is a proper map $g\colon Y \to X$ with $f \circ g$ properly homotopic to id_Y) is M_p^{τ}-placid. The same is true if the proper map has a right Sh_p-inverse.

At last, for classes $\mathcal{B}$ and $\mathcal{C}$ of spaces, a proper map is an $M_p^{\mathcal{B},\mathcal{C}}$-*bijection* if it is both an $M_p^{\mathcal{B}}$-injection and an $M_p^{\mathcal{C}}$-surjection. We shall use a shorter name $M_p^{\mathcal{B}}$-*bijection* for an $M_p^{\mathcal{B},\mathcal{B}}$-bijection.

3.10 Theorem. *If a map $f\colon X \to Y$ is M_p^{τ}-placid and X is $M_p^{\mathcal{B},\mathcal{C}}$-smooth, then Y is also $M_p^{\mathcal{B},\mathcal{C}}$-smooth.*

Proof. Let a cover σ of Y be given. Let $\pi \in \sigma^*$ and $\alpha = f^{-1}(\pi)$. Since X is $M_p^{\mathcal{B},\mathcal{C}}$-smooth, there is a $\beta \in \mathrm{Cov}(X)$ such that for M_p^{β}-functions K and L from a member C of $\mathcal{C}$ into X the relation $K \overset{\beta}{\sim}_C L$ implies the relation $K \overset{\alpha}{\sim} L$. Now we utilize the fact that f is M_p^{τ}-placid to select an M_p^{β}-function $H\colon Y \to X$ with $\mathrm{id}_Y \overset{\pi}{\sim} f \circ H$. Let $M\colon Y \times I \to Y$ be an M_P^{π}-homotopy that realizes this relation and let $\zeta \in D(M, \pi)$. Let $\gamma \in S(H, \beta)$. Let a $\tau \in \mathrm{Cov}(Y)$ be a common refinement of ζ and γ. Then τ is the required cover of Y. Indeed, consider M_p^{τ}-functions F and G from a member B of $\mathcal{B}$ into Y and assume that $F \overset{\tau}{\sim}_C G$. Let K and L be $H \circ F$ and $H \circ G$. Then K and L are M_p^{β}-functions from B into X. From the last relation it follows that $K \overset{\beta}{\sim}_C L$ so that $K \overset{\alpha}{\sim} L$. Composing this relation with the map f we obtain $f \circ K \overset{\pi}{\sim} f \circ L$. Our choices imply the following chain of relations $F \overset{\pi}{\sim} f \circ H \circ F = f \circ K \overset{\pi}{\sim} f \circ L = f \circ H \circ G \overset{\pi}{\sim} G$. Hence, $F \overset{\sigma}{\sim} G$. $\Diamond$

It has been shown in [8, (3.1)] that another important example of M_p^{τ}-placid maps provide properly refinable maps. We call an onto proper map $f\colon X \to Y$ between spaces *properly refinable* provided for every cover τ of Y and every cover σ of X there is an onto proper map $g\colon X \to Y$ such that f and g are τ-close and g^{-1} is an M_p^{σ}-function. We call g a proper (σ, τ)-*refinement* of the map f. The notion of a refinable map between compact metric spaces was first defined by Jo Ford and James Rogers Jr.. The above extension to arbitrary spaces is particularly suitable for our theory.

The existence of a properly refinable map from a space X onto a space Y clearly implies that X is $M_p^{\{Y\}}$-like. Hence, as a consequence of Ths. 3.7 and 3.10 we obtain the following analogue of Th. 1.8 in [11] for $M_p^{\mathcal{B},\mathcal{C}}$-smooth spaces.

3.11 Corollary. *Let $f\colon X \to Y$ be a properly refinable map. Then the space X is $M_p^{\mathcal{B},\mathcal{C}}$-smooth if and only if Y is $M_p^{\mathcal{B},\mathcal{C}}$-smooth.*

3.12 Theorem. *If a map $f\colon X \to Y$ is an $M_p^{\mathcal{C},\mathcal{B}}$-bijection and the domain X is $M_p^{\mathcal{B},\mathcal{C}}$-smooth, then the codomain Y is also $M_p^{\mathcal{B},\mathcal{C}}$-smooth.*

Proof. Let a cover σ of Y be given. Let $\varrho \in \sigma^*$ and $\alpha = f^{-1}(\varrho)$. Since X is $M_p^{\mathcal{B},\mathcal{C}}$-smooth, there is a $\beta \in \mathrm{Cov}(X)$ such that for M_p^β-functions K and L from a member of $\mathcal{B}$ into Y the relation $K \overset{\beta}{\sim}_{\mathcal{C}} L$ implies the relation $K \overset{\alpha}{\sim} L$. We now use the assumption that f is an $M_p^{\mathcal{C}}$-injection to select a $\gamma \in \beta^+$ and a $\lambda \in \varrho^+$ such that for M_p^γ-functions P and Q from a member of $\mathcal{C}$ into X the relation $f \circ P \overset{\lambda}{\sim} f \circ Q$ implies the relation $P \overset{\beta}{\sim} Q$. Let $\mu \in \lambda^*$. At last, since f is also an $M_p^{\mathcal{B}}$-surjection, there is a $\tau \in \mu^+$ such that for every M_p^τ-function F from a member of $\mathcal{B}$ into Y there is an M_p^γ-function K with $F \overset{\mu}{\sim} f \circ K$. Then τ is the required cover of Y.

In order to verify this claim, assume that B is a member of $\mathcal{B}$ and $F, G\colon B \to Y$ are M_p^τ-functions with $F \overset{\tau}{\sim}_{\mathcal{C}} G$. In other words, suppose that there is a cover ξ of B such that $F \circ H \overset{\tau}{\sim} G \circ H$ for every M_p^ξ-function H from a member of $\mathcal{C}$ into B. Choose M_p^γ-functions $K, L\colon B \to X$ and M_p^μ-homotopies $V, W\colon B \times I \to Y$ such that $V_0 = F$, $W_0 = G$, $V_1 = f \circ K$, and $W_1 = f \circ L$. Let $\theta \in \xi^+$ be from the intersection of sets $S(K, \gamma)$, $S(L, \gamma)$, $D(V, \mu)$, and $D(W, \mu)$. Let C be a member of $\mathcal{C}$ and let $H\colon C \to B$ be an M_p^θ-function. Our choices imply the following extended chain of relations $f \circ K \circ H \overset{\mu}{\sim} F \circ H \overset{\tau}{\sim} G \circ H \overset{\mu}{\sim} f \circ L \circ H$. It follows that $f \circ K \circ H \overset{\lambda}{\sim} f \circ L \circ H$. Since $K \circ H$ and $L \circ H$ are the M_p^γ-functions from C into X, we get $K \circ H \overset{\beta}{\sim} L \circ H$. Thus, we have checked that $K \overset{\beta}{\sim}_{\mathcal{C}} L$. The way in which we selected the cover β implies that $K \overset{\alpha}{\sim} L$. Therefore, $F \overset{\mu}{\sim} f \circ K \overset{\varrho}{\sim} f \circ L \overset{\mu}{\sim} G$. Hence, $F \overset{\sigma}{\sim} G$. $\Diamond$

4. $M_p^{\mathcal{B}}$-calm spaces

In the present section we shall transfer from shape theory into proper shape theory the important invariant of calmness. This concept was invented by the author [6] for compact metric spaces. We shall define $M_p^{\mathcal{B}}$-calm spaces with respect to a class $\mathcal{B}$ of spaces in order to cover all possible variations of calmness (see [6]).

Let $\mathcal{B}$ be a class of spaces. A space X is *$M_p^{\mathcal{B}}$-calm* provided there is a cover σ of X with the property that for every cover τ of X we can find a cover ϱ of X such that M_p^ϱ-functions F and G from a member C

of $\mathcal{C}$ into X which are M_p^σ-homotopic are also M_p^τ-homotopic.

We shall first consider how this definition depends on the class $\mathcal{B}$. Once again the M_p-domination offers an answer.

