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COMMUTATIVITY RESULTS FOR RINGS THROUGH STREB'S CLASSIFICATION

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Abstract: An associative ring R is commutative if (and only if) for each $x, y \in R$, there exist integers $m > 0$, $n \geq 0$ and $f(X), g(X), h(X) \in X^2\mathbb{Z}\,[X]$ with $f(1) = \pm 1$ such that $[x, yx^m - f(y)x^n] = 0$ and $[x - g(x), y - h(y)] = 0$. Further, we extend this result for one sided s-unital rings.

Throughout this paper, R will denote an associative ring with center $Z(R)$, and $C(R)$ the commutator ideal of R. Let $N(R)$ be the set of nilpotent elements in R, and let $N^*(R)$ be the subset of $N(R)$ consisting of all elements in R which square to zero. A ring R is called *left* (resp. *right*) *s-unital* if $x \in Rx$ (resp. $x \in xR$) for every $x \in R$. Further, R is called *s-unital* if $x \in Rx \cap xR$ for all $x \in R$. If R is s-unital (resp. left or right s-unital), then for any finite subset F of R, there exists an element $e \in R$ such that $ex = xe = x$ (resp. $ex = x$ or $xe = x$) for all $x \in F$. Such an element e will be called a *pseudo* (resp. *pseudo left* or *pseudo right*) *identity* of F in R. We denote by $\mathbb{Z}\langle X, Y\rangle$ the polynomial ring over $\mathbb{Z}$ the ring of integers, in the non-cummuting indeterminates X, and Y. As usual $\mathbb{Z}[X]$ is the totality of polynomials in X with coefficients in $\mathbb{Z}$ and for any $x, y \in R$, $[x, y] = xy - yx$. For any positive integer d, we consider the following ring property:

$\mathbf{Q}(d)$: if $x, y \in R$, and $d[x,y] = 0$, then $[x,y] = 0$.

By $GF(q)$, we mean the Galois field (finite field) with q elements, and $(GF(q))_2$ the ring of all 2×2 matrices over $GF(q)$. Set $e_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $e_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and $e_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ in $(GF(p))_2$ for a prime p.

In [7, Prop. 2], Komatsuo et al. proved the following important result:

Proposition 1. *Let R be a ring generated by two elements such that the commutator ideal $C(R)$, is the heart of R and $C(R)R = RC(R) = 0$. Then R is nilpotent.*

In view of Prop. 1, we see that Streb's Theorem of [8] can be stated as follows:

Theorem S. *Let R be a non-commutative ring ($R \neq Z(R)$). Then there exists a factor subring of R which is of type* (a)$_{\text{i}}$, (a)$_{\text{ii}}$, (b), (c), (d), (e), (f) *or* (g):

(a)$_{\text{i}}$ $\begin{bmatrix} GF(p) & GF(p) \\ 0 & GF(p) \end{bmatrix}$, *$p$ a prime.*

(a)$_{\text{ii}}$ $\begin{bmatrix} 0 & GF(p) \\ 0 & GF(p) \end{bmatrix}$, *$p$ a prime.*

(b) $M_\eta(\mathbf{K}) = \left\{ \begin{bmatrix} a & b \\ 0 & \eta(a) \end{bmatrix} \mid a, b \in \mathbf{K} \right\}$, *where $\mathbf{K}$ is a finite field with a non-trivial automorphism η.*

(c) *A non-commutative division ring.*

(d) *A non-commutative ring with no non-zero divisors of zero.*

(e) *A finite nilpotent ring S such that $C(S)$ is the heart of S and $SC(S) = C(S)S = 0$.*

(f) *A ring S generated by two elements of finite additive order such that $C(S)$ is the heart of S, $SC(S) = C(S)S = 0$, and $N(S)$ is a commutative nilpotent ideal of S.*

(g) *A simple radical ring with no non-zero divisors of zero.*

Further, from the proof of [8, Korollar 1], we have the following:

Theorem ST. *Let R be a non-commutative ring with* 1. *Then there exists a factor subring of R which is of type* (a)$_{\text{i}}$, (b), (c), (d), (d)$'$, (e)$'$ *or* (e)$''$:

(a)$_{\text{i}}$ $\begin{bmatrix} GF(p) & GF(p) \\ 0 & GF(p) \end{bmatrix}$, *$p$ a prime.*

(b) $M_\eta(\mathbf{K}) = \left\{ \begin{bmatrix} a & b \\ 0 & \eta(a) \end{bmatrix} \mid a, b \in \mathbf{K} \right\}$, *where $\mathbf{K}$ is a finite field with a non-trivial automorphism η.*

(c) *A non-commutative division ring.*

(d) *A non-commutative ring with no non-zero divisors of zero.*

(d)′ $T = <1> + S$ *is a finite integral domain, where* S *is a simple radical ring.*

(d) *A non-commutative ring with no non-zero divisors of zero.*

(e)′ $T = <1> + S$, *where* S *is a finite nilpotent ring such that* $C(S)$ *is the heart of* S *and* $SC(S) = C(S)S = 0$.

(e)″ $T = < 1 > + S$, *where* S *is a non-commutative subring of* T *such that* $S[S,S] = [S,S]S = 0$.

Now Th. S and Th. ST give the following Meta Theorem which plays an important role in our subsequent study.

Lemma 1 (Meta Theorem). *Let* **P** *be a ring property which is inherited by factor subrings. If no rings of type* $(a)_i$, $(a)_{ii}$, (b), (c), (e), *or* (g), (f) *(resp.* $(a)_i$, (b), (c), (d), (d)′, (e)′ *or* (e)″) *satisfy* **P**, *then every ring (resp. every ring with unity* 1) *satisfying* **P** *is commutative.*

Our objective is to prove the following results.

Theorem 1. *Let* R *be a ring. Then* R *is commutative if (and only if) for each* $x, y \in R$, *there exist integers* $m > 0$, $n \geq 0$, *and* $f(X), g(X), h(X) \in X^2\mathbb{Z}[X]$ *with* $f(1) = \pm 1$ *such that* $[x, yx^m - f(y)x^n] = 0$ *and* $[x - g(x), y - h(y)] = 0$.

Theorem 2. *Let* R *be a right s-unital ring, and let* m *and* n *be non-negative integers. Assume that for each* $y \in R$, *there exists* $f(X) \in X^2\mathbb{Z}[X]$ *such that* $[x, yx^m - f(y)x^n] = 0$ *for all* $x \in R$. *Then* R *is commutative.*

Theorem 3. *Let* R *be a right (or left) s-unital ring. Then the following are equivalent:*

(i) R *is commutative.*

(ii) *For each* x, y *in* R, *there exist non-negative integers* $m > 0$, $n \geq 0$ *and* $f(X) \in X^2\mathbb{Z}[X]$ *with* $f(1) = \pm 1$ *such that* $[x, yx^m - f(y)x^n] = 0$, *and for each* $x \in R$, *either* $x \in Z(R)$, *or there exists* $g(X) \in X^2\mathbb{Z}[X]$ *such that* $x - g(x) \in N(R)$.

(iii) *For each* $y \in R$, *there exists* $f(X) \in X^2\mathbb{Z}[X]$ *with* $f(1) = \pm 1$ *such that* $[x, yx^m - f(y)x^n] = 0$ *for all* $x \in R$, *provided* m, n *are fixed non-negative integers.*

Theorem 4. *Let* R *be a right s-unital ring. Suppose that* R *satisfies a polynomial identity*

$$[f(X), Y]X^m + \lambda(X,Y)[X, g(Y)]\lambda^*(X,Y) = 0,$$

where m *is a non-negative integer,* $\lambda(X,Y)$ *and* $\lambda^*(X,Y)$ *are monic monomials in* $\mathbb{Z}<X,Y>$, $f(X)$ *and* $g(X)$ *are polynomials in* $X\mathbb{Z}[X]$ *with*

$f(1)=\pm 1$ and $g(1)=\pm 1$, and every monomial of $\lambda(X,Y)g(Y)\lambda^*(X,Y)$ has degree ≥ 2 in Y. Suppose that $n=(f'(1),g'(1))$ is non-zero, where $f'(X)$ and $g'(X)$ are the usual derivatives of $f(X)$ and $g(X)$ respectively. If R satisfies the property $\mathbf{Q}(n)$, then R is commutative.