4.1 Theorem. *If a class of spaces $\mathcal{B}$ is M_p-dominated by another such class $\mathcal{C}$ and a space X is $M_p^{\mathcal{C}}$-calm, then X is also $M_p^{\mathcal{B}}$-calm.*

Proof. Since X is $M_p^{\mathcal{C}}$-calm, there is a cover σ of X such that for every $\nu \in \mathrm{Cov}(X)$ there is a $\varrho \in \mathrm{Cov}(X)$ so that for M_p^ϱ-functions K and L from a member C of $\mathcal{C}$ into X the relation $K \overset{\sigma}{\sim} L$ implies the relation $K \overset{\nu}{\sim} L$. Then σ is the required cover. Indeed, let τ be an arbitrary cover of X. Let $\nu \in \tau^*$. Choose a cover ϱ as above. Let a $B \in \mathcal{B}$ and M_p^ϱ-functions F and G from B into X be given and assume that $F \overset{\sigma}{\sim} G$. Let H be an M_p^σ-homotopy joining F and G. Let $\beta \in D(H, \sigma)$. We can assume that β is so fine that both F and G are $M_p^{\beta,\varrho}$-functions. Since the class $\mathcal{B}$ is M_p-dominated by the class $\mathcal{C}$, there is a $C \in \mathcal{C}$ and an M_p^β-function $D\colon B \to C$ such that for every $\gamma \in \mathrm{Cov}(C)$ there is an M_p^γ-function $U\colon C \to B$ with $\mathrm{id}_B \overset{\beta}{\sim} D \circ U$. Let K and L be the compositions $F \circ D$ and $G \circ D$, respectively. Then K and L are M_p^ϱ-functions from C into X with $K \overset{\sigma}{\sim} L$. Our choices imply $K \overset{\nu}{\sim} L$. Let E be an M_p^ν-homotopy joining K and L. Let $\gamma \in \in D(E, \nu)$. Choose a U as above. Then we have the following chain of relations $F \overset{\nu}{\sim} F \circ D \circ U = K \circ U \overset{\nu}{\sim} L \circ U = G \circ D \circ U \overset{\nu}{\sim} G$. Hence, $F \overset{\tau}{\sim} G$. $\Diamond$

Our goal now is to show that $M_p^{\mathcal{B}}$-calmness is indeed a proper shape invariant. We can prove a far better result, namely that it is preserved under $\mathcal{S}h_p$-domination.

4.2 Theorem. *A space is $M_p^{\mathcal{C}}$-calm if and only if it is $\mathcal{S}h_p$-dominated by an $M_p^{\mathcal{C}}$-calm space.*

Proof. Let X be a space, let Y be an $M_p^{\mathcal{C}}$-calm space, and assume that $\varphi\colon X \to Y$ and $\psi\colon Y \to X$ are M_p-nets such that the composition $\psi \circ \varphi$ is M_p-homotopic to the identity M_p-net ι^X on X.

Since Y is $M_p^{\mathcal{C}}$-calm, there is a cover α of Y with the property that for every cover β of Y there is a $\gamma \in \mathrm{Cov}(Y)$ such that M_p^γ-functions K and L from a member C of $\mathcal{C}$ into Y which are M_p^α-homotopic are already M_p^β-homotopic. Let $\mu \in \alpha^*$.

Since φ is an M_p-net, there is an index $c \in \mathrm{Inc}(Y)$ such that $F_d \overset{\mu}{\sim} F_e$ for all $d, e > c$. Choose a $d > c$ and a cover σ of X so that

F_d is an $M_p^{\sigma,\mu}$-function. Then σ is the required cover. Indeed, let a cover τ of X be given. Let $\nu \in \tau^*$. By assumption, there is an index $a \in \mathrm{Inc}(X)$ such that $a > \{\nu\}$ and $G_x \circ F_z \overset{\nu}{\sim} \mathrm{id}_X$, where $x = \psi(a)$, $\delta = \psi^*(a)$, $y = \{\delta\}$, and $z = \varphi(y)$. Notice that G_x is an $M_p^{\delta,\nu}$-function. Let $M\colon X \times I \to X$ be an M_p^{ν}-homotopy that realizes the last relation. Let $\varepsilon \in D(M, \nu)$ and $\beta \in \delta^*$. Select an index $w > z$ such that $F_b \overset{\beta}{\sim} F_w$ for every $b > w$. Observe that the condition $w > z$ implies that F_w and F_z are joined by an M_p^{δ}-homotopy $N\colon X \times I \to Y$. Let $\xi \in D(N, \delta)$. Pick a cover γ of Y with respect to α and β as above. Finally, we select an index $b > w$ and a $\pi \in \xi^+$ such that F_b is an $M_p^{\pi,\gamma}$-function.

Let $P\colon X \times I \to Y$ be an M_p^{β}-homotopy joining F_b and F_w and let $R\colon X \times I \to Y$ be an M_p^{μ}-homotopy joining F_b and F_d. Let ϱ be from the intersection of sets $D(P, \beta)$ and $D(R, \mu)$. Consider M_p^{ϱ}-functions F and G from a member C of $\mathcal{C}$ into X and assume that $F \overset{\sigma}{\sim} G$. Let K and L denote compositions $F_b \circ F$ and $F_b \circ G$, respectively. These are M_p^{γ}-functions and $K = F_b \circ F \overset{\mu}{\sim} F_d \circ F \overset{\mu}{\sim} F_d \circ G = L$, i.e., $K \overset{\alpha}{\sim} L$. By assumption, it follows that $K \overset{\beta}{\sim} L$. This relation implies the following chain $F_w \circ F \overset{\beta}{\sim} F_b \circ F = K \overset{\beta}{\sim} L = F_b \circ G \overset{\beta}{\sim} F_w \circ G$. Hence, $F_w \circ F \overset{\delta}{\sim} F_w \circ G$ so that we get $G_x \circ F_w \circ F \overset{\nu}{\sim} G_x \circ F_w \circ G$. But, we also have relations $G_x \circ F_z \circ F \overset{\nu}{\sim} G_x \circ F_w \circ F$, $F \overset{\nu}{\sim} G_x \circ F_z \circ F$, $G \overset{\nu}{\sim} G_x \circ F_z \circ G$, and $G_x \circ F_w \circ G \overset{\nu}{\sim} G_x \circ F_z \circ G$. Together these relations imply the desired conclusion $F \overset{\tau}{\sim} G$. $\Diamond$

The next result is typical for shape theory. It shows the role of $M_p^{\mathcal{B},\mathcal{C}}$-smooth spaces and is similar to the author's theorem that a $(\mathcal{B}, \mathcal{C})$-smooth and $\mathcal{C}$-calm compactum is $\mathcal{B}$-calm [6].

4.3 Theorem. *Let $\mathcal{B}$ and $\mathcal{C}$ be classes of topological spaces. If a space X is both $M_p^{\mathcal{B},\mathcal{C}}$-smooth and $M_p^{\mathcal{C}}$-calm, then it is also $M_p^{\mathcal{B}}$-calm.*

Proof. Since X is $M_p^{\mathcal{C}}$-calm, there is a cover σ of X such that for every $\tau \in \mathrm{Cov}(X)$ there is a $\varrho \in \mathrm{Cov}(X)$ with the property that for M_p^{ϱ}-functions K and L from a member of $\mathcal{C}$ into X the relation $K \overset{\sigma}{\sim} L$ implies the relation $K \overset{\tau}{\sim} L$.

Let $\beta \in \mathrm{Cov}(X)$. We utilize the assumption that X is $M_p^{\mathcal{B},\mathcal{C}}$-smooth to select a $\tau \in \mathrm{Cov}(X)$ such that for M_p^{τ}-functions F and G from a member C into X the relation $F \overset{\tau}{\sim}_C G$ implies the relation $F \overset{\beta}{\sim} G$. Finally, choose a cover $\varrho \in \tau^+$ as above.

Consider M_p^{ϱ}-functions F and G from a member B of $\mathcal{B}$ into X and assume that $F \overset{\sigma}{\sim} G$. Let $K\colon B \times I \to X$ be an M_p^{σ}-homotopy joining F and G. Let $\gamma \in D(H, \sigma)$. For every M_p^{γ}-function H from a member C of $\mathcal{C}$ into B the compositions K and L of H and F and H and G, respectively, satisfy $K \overset{\sigma}{\sim} L$. It follows that $K \overset{\tau}{\sim} L$. Hence, $F \overset{\tau}{\sim}_{\mathcal{C}} G$, and we get the desired conclusion $F \overset{\beta}{\sim} G$. $\Diamond$

In the rest of this section we shall consider the question of identifying those proper maps which will preserve or inversely preserve $M_p^{\mathcal{B}}$-calm spaces. The answer provide proper maps studied in [8] whose definitions have been recalled in §3. The following result resembles Th. 3.8.

4.4 Theorem. *If $f\colon X \to Y$ is an $M_p^{\mathcal{C}}$-injection and Y is $M_p^{\mathcal{C}}$-calm, then X is also $M_p^{\mathcal{C}}$-calm.*

Proof. Since Y is $M_p^{\mathcal{C}}$-calm, there is a cover α of Y such that for every $\beta \in \operatorname{Cov}(Y)$ there is a $\gamma \in \operatorname{Cov}(Y)$ with the property that for every M_p^{γ}-functions K and L from a member of $\mathcal{C}$ into Y the relation $K \overset{\alpha}{\sim} L$ implies the relation $K \overset{\beta}{\sim} L$. Let $\sigma = f^{-1}(\alpha)$. Then σ is the required cover of X.