Following [4], let $\mathbf{P}$ be a ring property. If $\mathbf{P}$ is inherited by every subring and every homomorphic image, then $\mathbf{P}$ is called an $\mathbf{h}$-property. More weakly, if $\mathbf{P}$ is inherited by every finitely generated subring and every natural homomorphic image modulo the annihilator of a central element, then $\mathbf{P}$ is called an $\mathbf{H}$-property.

A ring property $\mathbf{P}$ such that a ring R has the property $\mathbf{P}$ if and only if all its finitely generated subrings have $\mathbf{P}$, is called an $\mathbf{F}$-property.

Lemma 2 ([4, Prop. 1]). *Let $\mathbf{P}$ be an $\mathbf{H}$-property, and let $\mathbf{P}'$ be an $\mathbf{F}$-property. If every ring R with unity 1 having the property $\mathbf{P}$ has the property $\mathbf{P}'$, then every s-unital ring having $\mathbf{P}$ has $\mathbf{P}'$.*

Lemma 3 ([3, Th.]). *If for every x, y in a ring R, we can find a polynomial $p_{x,y}(t)$ with integer coefficients which depend on x and y such that $[x^2p_{x,y}(x)-x,y]=0$, then R is commutative.*

Lemma 4 ([1, Lemma]). *Let R be a ring with unity 1. If for each $x,y\in$ $\in R$, there exists an integer $m=m(x,y)\geq 1$ such that $x^m[x,y]=0$, or $[x,y]x^m=0$, then necessarily $[x,y]=0$.*

Lemma 5 ([5, Th.]). *Let f be a polynomial in non-commuting indeterminates $x_1,x_2,\ldots,x_n$ with coprime integer coefficients. Then the following statements are equivalent:*

(1) *For any ring R satisfying $f=0$, $C(R)$ is a nil ideal.*

(2) *For every prime p, $(GF(p))_2$ fail to satisfy $f=0$.*

In [2], Chacron defined the cohypercenter $C'(R)$ of a ring R as the set of all elements $a\in R$ such that for each $x\in R$ there holds $[a,x-f(x)]=0$ with some $f(X)\in X^2\mathbb{Z}[X]$, which is a commutative subring of R ([2, Remark 12]). Indeed Chacron proved the following result:

Theorem C (Chacron, [2]). *Suppose that R satisfies the following condition:*

(C) *For each $x,y\in R$, there exist $f(X),g(X)\in X^2\mathbb{Z}[X]$ such that $[x-f(x),y-g(y)]=0$.*

Then we have the following:

(1) *$C'(R)$ is a commutative subring of R containing $N(R)$;*

(2) *$N(R)$ is a commutative ideal of R containing $C(R)$;*

(3) *$N(R)[C'(R),R]=[C'(R),R]N(R)=0$ and $[C'(R),R]\subseteq N^*(R)$.*

In this paper, we hall study rings satisfying condition (C) of Th. C by making use of the recent result of W. Streb [8], which we called *Streb's classification.*

Theorem SC (Streb [8]). *Suppose that a ring R satisfies the following condition:*

(SC) *For each $x, y \in R$, there exists a polynomial $f(X,Y) \in \in \mathbb{Z}<X,Y>[X,Y]\mathbb{Z}<X,Y>$ each of whose monomial terms is of length ≥ 3 such that $[x,y] = f(x,y)$.*

Then there exists no factor subring of R which is of type (e) *or* (f). *Therefore, if R is non-commutative, then there exists a factor subring of R which is of type* (a), (b), (c) *or* (d).

The next result is crucial in our subsequent study is immediate by Th. C, and Th. SC.

Theorem KT. *Suppose that a ring R satisfies* (C). *Then there exists no factor subring of R which is of type* (c), (d), (e) *or* (f). *Therefore, if R is non-commutative, then there exists a factor subring of R which is of type* (a) *or* (b).

Proof of Th. 1. Let p be prime. Consider the ring $\begin{bmatrix} GF(p) & GF(p) \\ 0 & GF(p) \end{bmatrix}$. Set $x = e_{22}$ and $y = e_{12}$ in our hypothesis to obtain

$$[e_{22}, e_{12}e_{22}^m - f(e_{12})e_{22}^n] \neq 0$$

for all integers $m > 0$, $n \geq 0$ and $f(X) \in X^2\mathbb{Z}[X]$ with $f(1) = \pm 1$. Further, consider the ring $M_\eta(\mathbf{K})$, a ring of type (b). Let $x = \begin{bmatrix} \gamma & 0 \\ 0 & \eta(\gamma) \end{bmatrix}$, ($\eta(\gamma) \neq \gamma$) and $y = e_{12}$. Then

$$[x, yx^m - f(y)x^n] = [x,y]x^m = y(\gamma - \eta(\gamma))\gamma^m \neq 0$$

for all integers $m > 0$, $n \geq 0$ and $f(X) \in X^2\mathbb{Z}[X]$. Hence, R is commutative by Th. KT. $\Diamond$

Corollary 1. *Suppose that for each $x, y \in R$, there exist integers $l > 1$, $m > 0$, $n \geq 0$, and $f(X), g(X) \in X^2\mathbb{Z}[X]$ such that $[x, yx^m - y^l x^n] = 0$ and $[x - f(x), y - g(y)] = 0$. Then R is commutative.*

Lemma 6. *If R is a right s-unital and not left s-unital, then R has a factor subring of type* $(\mathrm{a})_\mathrm{i}$.

Proof. There exists $x \in R$ such that $x \notin xR$, (R is not left s-unital). Let $e, f \in R$ such that $xe = x$ and $ef = e$. Then $xf = x$. Put $y = x - - fx$. Then $y \neq 0$, $y^2 = 0$, $ye = y$ and $ey = 0$. Let M be an ideal of $< e, y >$ which is maximal with respect to $y \notin M$. Put $I =< e, y > /M$,

$\bar{e} = e + M$, $\bar{y} = y + M$. Thus $\bar{y}\bar{e} = \bar{y}$ and $\bar{e}\bar{y} = 0 = \bar{y}^2$. So we have $I = \langle \bar{e} \rangle + \bar{y}\mathbb{Z}$ and $\bar{y}\mathbb{Z}$ is the smallest non-zero ideal of I. Hence $\bar{y}\mathbb{Z}$ is an irreducible right $\langle \bar{e} \rangle$-module. Next, we can see that $A = \{s \in \langle \bar{e} \rangle \mid \bar{y}s = 0\}$ is an ideal of I which does not contain $\bar{y}$, so $A = 0$. Therefore $\langle \bar{e} \rangle$ is a commutative primitive ring and so a field. Since $\bar{e}^2 - \bar{e} \in A = 0$, $I = \bar{e}\mathbb{Z} \oplus \bar{y}\mathbb{Z}$ is of type $(\mathrm{a})_{\mathrm{i}}$. ◊

Proof of Th. 2. Trivially, we can check that no rings of type $(\mathrm{a})_{\mathrm{i}}$ or (b) satisfy our hypothesis. In view of Lemma 6, R is s-unital. Hence, by Lemma 2, we may assume that R with 1. If $m = n = 0$, then $[x, y - f(y)] = 0$. Therefore, R is commutative by Lemma 3. Henceforth, we may assume that $m > 0$, or $n > 0$. Then $x = e_{22}$ and $y = e_{12}$ in $(GF(p))_2$ p prime, fails to satisfy $[x, y]x^m = [x, f(y)]x^n$. Hence, by Lemma 5, R has no factor subrings of type (d). Further, suppose that R has a factor subring T of type (e)$'$. Take $s, t \in S$ such that $[s, t] \neq 0$. Then there exists $f(X) \in X^2\mathbb{Z}[X]$ such that $[s, t] = [s, t](s + 1)^m - [s, f(t)](s + 1)^n = 0$, which is a contradiction. Therefore, R is commutative by Lemma 1. ◊