In order to check this, assume that τ is a cover of X. Since f is an $M_p^{\mathcal{C}}$-injection, there is a $\pi \in \operatorname{Cov}(X)$ and a $\beta \in \operatorname{Cov}(Y)$ such that for M_p^{π}-functions F and G from a member of $\mathcal{C}$ into X the relation $f \circ F \overset{\beta}{\sim} f \circ G$ implies the relation $F \overset{\tau}{\sim} G$. Pick a γ as above. Let $\varrho \in \operatorname{Cov}(X)$ be a common refinement of π and $f^{-1}(\gamma)$.

Let $C \in \mathcal{C}$ and assume that M_p^{ϱ}-functions $F, G\colon C \to X$ satisfy $F \overset{\sigma}{\sim} G$. Let K and L be the compositions $f \circ F$ and $f \circ G$, respectively. Then K and L are M_p^{γ}-functions from C into Y and we have $K \overset{\alpha}{\sim} L$. It follows that $f \circ F \overset{\beta}{\sim} f \circ G$ and therefore that $F \overset{\tau}{\sim} G$. $\Diamond$

The following result gives a partial converse to Theorem 4.4.

4.5 Theorem. *If a proper map $f\colon X \to Y$ is properly refinable and the codomain Y is $M_p^{\mathcal{C}}$-calm, then f is an $M_p^{\mathcal{C}}$-injection.*

Proof. Let a cover α of X be given. Since Y is $M_p^{\mathcal{C}}$-calm, there is a $\sigma \in \operatorname{Cov}(Y)$ with the property that for every $\tau \in \operatorname{Cov}(Y)$ we can find a $\varrho \in \tau^{+}$ such that for M_p^{ϱ}-functions K and L from a member of $\mathcal{C}$ into Y the relation $K \overset{\sigma}{\sim} L$ implies the relation $K \overset{\tau}{\sim} L$. Let $\xi \in \sigma^{*2}$ and $\beta \in \alpha^{*}$. Since f is properly refinable, there is a proper map g from X onto Y such that f and g are ξ-close and g^{-1} is an M_p^{β}-function. Let

$\tau \in S(g^{-1}, \beta)$. Next, we select a ϱ as above and let η be a common refinement of β and $g^{-1}(\varrho)$.

Consider M_p^η-functions F and G from a member C of $\mathcal{C}$ into X and assume that $f \circ F \overset{\xi}{\sim} f \circ G$. Let K and L be the compositions $g \circ F$ and $g \circ G$, respectively. Then K and L are M_p^ϱ-functions from C into Y. From the previous selections we get $K = g \circ F \overset{st(\xi)}{\sim} f \circ F \overset{\xi}{\sim} f \circ G \overset{st(\xi)}{\sim} g \circ G = L$ and thus $K \overset{\sigma}{\sim} L$. Our choices now imply that $K \overset{\tau}{\sim} L$. It follows that $F \overset{\beta}{\sim} g^{-1} \circ g \circ F = g^{-1} \circ K \overset{\beta}{\sim} g^{-1} \circ L = g^{-1} \circ g \circ G \overset{\beta}{\sim} G$. Hence, $F \overset{\alpha}{\sim} G$. $\Diamond$

4.6 Corollary. *The image Y of an $M_p^\mathcal{C}$-calm space X under a properly refinable proper map $f\colon X \to Y$ is $M_p^\mathcal{C}$-calm if and only if the map f is an $M_p^\mathcal{C}$-injection.*

In an attempt to prove an analogue of Theorem 3.10 for $M_p^\mathcal{B}$-calm spaces instead of M_p^r-placid maps we must use the following stronger form of this notion. A proper map $f\colon X \to Y$ between spaces is M_q^r-*placid* provided for every cover σ of X there is a cover α of Y such that for every cover ϱ of X and every cover β of Y there is an M-function $J\colon Y \to X$ which is both an M_p^ϱ-function and an $M_p^{\alpha,\sigma}$-function and $f \circ J$ and id_Y are M_p^β-homotopic.

4.7 Theorem. *If a proper map $f\colon X \to Y$ is M_q^r-placid and the domain X is $M_p^\mathcal{C}$-calm, then the codomain Y is also $M_p^\mathcal{C}$-calm.*

Proof. Since X is $M_p^\mathcal{C}$-calm, there is an $\sigma \in \mathrm{Cov}(X)$ such that for every $\tau \in \mathrm{Cov}(X)$ we can find a $\varrho \in \mathrm{Cov}(X)$ with the property that for M_p^ϱ-functions K and L from a member of $\mathcal{C}$ into X the relation $K \overset{\sigma}{\sim} L$ implies the relation $K \overset{\tau}{\sim} L$. Since f is M_q^r-placid there is a cover α of Y such that for every cover ϱ of X and every cover δ of Y there is an M-function $J\colon Y \to X$ which is both an M_p^ϱ-function and an $M_p^{\alpha,\sigma}$-function and there is an M_p^δ-homotopy H joining $f \circ J$ and id_Y. Then α is the required cover of Y.

To check this, let a cover β of Y be given. Let $\delta \in \beta^*$ and let $\tau = f^{-1}(\delta)$. Pick a ϱ and a J as above. Let $\gamma \in S(J, \varrho)$. Then γ has the required property. Indeed, let D and E be M_p^γ-functions from a member C of $\mathcal{C}$ into Y and assume that $D \overset{\alpha}{\sim} E$. Let K and L be the compositions $J \circ D$ and $J \circ E$, respectively. Then K and L are M_p^γ-functions from C into Y and since J is an $M_p^{\alpha,\sigma}$-function we obtain

that $K \overset{\alpha}{\sim} L$. It follows from our selections that $K \overset{\tau}{\sim} L$ so that after composing with f we get $f \circ K \overset{\mu}{\sim} f \circ L$. Thus, we have the following chain of relations $D \overset{\delta}{\sim} f \circ J \circ D = f \circ K \overset{\delta}{\sim} f \circ L = f \circ J \circ E \overset{\delta}{\sim} E$. Hence, $D \overset{\beta}{\sim} E$. $\Diamond$

4.8 Theorem. *Let $\mathcal{C}$ be a class of spaces. Let X be an $M_p^{\mathcal{C}}$-calm space. If a map $f\colon X \to Y$ is an $M_p^{\mathcal{C}}$-bijection, then the space Y is also $M_p^{\mathcal{C}}$-calm.*

Proof. Since X is $M_p^{\mathcal{C}}$-calm, there is an $\alpha \in \mathrm{Cov}(X)$ such that for every $\beta \in \mathrm{Cov}(X)$ we can find a $\gamma \in \mathrm{Cov}(X)$ with the property that for M_p^{γ}-functions K and L from a member of $\mathcal{C}$ into X the relation $K \overset{\alpha}{\sim} L$ implies the relation $K \overset{\beta}{\sim} L$.

Since f is an $M_p^{\mathcal{C}}$-injection, there is a $\xi \in \mathrm{Cov}(Y)$ and an $\eta \in \in \mathrm{Cov}(X)$ such that for M_p^{η}-functions K and L from a member of $\mathcal{C}$ into X the relation $f \circ K \overset{\xi}{\sim} f \circ L$ implies the relation $K \overset{\alpha}{\sim} L$. Let $\sigma \in \xi^*$. Then σ is the required cover of Y.

In order to check this, let a $\tau \in \mathrm{Cov}(Y)$ be given. Let a $\mu \in \tau^*$ refines σ. Put $\beta = f^{-1}(\mu)$. Choose a cover γ as above. Since f is an $M_p^{\mathcal{C}}$-surjection, there is a $\varrho \in \eta^+$ such that for every M_p^{ϱ}-function F from a member C of $\mathcal{C}$ into Y there is an M_p^{γ}-function $K\colon C \to X$ with $F \overset{\mu}{\sim} f \circ K$.

Consider M_p^{ϱ}-functions F and G from a member C of $\mathcal{C}$ into Y and assume that $F \overset{\sigma}{\sim} G$. Choose M_p^{γ}-functions K and L from C into X such that $F \overset{\mu}{\sim} f \circ K$ and $G \overset{\mu}{\sim} f \circ L$. From the previous two relations we obtain $f \circ K \overset{\xi}{\sim} f \circ L$. It follows that $K \overset{\alpha}{\sim} L$ and therefore $K \overset{\beta}{\sim} L$ and $f \circ K \overset{\mu}{\sim} f \circ L$. Combining the last two relations, this time we shall get the conclusion $F \overset{\tau}{\sim} G$. $\Diamond$

5. $N_p^{\mathcal{B},\mathcal{C}}$-smooth and $P_p^{\mathcal{B},\mathcal{C}}$-smooth classes

The notion of an $M_p^{\mathcal{B},\mathcal{C}}$-smooth class of spaces allow us to obtain two new properties that are preserved under M_p-domination. They could be considered as dual to the notion of an $M_p^{\mathcal{B},\mathcal{C}}$-smooth space. While in the previous three sections we investigated a space X by looking at small proper multi-valued functions from members of a given class of spaces $\mathcal{B}$ into X, we now change our point of view by concen-

trating on small proper multi-valued functions from X into members of $\mathcal{B}$.