Lemma 7. *Let R be a ring with 1. Suppose that for each $x, y \in R$, there exists non-negative integers m, n and $f(X) \in X^2\mathbb{Z}[X]$ such that $[x, yx^m - f(y)x^n] = 0$. Then $N(R) \subseteq Z(R)$.*

Proof. Suppose that $a \in N(R)$, and $a \in R$. Then $[x, a]x^{m_1} = [x, f_1(a)]x^{n_1}$, for $m_1 \geq 0$, $n_1 \geq 0$, and some $f_1(X) \in X^2\mathbb{Z}[X]$. Also, $[x, f_1(a)]x^{m_2} = [x, f_2(f_1(a))]x^{n_2}$, for some $m_2 \geq 0$, $n_2 \geq 0$, and some $f_2(X) \in X^2\mathbb{Z}[X]$. Thus

$$[x, a]x^{m_1+m_2} = [x, f_2(f_1(a))]x^{n_1+n_2}.$$

Continuing this process, we can see that

$$[x, a]x^{m_1+\ldots+m_t} = [x, f_t(\cdots f_1(a)\cdots)]x^{n_1+\cdots+n_t},$$

for some $m_k \geq 0$, $n_k \geq 0$ and some $f_k(X) \in X^2\mathbb{Z}[X]$, $k = 1, \cdots, t$. Since $a \in N(R)$, for sufficiently large t, we get

$$[x, a]x^{m_1+\cdots+m_t} = 0,$$

and so

$$[x, a](x + 1)^{m_1+\cdots+m_t} = 0,$$

for $m_1 + \cdots + m_t \geq 0$. By Lemma 4, $[x, a] = 0$. Thus, $N(R) \subseteq Z(R)$. ◊

Proof of Theorem 3. It suffices to show that each of (ii) and (iii) implies (i).

(ii)⇒(i): Consider the ring $(GF(p))_2$, p a prime. Then we see that $[e_{22}, e_{12}e_{22}^m - f(e_{12})e_{22}^n] = e_{12} \neq 0$, for any integers $m > 0$, $n \geq 0$ and $f(X) \in X^2\mathbb{Z}[X]$ with $f(1) = \pm 1$. Accordingly, R has no factor subrings of type $(a)_i$. Thus in view of Lemma 6 and its dual, R is s-unital. By Lemma 2, we may assume that R has unity 1. Since $N(R) \subseteq Z(R)$, by Lemma 7, R satisfies all the hypotheses of Th. 1. Therefore, R is commutative.

(iii)⇒(i): In case $m > 0$, we have shown above that R has no factor subrings of type $(a)_{iii}$. If $m = 0$, then we consider in $(GF(p))_2$, p a prime, $x = e_{22}$ and $y = e_{12}$ in our hypotheses to obtain $[e_{22}, e_{12}e_{22}^m - f(e_{12})e_{22}^n] \neq 0$ for any integer $n \geq 0$ and $f(X) \in X^2\mathbb{Z}[X]$. Hence, R has no factor subrings of type $(a)_i$. In view of the dual of Lemma 6, if R is left s-unital, then R is also right s-unital. By Th. 2, R is commutative. ◊

Corollary 2. *If R is a right (or left) s-unital ring, then the following conditions are equivalent:*

(1) *R is commutative.*

(2) *For each $x, y \in R$, there exist integers $l > 1$, $m > 0$, $n \geq 0$ such that $[x, yx^m - y^l x^n] = 0$, and for each $x \in R$, either $x \in Z(R)$ or there exists $f(X) \in X^2\mathbb{Z}[X]$ such that $x - f(x) \in N(R)$.*

(3) *For each $y \in R$, there exists an integer $l > 1$ such that $[x, yx^m - y^l x^n] = 0$, for all $x \in R$, where m, n are fixed non-negative integers.*

Following Kobayashi [6], let Θ be the additive mapping of $\mathbb{Z}\langle X, Y\rangle$ to $\mathbb{Z}$ defined as follows: For each monic monomial $X_1, \cdots, X_t$, (X_i is either X or Y), $\Theta(X_1, \cdots, X_t)$ is the number of pairs (i, j) such that $1 \leq i < j \leq t$ and $X_i = X$, $X_j = Y$. Trivially, one can see that, for any $f(X, Y) \in \mathbb{Z}\langle X, Y\rangle$, $\Theta(f(X, Y))$ equals the coefficient of XY occurring in $f(X + 1, Y + 1)$.