Let $\mathcal{B}$ and $\mathcal{C}$ be classes of spaces. A class of spaces $\mathcal{X}$ is (1) $N_p^{\mathcal{B},\mathcal{C}}$-*smooth* and (2) $P_p^{\mathcal{B},\mathcal{C}}$-*smooth* provided the class $\mathcal{B}$ is (1) $M_p^{\mathcal{X},\mathcal{C}}$-smooth and (2) $M_p^{\mathcal{C},\mathcal{X}}$-smooth, respectively. In other words, provided that

(1) for every $B \in \mathcal{B}$ and every $\sigma \in \mathrm{Cov}(B)$ there is a $\tau \in \mathrm{Cov}(B)$ with the property that for M_p^τ-functions F and G from a member of $\mathcal{X}$ into B the relation $F \overset{\tau}{\sim}_{\mathcal{C}} G$ implies the relation $F \overset{\sigma}{\sim} G$;

(2) for every $B \in \mathcal{B}$ and every $\sigma \in \mathrm{Cov}(B)$ there is a $\tau \in \mathrm{Cov}(B)$ with the property that for M_p^τ-functions F and G from a member of $\mathcal{C}$ into B the relation $F \overset{\tau}{\sim}_{\mathcal{X}} G$ implies the relation $F \overset{\sigma}{\sim} G$.

We shall say that a space X has one of the above properties provided the class $\{X\}$ consisting just of a space X has this property.

The three versions of proper smoothness share many properties. We shall now state and prove the N and the P versions of most results from §3.

5.1 Theorem. *A class $\mathcal{X}$ of topological spaces is $N_p^{\mathcal{B},\mathcal{C}}$-smooth if and only if it is M_p-dominated by an $N_p^{\mathcal{B},\mathcal{C}}$-smooth class of spaces.*

Proof. Suppose that $\mathcal{X}$ is M_p-dominated by an $N_p^{\mathcal{B},\mathcal{C}}$-smooth class $\mathcal{Y}$. Then the class $\mathcal{B}$ is $M_p^{\mathcal{Y},\mathcal{C}}$-smooth so that $\mathcal{B}$ is $M_p^{\mathcal{X},\mathcal{C}}$-smooth by Theorem 3.5. Hence, $\mathcal{X}$ is $N_p^{\mathcal{B},\mathcal{C}}$-smooth. ◊

5.2 Theorem. *A class $\mathcal{X}$ of spaces is $P_p^{\mathcal{B},\mathcal{C}}$-smooth if and only if it M_p-dominates a $P_p^{\mathcal{B},\mathcal{C}}$-smooth class of spaces.*

Proof. Similar to the proof of Th. 5.1. ◊

5.3 Theorem. *Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be classes of spaces. If a class $\mathcal{X}$ of spaces is both $N_p^{\mathcal{B},\mathcal{A}}$-smooth and $M_p^{\mathcal{A},\mathcal{C}}$-tame, then $\mathcal{X}$ is also $N_p^{\mathcal{B},\mathcal{C}}$-smooth.*

Proof. Let a member B of $\mathcal{B}$ and a cover σ of B be given. Since X is $N_p^{\mathcal{B},\mathcal{A}}$-smooth, there is a $\pi \in \mathrm{Cov}(B)$ such that for M_p^π-functions F and G from a member of $\mathcal{X}$ into B the relation $F \overset{\pi}{\sim}_{\mathcal{A}} G$ implies the relation $F \overset{\sigma}{\sim} G$. Let $\tau \in \pi^*$. Then τ is the cover we have been looking for.

Indeed, let $X \in \mathcal{X}$ and let $F, G\colon X \to B$ be M_p^τ-functions such that $F \overset{\tau}{\sim}_{\mathcal{C}} G$. By definition, this means that there is a cover $\alpha \in \in \mathrm{Cov}(X)$ such that α belongs to both $S(F, \tau)$ and $S(G, \tau)$ and the compositions $F \circ K$ and $G \circ K$ are M_p^τ-homotopic for every M_p^α-function $K\colon C \to X$ from a member of $\mathcal{C}$ into X. Now we utilize the fact

that $\mathcal{X}$ is also $M_p^{\mathcal{A},\mathcal{C}}$-tame to select a cover β of X such that for every $A \in \mathcal{A}$ and every M_p^{β}-function $H\colon A \to X$ there is a $C \in \mathcal{C}$ and an M_p^{α}-function $K\colon C \to X$ so that for every $\gamma \in \mathrm{Cov}(C)$ there is an M_p^{γ}-function $D\colon A \to C$ with $H \overset{\alpha}{\sim} K \circ D$.

Consider an $A \in \mathcal{A}$ and an M_p^{β}-function $H\colon A \to X$. Choose a C and then a K as above. By assumption, the compositions $F \circ K$ and $G \circ K$ are joined by an M_p^{τ}-homotopy W. Let $\delta \in S(W, \tau)$ and $\gamma \in D(C, \delta)$. Pick an M_p^{γ}-function D as above. Then we obtain the following chain of relations $F \circ H \overset{\tau}{\sim} F \circ K \circ D \overset{\mu}{\sim} G \circ K \circ D \overset{\tau}{\sim} G \circ H$. It follows that $F \overset{\pi}{\sim}_{\mathcal{A}} G$. Hence, $F \overset{\sigma}{\sim} G$. $\Diamond$

5.4 Theorem. *Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be classes of spaces. If a class $\mathcal{X}$ of spaces is $P_p^{\mathcal{B},\mathcal{A}}$-smooth and the class $\mathcal{C}$ is $M_p^{\mathcal{C},\mathcal{A}}$-smooth, then $\mathcal{X}$ is also $P_p^{\mathcal{B},\mathcal{C}}$-smooth.*

Proof. Similar to the proof of Th. 5.3. $\Diamond$

5.5 Theorem. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ be classes of spaces such that $\mathcal{B}$ and $\mathcal{D}$ are M_p-dominated by $\mathcal{A}$ and $\mathcal{C}$, respectively. If a class $\mathcal{X}$ of spaces is $N_p^{\mathcal{A},\mathcal{D}}$-smooth, then it is also $N_p^{\mathcal{B},\mathcal{C}}$-smooth.*

Proof. The assumption that $\mathcal{X}$ is $N_p^{\mathcal{A},\mathcal{D}}$-smooth means that $\mathcal{A}$ is $M_p^{\mathcal{X},\mathcal{D}}$-smooth. Since $\mathcal{B}$ is M_p-dominated by $\mathcal{A}$, it follows from (3.1) that $\mathcal{B}$ is $M_p^{\mathcal{X},\mathcal{D}}$-smooth. But, since $\mathcal{D}$ is M_p-dominated by $\mathcal{C}$, we get that $\mathcal{B}$ is $M_p^{\mathcal{X},\mathcal{C}}$-smooth and therefore that $\mathcal{X}$ is $N_p^{\mathcal{B},\mathcal{C}}$-smooth. $\Diamond$

5.6 Theorem. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ be classes of spaces such that $\mathcal{B}$ and $\mathcal{C}$ are M_p-dominated by $\mathcal{A}$ and $\mathcal{D}$, respectively. If a class $\mathcal{X}$ of spaces is $P_p^{\mathcal{A},\mathcal{D}}$-smooth, then it is also $P_p^{\mathcal{B},\mathcal{C}}$-smooth.*

Proof. See the proof of Th. 5.5. $\Diamond$

There seems to be no analogue of (3.7) for $N_p^{\mathcal{B},\mathcal{C}}$-smooth and $P_p^{\mathcal{B},\mathcal{C}}$-smooth spaces. In order to state versions of (3.8) we need the following dual form of the notion of an $M_p^{\mathcal{B}}$-injection.

Let $\mathcal{B}$ be a class of spaces. A class $\mathcal{F}$ of proper maps is $N_p^{\mathcal{B}}$-*injective* provided for every $B \in \mathcal{B}$ and every $\sigma \in \mathrm{Cov}(B)$ there is a $\tau \in \mathrm{Cov}(B)$ such that for every $f\colon X \to Y$ from $\mathcal{F}$ and for every M_p^{τ}-functions F and G from Y into B the relation $F \circ f \overset{\tau}{\sim} G \circ f$ implies the relation $F \overset{\sigma}{\sim} G$. A proper map $f\colon X \to Y$ is an $N_p^{\mathcal{B}}$-*injection* provided the class $\{f\}$ is $N_p^{\mathcal{B}}$-injective.