Let $\mathbf{N}$ be the set of all non-negative integers, $F(X, Y) \in \mathbb{Z}\langle X, Y\rangle$, and $(m, n) \in \mathbf{N} \times \mathbf{N}$. Then (m, n)-component of F, the sum of all monomials of degree (m, n), that is, of degree m with respect to X, and of degree n with respect to Y, is denoted by $F_{m,n}$.

Using the above definition, we state the following:

Lemma 8 ([6, Th.]). *Let R be a ring with unity 1, and let $F(X, Y)$ be a polynomial in $\mathbb{Z}\langle X, Y\rangle$ of total degree d. Suppose that the greatest common divisor of $\{(m-1)!(n-1)!\Theta(F_{m,n}) \mid m + n = d,\ m, n > 0\}$ is positive. If R satisfies the identity $F(X, Y) = 0$, then R satisfies the* *identity* $l(XY - YX) = 0$. *Therefore, if moreover* R *has* $\mathbf{Q}(l)$, *then* R *is commutative.*

Proof of Th. 4. By Lemma 1, it is enough to show that R has no factor subrings of type $(\mathrm{a})_{\mathrm{ii}}$, (b), (d) or (f). It is easy to see that no rings of type $(\mathrm{a})_{\mathrm{ii}}$ satisfy

$$[f(X), Y]X^m + \lambda(X,Y)[X, g(Y)]\lambda^*(X,Y) = 0,$$

where m is a non-negative integer. In view of Lemma 5, we also see that R has no factor subrings of type (d). Further, by Lemma 6, R is s-unital. Hence, in view of Lemma 2, we may assume that R has unity 1.

The sum of all monomials which have the maximal degree in

$$[f(X), Y]X^m + \lambda(X,Y)[X, g(Y)]\lambda^*(X,Y)$$

is one of the following:

$$a[X^k, Y]X^m, \qquad b\lambda(X,Y)[X, Y^l]\lambda^*(X,Y),$$

and

$$a[X^k, Y]X^m + b\lambda(X,Y)[X, Y^l]\lambda^*(X,Y),$$

where aX^k and bY^l are the leading terms of $f(X)$ and $g(Y)$, respectively. Now it is easy to see that

$$\Theta(a[X^k, Y]X^m) = ak \;\text{ and }\; \Theta(b\lambda(X,Y)[X, Y^l]\lambda^*(X,Y) = bl.$$

Hence, by Lemma 8 there exists a positive integer, n such that $n[x,y] = 0$ for all $x, y \in R$. Since R satisfies $\mathbf{Q}(d)$, we may assume that $(n,d) = 1$. If T is any factor subring of R, then T inherits the property that $n[x,y] = 0$ for all $x, y \in T$. Thus T satisfies $\mathbf{Q}(d)$.

Next, suppose that $R = M_\eta(\mathbf{K})$. Let $c = \begin{bmatrix} a & 0 \\ 0 & \eta(a) \end{bmatrix}$, $(\eta(a) \neq a)$, $e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then, by our assumption, we get $[f(c), e]c^m = -\lambda(c,e)[c, g(e)]\lambda^*(c,e) = 0$. But c is invertible, so we have $[f(c), e] = 0$. So $[f(c), 1+e]c^m = -\lambda(c, 1+e)[c, g(1+e)]\lambda^*(c, 1+e) = 0$. Therefore, $g'(1)[c,e] = [c, g(1+e)] = 0$. Now, $[f(c), c+e]c^m = -\lambda(c, c+e)[c, g(c+e)]\lambda^*(c, c+e)$ and both c and $c+e$ are invertibe, then we obtain $[c, g(c+e)] = 0$. We have

$$g(c+e) = \begin{bmatrix} g(a) & (\eta(g(a)) - g(a))(\eta(a) - a)^{-1} \\ 0 & \eta(g(a)) \end{bmatrix}.$$