For a class $\mathcal{F}$ of maps let $\mathcal{F}'$ and $\mathcal{F}''$ denote collections of all domains and of all codomains of members of $\mathcal{F}$, respectively.

5.7 Theorem. *If $\mathcal{F}$ is an $N_p^{\mathcal{B}}$-injective class of proper maps and the*

class $\mathcal{F}'$ is $N_p^{\mathcal{B},\mathcal{C}}$-smooth, then the class $\mathcal{F}''$ is also $N_p^{\mathcal{B},\mathcal{C}}$-smooth.

Proof. Let a member B of $\mathcal{B}$ and a cover σ of B be given. Since the class $\mathcal{F}$ is $M_p^{\mathcal{B}}$-injective, there is a $\mu \in \mathrm{Cov}(B)$ such that for every proper map $f\colon X \to Y$ from $\mathcal{F}$ and all M_p^{μ}-functions $F, G\colon Y \to B$ the relation $F \circ f \overset{\mu}{\sim} G \circ f$ implies the relation $F \overset{\sigma}{\sim} G$. We utilize now the assumption that the class $\mathcal{F}'$ is $M_p^{\mathcal{B},\mathcal{C}}$-smooth to select the required cover τ of B such that for every member X of $\mathcal{F}'$ and all M_p^{τ}-functions $P, Q\colon X \to B$ the relation $P \overset{\tau}{\sim}_{\mathcal{C}} Q$ implies the relation $P \overset{\mu}{\sim} Q$.

Let Y be a member of the class $\mathcal{F}''$ and let $F, G\colon Y \to B$ be M_p^{τ}-functions and assume that $F \overset{\tau}{\sim}_{\mathcal{C}} G$. Let $f\colon X \to Y$ be from the class $\mathcal{F}$. Let P and Q be the compositions $F \circ f$ and $G \circ f$. It is easy to check that $P \overset{\tau}{\sim}_{\mathcal{C}} Q$. It follows that $P \overset{\mu}{\sim} Q$ and therefore that $F \overset{\sigma}{\sim} G$. $\Diamond$

In a similar way one can prove the following dual result for the $P_p^{\mathcal{B},\mathcal{C}}$-smooth classes of spaces.

5.8 Theorem. *If $\mathcal{F}$ is an $N_p^{\mathcal{B}}$-injective class of proper maps and the class $\mathcal{F}''$ is $P_p^{\mathcal{B},\mathcal{C}}$-smooth, then the class $\mathcal{F}'$ is also $P_p^{\mathcal{B},\mathcal{C}}$-smooth.*

5.9 Theorem. *If $\mathcal{F}$ is a class of M_p^l-placid proper maps and the class $\mathcal{F}''$ is $N_p^{\mathcal{B},\mathcal{C}}$-smooth, then the class $\mathcal{F}'$ is also $N_p^{\mathcal{B},\mathcal{C}}$-smooth*

Proof. Let a member B of $\mathcal{B}$ and a cover σ of B be given. Let $\mu \in \in \sigma^*$. Since the class $\mathcal{F}''$ is $N_p^{\mathcal{B},\mathcal{C}}$-smooth, there is a $\tau \in \mu^+$ such that for every member Y of $\mathcal{F}''$ and all M_p^{τ}-functions $P, Q\colon Y \to B$ the relation $P \overset{\tau}{\sim}_{\mathcal{C}} Q$ implies the relation $P \overset{\mu}{\sim} Q$. Then τ is the required cover. Indeed, let X be from the class $\mathcal{F}'$ and let $F, G\colon X \to B$ be M_p^{τ}-functions and assume that $F \overset{\tau}{\sim}_{\mathcal{C}} G$. Let $f\colon X \to Y$ be a proper map from the class $\mathcal{F}$. Let $\theta \in \mathrm{Cov}(X)$ be from the intersection of sets $S(F, \tau)$ and $S(G, \tau)$ and have the property that $F \circ H \overset{\tau}{\sim} G \circ H$ for every M_p^{θ}-function H from a member of the class $\mathcal{C}$ into X. Since f is M_p^l-placid, there is an M_p^{θ}-function $J\colon Y \to X$ with $J \circ f \overset{\theta}{\sim} \mathrm{id}_X$. Let P and Q be the compositions $F \circ J$ and $G \circ J$. Then P and Q are M_p^{τ}-functions and $P \overset{\tau}{\sim}_{\mathcal{C}} Q$. It follows that $P \overset{\mu}{\sim} Q$. Our choices imply $F \overset{\tau}{\sim} F \circ J \circ f = P \circ f \overset{\mu}{\sim} Q \circ f = G \circ J \circ f \overset{\tau}{\sim} G$. Hence, $F \overset{\sigma}{\sim} G$. $\Diamond$

5.10 Corollary. *An M_p-retract of an $N_p^{\mathcal{B},\mathcal{C}}$-smooth space is itself $N_p^{\mathcal{B},\mathcal{C}}$-smooth.*

5.11 Theorem. *If $\mathcal{F}$ is a class of M_p^{τ}-placid proper maps and the class $\mathcal{F}''$ is $N_p^{\mathcal{B},\mathcal{C}}$-smooth, then the class $\mathcal{F}'$ is also properly $N_p^{\mathcal{B},\mathcal{C}}$-smooth.*

Proof. The proof is similar to the proof of Th. 5.9. $\Diamond$

In the next result that corresponds to Th. 3.12 we shall use a notion of $N_p^{\mathcal{B}}$-surjective class of proper maps from [8] whose definition we now recall. Let $\mathcal{B}$ be a class of spaces. A class $\mathcal{F}$ of proper maps is *$N_p^{\mathcal{B}}$-surjective* provided for every $B \in \mathcal{B}$ and every $\sigma \in \mathrm{Cov}(B)$ there is a $\tau \in \mathrm{Cov}(B)$ such that for every $f\colon X \to Y$ from $\mathcal{F}$ and every M_p^{τ}-function $F\colon X \to B$ there is an M_p^{σ}-function $G\colon Y \to B$ with $F \overset{\sigma}{\sim} G \circ f$. A proper map $f\colon X \to Y$ is an *$N_p^{\mathcal{B}}$-surjection* provided the class $\{f\}$ is $N_p^{\mathcal{B}}$-surjective. Also, a class of proper maps which is both $N_p^{\mathcal{B}}$-injective and $N_p^{\mathcal{C}}$-surjective is called *$N_p^{\mathcal{B},\mathcal{C}}$-bijective*. We shall use *$N_p^{\mathcal{B}}$-bijective* for an $N_p^{\mathcal{B},\mathcal{B}}$-bijective class of proper maps. A proper map f is an *$N_p^{\mathcal{B},\mathcal{C}}$-bijection* provided the class $\{f\}$ made up of f alone is $N_p^{\mathcal{B},\mathcal{C}}$-bijective. An *$N_p^{\mathcal{B}}$-bijection* is defined analogously.

5.12 Theorem. *If $\mathcal{F}$ is an $N_p^{\mathcal{B}}$-surjective class of $M_p^{\mathcal{C}}$-surjections and the class $\mathcal{F}''$ is $N_p^{\mathcal{B},\mathcal{C}}$-smooth, then the class $\mathcal{F}'$ will be also $N_p^{\mathcal{B},\mathcal{C}}$-smooth.*

Proof. Let a member B of $\mathcal{B}$ and a cover σ be given. Let $\lambda \in \sigma^*$. Since the class $\mathcal{F}''$ is $N_p^{\mathcal{B},\mathcal{C}}$-smooth, there is a $\nu \in \mathrm{Cov}(B)$ such that for every member Y of $\mathcal{F}''$ and all M_p^{ν}-functions $K, L\colon Y \to B$ the relation $K \overset{\nu}{\sim}_{\mathcal{C}} L$ implies the relation $K \overset{\lambda}{\sim} L$. Let $\mu \in \nu$. We utilize now the assumption that the class $\mathcal{F}$ is $M_p^{\mathcal{B}}$-surjective to select the required cover $\tau \in \mu^+$ of B such that for every map $f\colon X \to Y$ from the class $\mathcal{F}$ and every M_p^{τ}-function $F\colon X \to B$ there is an M_p^{μ}-function $K\colon Y \to B$ with $F \overset{\mu}{\sim} K \circ f$.

Consider a member X of $\mathcal{F}'$ and M_p^{τ}-functions $F, G\colon X \to B$ and assume that $F \overset{\tau}{\sim}_{\mathcal{C}} G$. In other words, assume that there is a cover $\theta \in \in \mathrm{Cov}(X)$ such that $F \circ H \overset{\tau}{\sim} G \circ H$ for every M_p^{θ}-function H from a member of the class $\mathcal{C}$ into X. Let $f\colon X \to Y$ be from the class $\mathcal{F}$. Pick M_p^{μ}-functions $K, L\colon Y \to B$ such that $F \overset{\mu}{\sim} K \circ f$ and $G \overset{\mu}{\sim} L \circ f$. Let V and W be M_p^{μ}-homotopies which realize the last two relations. Let $\alpha \in \theta^+$ be from the intersection of sets $D(V, \mu)$ and $D(W, \mu)$ and let $\xi \in \mathrm{Cov}(Y)$ be from the intersection of sets $S(K, \mu)$ and $S(L, \mu)$. Since f is an $M_p^{\mathcal{C}}$-surjection, there is a $\zeta \in \mathrm{Cov}(Y)$ with the property that for every $C \in \mathcal{C}$ and every M_p^{ζ}-function $M\colon C \to Y$ there is an M_p^{α}-function $H\colon C \to X$ with $M \overset{\xi}{\sim} f \circ H$.

Let C be a member of the class $\mathcal{C}$ and let $M\colon C \to Y$ be an M_p^{ζ}-function. Choose an H as above. Then we obtain the following chain

of relations: $K \circ M \overset{\mu}{\sim} K \circ f \circ H \overset{\mu}{\sim} F \circ H \overset{\tau}{\sim} G \circ H \overset{\mu}{\sim} L \circ f \circ H \overset{\mu}{\sim} L \circ M$. It follows that $K \circ M \overset{\nu}{\sim} L \circ M$. In other words, we checked that $K \overset{\nu}{\sim}_{\mathcal{C}} L$. Now, we conclude that $K \overset{\lambda}{\sim} L$. This time we have $F \overset{\mu}{\sim} K \circ f \overset{\lambda}{\sim} L \circ f \overset{\mu}{\sim} G$. Hence, $F \overset{\sigma}{\sim} L$. $\Diamond$

The situation with $P_p^{\mathcal{B},\mathcal{C}}$-smooth classes of spaces is much simpler as the following theorem shows. The proof of it is left to the reader.

5.13 Theorem. *If $\mathcal{F}$ is an $N_p^{\mathcal{B}}$-surjective class of proper maps and the class $\mathcal{F}'$ is $P_p^{\mathcal{B},\mathcal{C}}$-smooth, then the class $\mathcal{F}''$ is also $P_p^{\mathcal{B},\mathcal{C}}$-smooth.*

6. $N_p^{\mathcal{B}}$-calm classes

In this section we shall do for $M_p^{\mathcal{C}}$-calm spaces what we have done in §5 for $M_p^{\mathcal{B},\mathcal{C}}$-smooth spaces. In other words, we shall introduce a dual notion called $N_p^{\mathcal{B}}$-calmness. It applies to classes of spaces and it satisfies five theorems which are analogues of results in §4.

Let $\mathcal{B}$ and $\mathcal{X}$ be classes of spaces. The class $\mathcal{X}$ is *$N_p^{\mathcal{B}}$-calm* provided the class $\mathcal{B}$ is $M_p^{\mathcal{X}}$-calm, i. e., provided for every $B \in \mathcal{B}$ there is a cover σ of B with the property that for every cover τ of B we can find a cover ϱ of B such that for every member X of $\mathcal{X}$ and M_p^{ϱ}-functions $F, G\colon X \to B$ the relation $F \overset{\sigma}{\sim} G$ implies the relation $F \overset{\tau}{\sim} G$. A space X is *$N_p^{\mathcal{B}}$-calm* provided the class $\{X\}$ consisting of X alone is $N_p^{\mathcal{B}}$-calm.

The following two theorems are easy consequences of Ths. 4.2 and 4.1, respectively.

6.1 Theorem. *If a class of spaces $\mathcal{B}$ is Sh_p-dominated by another such class $\mathcal{C}$ and a class of spaces $\mathcal{X}$ is $N_p^{\mathcal{C}}$-calm, then $\mathcal{X}$ is also $N_p^{\mathcal{B}}$-calm.*

6.2 Theorem. *A class of spaces $\mathcal{X}$ is $N_p^{\mathcal{B}}$-calm if and only if it is M_p-dominated by an $N_p^{\mathcal{B}}$-calm class of spaces $\mathcal{Y}$.*

6.3 Theorem. *If a class of proper maps $\mathcal{F}$ is $N_p^{\mathcal{B}}$-injective and the class $\mathcal{F}'$ is $N_p^{\mathcal{B}}$-calm, then the class $\mathcal{F}''$ is also $N_p^{\mathcal{B}}$-calm.*

Proof. Let a member B of $\mathcal{B}$ be given. Since the class $\mathcal{F}'$ is $N_p^{\mathcal{B}}$-calm, there is a cover σ of B such that for every $\theta \in \mathrm{Cov}(B)$ there is a $\varrho \in$ $\in \mathrm{Cov}(B)$ with the property that for every M_p^{ϱ}-function K and L from a member X of $\mathcal{F}'$ into B the relation $K \overset{\sigma}{\sim} L$ implies the relation $K \overset{\theta}{\sim} L$. Then σ is the required cover of B.

In order to check this, assume that τ is a cover of B. Since the class $\mathcal{F}$ is $N_p^{\mathcal{B}}$-injective, there is a $\theta \in \mathrm{Cov}(B)$ such that for every

proper map $f\colon X \to Y$ from $\mathcal{F}$ and all M_p^θ-functions F and G from Y into B the relation $F \circ f \overset{\theta}{\sim} G \circ f$ implies the relation $F \overset{\tau}{\sim} G$. Pick a ϱ as above. We can assume that ϱ refines θ.

Let Y be a member of the class $\mathcal{F}''$ and let M_p^ϱ-functions $F, G\colon Y \to B$ satisfy $F \overset{\sigma}{\sim} G$. Let $f\colon X \to Y$ be from the class $\mathcal{F}$. Let K and L be the compositions $F \circ f$ and $G \circ f$, respectively. Then K and L are M_p^ϱ-functions from X into B and we have $K \overset{\sigma}{\sim} L$. It follows that $F \circ f \overset{\theta}{\sim} G \circ f$ and therefore that $F \overset{\tau}{\sim} G$. $\Diamond$

The $N_p^{\mathcal{B}}$-calm classes of spaces are inversely preserved under M_p^l-placid maps.

6.4 Theorem. *If $\mathcal{F}$ is a class of M_p^l-placid maps and the class $\mathcal{F}''$ is $N_p^{\mathcal{B}}$-calm, then the class $\mathcal{F}'$ is also $N_p^{\mathcal{B}}$-calm.*

Proof. Let a member B of $\mathcal{B}$ be given. Since the class $\mathcal{F}''$ is $N_p^{\mathcal{B}}$-calm, there is a $\sigma \in \mathrm{Cov}(B)$ such that for every $\mu \in \mathrm{Cov}(B)$ we can find a $\varrho \in \mathrm{Cov}(B)$ with the property that for M_p^ϱ-functions K and L from a member Y of $\mathcal{F}''$ into B the relation $K \overset{\sigma}{\sim} L$ implies the relation $K \overset{\mu}{\sim} L$. Then σ is the required cover of B.

To check this, let a cover τ of B be given. Let $\mu \in \tau^*$. Pick a ϱ as above. We can assume that ϱ refines μ. Let F and G be M_p^ϱ-functions from a member X of $\mathcal{F}'$ into B and assume that $F \overset{\sigma}{\sim} G$. Let W be an M_p^σ-homotopy joining F and G. Let $\theta \in \mathrm{Cov}(X)$ be from the intersection of sets $D(W, \sigma)$, $S(F, \varrho)$, and $S(G, \varrho)$.