Therefore, $[c, g(c+e)] = 0$ means that $\eta(g(a)) = g(a)$, and this implies that $[g(c), e] = 0$. Hence, it follows that

$$[f(1+e), c](1+e)^m = -\lambda(1+e, c)[1+e, g(c)]\lambda^*(1+e, c) = 0,$$

and hence $[e, c]f'(1) = [f(1+e), c] = 0$. This together with $[c, e]g'(1) = 0$ implies that $d[c, e] = 0$. By $\mathbf{Q}(d)$, we get $[c, e] = 0$. Thus we have a contradiction.

Finally, we suppose that R is of type (e)$'$. Choose $s, t \in S$ with $[s, t] \neq 0$. Then

$$[s, t]f'(1) = [f(1+s), t](1+s)^m = -\lambda(1+s, t)[1+s, g(t)]\lambda^*(1+s, t) = 0.$$

So $0 = [s, t]f'(1) = [f(1+s), 1+t](1+s)^m = -\lambda(1+s, 1+t)[1+s, g(1+t)]\lambda^*(1+s, 1+t) = -[s, t]g'(1)$. Hence $d[s, t] = 0$. By $\mathbf{Q}(d)$, we have $[s, t] = 0$ which is a contradiction. ◊

Corollary 3. *Let R be a right or left s-unital ring. Suppose that R satisfies the polynomial identity $[f(X), Y]X^m + [X, g(Y)]\lambda^*(X, Y) = 0$, where m is a non-negative integer, $\lambda^*(X, Y)$ is a monic monomial in $\mathbb{Z}<X, Y>$, $f(X)$, $g(X)$ are polynomials in $X\mathbb{Z}[X]$ with $f(1) = \pm 1$, $g(1) = \pm 1$, and every monomial of $g(Y)\lambda^*(X, Y)$ has degree ≥ 2 in Y. Suppose that $d = (f'(1), g'(1))$ is non-zero. If R satisfies $\mathbf{Q}(d)$, then R is commutative.*

Proof. As in the proof of Th. 3, we can see that R has no factor subrings of type $(a)_1$ and R is s-unital. Therefore, R is commutative by Th. 4. ◊

Corollary 5. *Let R be a right or left s-unital ring. Suppose that R satisfies the polynomial identity $[X^k, Y]X^m - [X, Y^l]X^n = 0$, where $k > 0$, $l > 1$, $m \geq 0$, and $n \geq 0$. Let $d = (k, l)$. If R satisfies $\mathbf{Q}(d)$, then R is commutative.*

References

[1] BELL, H. E.: The identity $(xy)^n = x^n y^n$: does it buy commutativity, *Math. Mag.* **55** (1982), 165–170.

[2] CHACRON, M.: A commutativity theorem for rings, *Proc. Math. Soc.* **59** (1976), 211–216.

[3] HERSTEIN, I. N.: The structure of a certain class of rings, *Amer. J. Math.* **75** (1953), 864–871.

[4] HIRANO, Y., KOBAYASHI, Y. and TOMINAGA, H.: Some polynomial identities and commutativity of s-unital rings, *Math. J. Okayama Univ.* **24** (1982), 7–13.

[5] KEZLAN, T. P.: A note on commutativity of semi-prime PI-rings, *Math. Japon.* **27** (1982), 267–268.

[6] KOBAYASHI, Y.: A note on commutativity of rings, *Math. J. Okayama Univ.* **23** (1981), 141–145.

[7] KOMATSU, H., NISHINAKA, T. and TOMINAGA, H.: A commutativity theorem for rings, *Bull. Austral. Math. Soc.* **44** (1991), 387–389.

[8] STREB, W.: Zur Struktur nichtkommutativer Ringe, *Math. J. Okayama Univ.* **31** (1989), 135–140.