Let $f\colon X \to Y$ be a map from $\mathcal{F}$. Since f is M_p^l-placid, there is an M_p^θ-function $J\colon Y \to X$ such that $\mathrm{id}_X \overset{\mu}{\sim} J \circ f$. Let K and L be $F \circ J$ and $G \circ J$. Then K and L are M_p^ϱ-functions from Y into B and we have $K \overset{\sigma}{\sim} L$. It follows from our selections that $K \overset{\mu}{\sim} L$ so that we have the following chain of relations. $F \overset{\varrho}{\sim} F \circ J \circ f = K \circ f \overset{\mu}{\sim} L \circ f = G \circ J \circ f \overset{\varrho}{\sim} G$. Hence, $F \overset{\tau}{\sim} G$. $\Diamond$

6.5 Corollary. *An M_p-retract of an $N_p^{\mathcal{B}}$-calm space is itself $N_p^{\mathcal{B}}$-calm.*

6.6 Theorem. *Let $\mathcal{B}$ be a class of spaces. If $\mathcal{F}$ is an $N_p^{\mathcal{B}}$-bijective class of proper maps and the class $\mathcal{F}''$ is $N_p^{\mathcal{B}}$-calm, then the class $\mathcal{F}'$ is also $N_p^{\mathcal{B}}$-calm.*

Proof. Let a member B of $\mathcal{B}$ be given. Since the class $\mathcal{F}''$ is $N_p^{\mathcal{B}}$-calm, there is an $\alpha \in \mathrm{Cov}(B)$ such that for every $\mu \in \mathrm{Cov}(B)$ we can find a $\pi \in \mathrm{Cov}(X)$ with the property that for M_p^π-functions K and L from

a member Y of $\mathcal{F}''$ into B the relation $K \overset{\alpha}{\sim} L$ implies the relation $K \overset{\mu}{\sim} L$.

Since the class $\mathcal{F}$ is $N_p^{\mathcal{B}}$-injective, there is a $\lambda \in \mathrm{Cov}(B)$ such that for every proper map $f\colon X \to Y$ from $\mathcal{F}$ and all M_p^{λ}-functions K and L from Y into B the relation $K \circ f \overset{\lambda}{\sim} L \circ f$ implies the relation $K \overset{\alpha}{\sim} L$. Let $\sigma \in \lambda^*$. Then σ is the required cover of B.

In order to check this, let a $\tau \in \mathrm{Cov}(B)$ be given. Let $\mu \in \tau^*$. Choose a cover π as above. We can assume that π refines both σ and μ. Since the class $\mathcal{F}$ is also $N_p^{\mathcal{B}}$-surjective, there is a $\varrho \in \pi^+$ such that for every proper map $f\colon X \to Y$ from $\mathcal{F}$ and every M_p^{ϱ}-function $F\colon X \to B$ there is an M_p^{π}-function $K\colon Y \to B$ with $F \overset{\pi}{\sim} K \circ f$.

Consider a member X of $\mathcal{F}'$ and M_p^{ϱ}-functions F and G from X into B and assume that $F \overset{\sigma}{\sim} G$. Let $f\colon X \to Y$ be a map from the class $\mathcal{F}$. Choose two M_p^{π}-functions K and L from Y into B such that $F \overset{\pi}{\sim} K \circ f$ and $G \overset{\pi}{\sim} L \circ f$. The last two relations imply the relation $K \circ f \overset{\lambda}{\sim} L \circ f$. It follows that $K \overset{\alpha}{\sim} L$ and therefore that $K \overset{\mu}{\sim} L$. Thus, we obtain the following chain of relations : $F \overset{\pi}{\sim} K \circ f \overset{\tau}{\sim} L \circ f \overset{\pi}{\sim} G$. From here we conclude that $F \overset{\tau}{\sim} G$. $\Diamond$

7. Covered and extended classes

In this section we shall explore dependence of all proper shape invariants which were defined on classes of spaces involved under the assumption that these classes are connected by either surjections or injections. The connection can be through one of the following two notions.

Let $\mathcal{F}$ be a class of proper maps and let $\mathcal{B}$ and $\mathcal{C}$ be classes of spaces. We shall say that the class $\mathcal{C}$ is *$\mathcal{F}$-covered* by $\mathcal{B}$ provided for every $C \in \mathcal{C}$ there is a $B \in \mathcal{B}$ and an $h\colon B \to C$ from $\mathcal{F}$. Similarly, the class $\mathcal{C}$ is *$\mathcal{F}$-extended* by $\mathcal{B}$ provided for every $C \in \mathcal{C}$ there is a $B \in \mathcal{B}$ and a $k\colon C \to B$ from $\mathcal{F}$.

For a class of spaces $\mathcal{B}$ we shall use $\mathcal{B}_i$, $\mathcal{B}_s$, and $\mathcal{B}_b$ to denote the classes of all $M_p^{\mathcal{B}}$-injections, $M_p^{\mathcal{B}}$-surjections, and $M_p^{\mathcal{B}}$-bijections. Also, $\mathcal{B}^i$, $\mathcal{B}^s$, and $\mathcal{B}^b$ denote the classes of all $N_p^{\mathcal{B}}$-injections, $N_p^{\mathcal{B}}$-surjections, and $N_p^{\mathcal{B}}$-bijections. Moreover, if $\mathcal{F}$ and $\mathcal{G}$ are classes of maps we let $\mathcal{FG}$ denote the intersection $\mathcal{F} \bigcap \mathcal{G}$.

We begin with the result on $M_p^{\mathcal{B},\mathcal{C}}$-smooth spaces and continue to cover all our proper shape invariants. The proofs are mostly omitted.

7.1 Theorem. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ be classes of topological spaces. If a space X is $M_p^{\mathcal{A},\mathcal{D}}$-smooth and either*

(cc) *$\mathcal{B}$ is $\{X\}^i$-covered by $\mathcal{A}$ and $\mathcal{D}$ is $\{X\}^i$-covered by $\mathcal{C}$,*

(ce) *$\mathcal{B}$ is $\{X\}^i$-covered by $\mathcal{A}$ and $\mathcal{D}$ is $\mathcal{A}^s$-extended by $\mathcal{C}$,*

(ec) *$\mathcal{B}$ is $\mathcal{D}_s\{X\}^s$-extended by $\mathcal{A}$ and $\mathcal{D}$ is $\{X\}^i$-covered by $\mathcal{C}$, or*

(ee) *$\mathcal{B}$ is $\mathcal{C}_s\{X\}^s$-extended by $\mathcal{A}$ and $\mathcal{D}$ is $\mathcal{A}^s$-extended by $\mathcal{C}$,*

then X is also $M_p^{\mathcal{B},\mathcal{C}}$-smooth.

7.2 Theorem. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{X}$ be classes of spaces. If $\mathcal{X}$ is $N_p^{\mathcal{A},\mathcal{D}}$-smooth and either*

(cc) *$\mathcal{B}$ is $\mathcal{X}^i$-covered by $\mathcal{A}$ and $\mathcal{D}$ is $\mathcal{X}^i$-covered by $\mathcal{C}$,*

(ce) *$\mathcal{B}$ is $\mathcal{X}^i$-covered by $\mathcal{A}$ and $\mathcal{D}$ is $\mathcal{A}_s$-extended by $\mathcal{C}$,*

(ec) *$\mathcal{B}$ is $\mathcal{D}_s\mathcal{X}^b$-extended by $\mathcal{A}$ and $\mathcal{D}$ is $\mathcal{X}^i$-covered by $\mathcal{C}$, or*

(ee) *$\mathcal{B}$ is $\mathcal{D}_s\mathcal{X}^s$-extended by $\mathcal{A}$ and $\mathcal{D}$ is $\mathcal{B}^s$-extended by $\mathcal{C}$, then $\mathcal{X}$ is also $N_p^{\mathcal{B},\mathcal{C}}$-smooth.*

7.3 Theorem. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{X}$ be classes of topological spaces such that $\mathcal{B}$ is $\mathcal{X}_b$-covered by $\mathcal{A}$ and $\mathcal{X}$ is both $N_p^{\mathcal{A},\mathcal{D}}$-smooth and $M_p^{\mathcal{D},\mathcal{C}}$-tame. Then $\mathcal{X}$ is also $N_p^{\mathcal{B},\mathcal{C}}$-smooth.*

Proof. Let a member B of $\mathcal{B}$ and a cover σ of B be given. Let $\mu \in \in \sigma^*$. Since the class $\mathcal{B}$ is $\mathcal{X}_b$-covered by the class $\mathcal{A}$, there is an $A \in \in \mathcal{A}$ and an $M_p^{\mathcal{X}}$-bijection $h\colon A \to B$. Let $\delta = h^{-1}(\mu)$. We utilize now the assumption that the class $\mathcal{X}$ is $N_p^{\mathcal{A},\mathcal{D}}$-smooth to select an $\varepsilon \in \in \mathrm{Cov}(A)$ such that for M_p^{ε}-functions P and Q from a member X of $\mathcal{X}$ into A the relation $P \overset{\varepsilon}{\sim}_{\mathcal{D}} Q$ implies the relation $P \overset{\delta}{\sim} Q$. Since h is an $M_p^{\mathcal{X}}$-injection, there is a $\lambda \in \mathrm{Cov}(A)$ and a $\nu \in \mu^+$ such that for M_p^{λ}-functions P and Q from a member of $\mathcal{X}$ into A the relation $h \circ P \overset{\nu}{\sim} h \circ Q$ implies the relation $P \overset{\varepsilon}{\sim} Q$. Let $\kappa \in \nu^*$. At last, choose the required cover $\tau \in \kappa^+$ of B using the fact that h is an $M_p^{\mathcal{X}}$-surjection such that for every M_p^{τ}-function F from a member X of $\mathcal{X}$ into B there is an M_p^{λ}-function $P\colon X \to A$ with $F \overset{\kappa}{\sim} h \circ P$.

Consider an $X \in \mathcal{X}$ and M_p^{τ}-functions F, $G\colon X \to B$ and assume that $F \overset{\tau}{\sim}_{\mathcal{C}} G$. Pick M_p^{λ}-functions P, $Q\colon X \to A$ and M_p^{κ}-homotopies V and W joining F and $h \circ P$ and G and $h \circ Q$, respectively.

Our goal now is to show that $P \overset{\varepsilon}{\sim}_{\mathcal{D}} Q$. In order to do this, we must find a cover ξ of X so that $P \circ N \overset{\varepsilon}{\sim} Q \circ N$ for every M_p^{ξ}-function N from a member of $\mathcal{D}$ into X. First, observe that the assumption

about F implies the existence of a $\theta \in \mathrm{Cov}(X)$ such that the relation $F \circ M \overset{\tau}{\sim} G \circ M$ holds for every M_p^θ-function M from a member of $\mathcal{C}$ into X. Let $\zeta \in \theta^+$ be from the intersection of sets $D(V, \kappa)$, $D(W, \kappa)$, $S(F, \tau)$, and $S(G, \tau)$. Since X is $M_p^{\mathcal{D}, \mathcal{C}}$-tame, there is a $\xi \in \mathrm{Cov}(X)$ with the property that for every M_p^ξ-function N from a member D of $\mathcal{D}$ into X there is a $C \in \mathcal{C}$ and an M_p^ζ-function $M\colon C \to X$ such that for every $\gamma \in \mathrm{Cov}(C)$ we can find an M_p^γ-function $K\colon D \to C$ with $N \overset{\zeta}{\sim} M \circ K$.

Let $D \in \mathcal{D}$ and let $N\colon D \to X$ be an M_p^ξ-function. Pick a C and an N as above. By assumption, there is an M_p^τ-homotopy Z joining $F \circ M$ and $G \circ M$. Let $\gamma \in D(Z, \tau)$. Choose a K as above. Our choices imply that $h \circ (P \circ N) \overset{\kappa}{\sim} F \circ N \overset{\tau}{\sim} F \circ M \circ K \overset{\tau}{\sim} G \circ M \circ K \overset{\tau}{\sim} G \circ N \overset{\kappa}{\sim} h \circ (Q \circ N)$. It follows that $h \circ (P \circ N) \overset{\nu}{\sim} h \circ (Q \circ N)$ so that $P \circ N \overset{\varepsilon}{\sim} Q \circ N$ and our claim has been verified.

Now, we conclude that $P \overset{\delta}{\sim} Q$ and therefore $h \circ P \overset{\mu}{\sim} h \circ Q$. Thus, we obtain now $F \overset{\kappa}{\sim} h \circ P \overset{\mu}{\sim} h \circ Q \overset{\kappa}{\sim} G$. Hence, $F \overset{\sigma}{\sim} G$. $\Diamond$

7.4 Theorem. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{X}$ be classes of spaces. If $\mathcal{X}$ is $N_p^{\mathcal{A}, \mathcal{D}}$-smooth and either*

(cc) *$\mathcal{B}$ is $\mathcal{C}_s\mathcal{X}_i$-covered by $\mathcal{A}$ and $\mathcal{C}$ is $\mathcal{A}^i$-covered by $\mathcal{D}$,*

(ce) *$\mathcal{B}$ is $\mathcal{C}_s\mathcal{X}_i$-covered by $\mathcal{A}$ and $\mathcal{C}$ is $\mathcal{A}^s$-extended by $\mathcal{D}$,*

(ec) *$\mathcal{B}$ is $\mathcal{D}_i$-extended by $\mathcal{A}$ and $\mathcal{C}$ is $\mathcal{B}^i$-covered by $\mathcal{D}$, or*

(ee) *$\mathcal{B}$ is $\mathcal{X}_i$-extended by $\mathcal{A}$ and $\mathcal{C}$ is $\mathcal{B}^s\mathcal{X}_s$-extended by $\mathcal{D}$,*

then $\mathcal{X}$ is also $P_p^{\mathcal{B}, \mathcal{C}}$-smooth.

7.5 Theorem. *Let $\mathcal{B}$ and $\mathcal{C}$ be classes of spaces. If a space X is $M_p^{\mathcal{B}}$-calm and the class $\mathcal{C}$ is either $\{X\}^i$-covered or $\{X\}^b$-extended by $\mathcal{B}$, then X is also $M_p^{\mathcal{C}}$-calm.*

7.6 Theorem. *Let $\mathcal{X}$, $\mathcal{B}$, and $\mathcal{C}$ be classes of spaces. If $\mathcal{X}$ is $N_p^{\mathcal{B}}$-calm and the class $\mathcal{C}$ is either $\mathcal{X}_b$-covered or $\mathcal{X}_i$-extended by $\mathcal{B}$, then $\mathcal{X}$ is also $N_p^{\mathcal{C}}$-calm.*

Proof. ($\mathcal{C}$ is $\mathcal{X}_i$-extended by $\mathcal{B}$). Let a member C of $\mathcal{C}$ be given. Since the class $\mathcal{C}$ is $\mathcal{X}_i$-extended by the class $\mathcal{B}$, there is a $B \in \mathcal{B}$ and an $M_p^{\mathcal{X}}$-injection $k\colon C \to B$. Since $\mathcal{X}$ is $N_p^{\mathcal{B}}$-calm, there is a cover $\sigma \in \mathrm{Cov}(B)$ such that for every $\tau \in \mathrm{Cov}(B)$ there is a $\varrho \in \mathrm{Cov}(B)$ with the property that for every $X \in \mathcal{X}$ and all M_p^ϱ-functions K and L from X into B the relation $K \overset{\sigma}{\sim} L$ implies the relation $K \overset{\tau}{\sim} L$. Let $\gamma = k^{-1}(\sigma)$. Then γ is the required cover of C.

Let δ be a cover of C. Since k is an $M_p^{\mathcal{X}}$-injection, there is a $\theta \in \in \mathrm{Cov}(C)$ and a $\tau \in \mathrm{Cov}(B)$ such that for M_p^{θ}-functions F and G from a member of $\mathcal{X}$ into C the relation $k \circ F \overset{\tau}{\sim} k \circ G$ implies the relation $F \overset{\delta}{\sim} G$. Pick a ϱ as above and let $\varepsilon \in \mathrm{Cov}(C)$ be a common refinement of θ and $k^{-1}(\varrho)$.

Consider M_p^{ε}-functions F and G from a member X of $\mathcal{X}$ into C and assume that $F \overset{\gamma}{\sim} G$. Let K and L be the compositions $k \circ F$ and $k \circ G$. Then K and L are M_p^{ϱ}-functions with $K \overset{\sigma}{\sim} L$. It follows that $K \overset{\tau}{\sim} L$ and therefore that $F \overset{\delta}{\sim} G$. $\Diamond$

References

[1] ALÓ, R. A. and SHAPIRO, M. L.: *Normal Topological Spaces*, Cambridge Univ. Press, Cambridge, 1972.

[2] BALL, B. J. and SHER, R. B.: A theory of proper shape for locally compact spaces, *Fund. Math.* **86** (1974), 163–192.

[3] BORSUK, K.: Some quantitative properties of shapes, *Fund. Math.* **93** (1976), 197–212.

[4] BORSUK, K.: Theory of Shape, PNW, Warszawa, 1975.

[5] ČERIN, Z.: Homotopy at infinity of proper maps, *Glasnik Mat.* **13** (1978), 135–154.

[6] ČERIN, Z.: Homotopy properties of locally compact spaces at infinity – calmness and smoothness, *Pacific J. Math.* **79** (1978), 69–91.

[7] ČERIN, Z.: Proper shape theory revisited, Preprint.

[8] ČERIN, Z.: Morphisms in proper shape theory, Preprint.

[9] ČERIN, Z.: Proper shape invariants: tameness and movability, Preprint.

[10] DOLD, A.: *Lectures on Algebraic Topology*, Springer-Verlag, Berlin, 1972.

[11] KATO, H.: Refinable maps in the theory of shape, *Fund. Math.* **113** (1981), 119–129.

[12] SANJURJO, V.: An intrinsic description of shape, *Trans. AMS* **329** (1992), 625–636.