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# CAUCHY'S EQUATION ON THE GRAPH OF AN INVOLUTION

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**Abstract:** Description of the form of solutions $f\colon \mathbb{R}\to\mathbb{R}$ of the functional equation $f(x+\alpha(x)) = f(x) + f(\alpha(x))$ is given in the case where $\alpha\colon \mathbb{R}^+\to\mathbb{R}^+$ is an involution ($\alpha\circ\alpha = \mathrm{id}$). When $\alpha(x) = \frac{1}{x}$, formal power series to the above equation are also determined.

The present paper was motivated by the following problem proposed by K. Lajkó on the XX. International Symposium on Functional Equations (Oberwolfach, 1982, cf. [3]): under what conditions the functions

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$$(*)\qquad f(x) = ax^2 + bx + 2a, \qquad a, b \in \mathbb{R},$$

are the only solutions of the equation

$$(1)\qquad f(x + 1/x) = f(x) + f(1/x), \qquad x \in \mathbb{R}^+ \quad (:= (0, +\infty))?$$

We give another proof of Lajkó's conjecture that $(*)$ is the only solution of (1) in the class of formal power series, some remarks on the general solution of the equation

$$(2)\qquad f(x + \alpha(x)) = f(x) + f(\alpha(x)), \quad \alpha(\alpha(x)) = x, \quad x \in \mathbb{R}^+,$$

and a theorem on the form of solutions of the equation

$$(3)\qquad f(\phi(x) + \psi(x)) = f(\phi(x)) + f(\psi(x)), \quad x \in \mathbb{R}^+,$$

with some specified $\phi$ and $\psi$.

The equations (1)–(3) are Cauchy's equations restricted to a graph of a given function which were recently studied by many authors, cf., e.g., W. Jarczyk [2] and the references quoted therein. In particular, equation (2) has been dealt with by J. Matkowski and M. Sablik [4], cf. the last section of our paper.

**1.** Let $\mathcal{F}$ be the linear space (over $\mathbb{R}$) of all formal power series

$$\mathcal{F} := \Big\{ f(x) = \sum_{k=-\infty}^{\infty} a_k x^k,\ a_k \in \mathbb{R} \Big\}$$

and consider the mapping $F\colon \mathcal{F} \to \mathcal{F}$, given by

$$[F(f)](x) = f(x + 1/x) - f(x) - f(1/x),\ f \in \mathcal{F},\ x \in \mathbb{R}^+.$$

The mapping $F$ is linear and solving (1) means determining ker $F$. We have $F(1) = -1$, $F(x) = 0$, $F(x^2) = 2$, so that the series $(*)$ belongs to ker $F$. Let us examine $F(x^k)$ where $k \in \mathbb{Z} \setminus \{0, 1, 2\}$. Since, for $k \in \mathbb{N} \setminus \{0, 1, 2\}$ we have

$$F(x^k) = 2x^k + 2x^{-k} + \sum_{i=1}^{k-1} \binom{k}{i} x^{2i-k}$$

and for $k = -m$, $m \in \mathbb{N}$,

$$F(x^{-m}) = \Big( \sum_{i=0}^{\infty} x^{2i+1} \Big)^m + x^{-m} + x^m,$$

we see that any system $\{F(x^{k_1}), \dots, F(x^{k_p})\}$, $k_i \in \mathbb{Z} \setminus \{0, 1, 2\}$ is linearly independent over $\mathbb{R}$. Thus the series $(*)$ are the only solutions of

(1) in $\mathcal{F}$.

**2.** We shall deal with equation (2) under the following hypotheses:

(H) The function $\alpha \colon \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous, strictly decreasing involution ($\alpha \circ \alpha = \mathrm{id}$), mapping bijectively $\mathbf{R}^+$ onto itself, and the function $g \colon \mathbb{R}^+ \to \mathbb{R}^+$, given by

$$g(x) := x + \alpha(x), \ x \in R^+,$$

is strictly increasing on $[c, +)$, where $c = \alpha(c)$.

Let us observe that $c$ is the only fixed point of $\alpha$, we have $\lim\limits_{x-0} \alpha(x) = +\infty$, $\lim\limits_{x-\infty} \alpha(x) = 0$, $g$ maps $[c, +\infty)$ bijectively onto $[2c, +\infty)$, and the sequence

$$x_n = g^n(c), \quad n \in \mathbb{N} \cup \{0\}$$

(where $g^n$ denotes the $n$-th iterate of the function $g$) is strictly increasing and unbounded, so that

$$[2c, +\infty) = \bigcup_{k=1} [x_k, x_{k+1}). \tag{4}$$

Let $g_i := g \mid_{[x_{i-1}, x_i)}$. Thus $g_i$ is an increasing continuous bijective function from $[x_{i-1}, x_i)$ onto $[x_i, x_{i+1})$. Let us put

$$h_i := g_i^{-1}.$$

Therefore $h_i \colon [x_{i-1}, x_i) \to [x_i, x_{i+1})$.

We have the following

**Theorem 1.** *Under hypotheses* (H), *if* $f_0 \colon (0, 2c) \to \mathbb{R}$ *is an arbitrary function, then the function*

$$f(x) = \begin{cases} f_0(x) & \text{if } x \in (0, 2c), \\ f_0 \circ h_1 \circ \ldots \circ h_i(x) + f_0 \circ \alpha \circ h_1 \circ \ldots \circ h_i(x) + f_0 \circ \alpha \circ h_2 \circ \\ \circ \ldots \circ h_i(x) + \ldots + f_0 \circ \alpha \circ h_i(x) & \text{if } x \in [x_i, x_{i+1}), \ i \in \mathbb{N} \end{cases} \tag{5}$$

*is a solution of equation* (2). *If* $f_0$ *is continuous then so is* $f$.

**Proof.** a) $x \in [c, +\infty)$. Let first $x \in [c, 2c)$. Then $g(x) = g_1(x) \in [x_1, x_2)$, $\alpha(x) \in (0, c)$ and from (4) we get

$$f(x + \alpha(x)) = f \circ g(x) = f_0 \circ h_1 \circ g(x) + f_0 \circ \alpha \circ h_1 \circ g(x) =$$
$$= f_0(x) + f_0(\alpha(x)) = f(x) + f(\alpha(x))$$

and (1) is satisfied. Suppose now that $f$ given by (4) satisfies (2) whenever $x \in [x_0, x_n)$ and let $x \in [x_n, x_{n+1})$. We have $g_{n+1}(x) = x + \alpha(x) \in [x_{n+1}, x_{n+2})$, and from (5) we obtain (since $h_{n+1} \circ g_{n+1} = \mathrm{id}$)

$$f(x+\alpha(x)) = f \circ g_{n+1}(x) = f_0 \circ h_1 \circ \ldots \circ h_n(x) + f_0 \circ \alpha \circ h_1 \circ$$
$$\circ \ldots \circ h_n(x) + \ldots + f_0 \circ \alpha \circ h_n(x) + f_0 \circ \alpha(x) = f(x) + f \circ \alpha(x).$$

Induction completes the proof in the case where $x \in [c, +\infty)$.

b) $x \in (0, c)$. Then $y = \alpha(x) \in (c, +\infty)$. Moreover $\phi(y) = \phi \circ \circ \phi(x) = x$, since $\phi$ is an involution. Thus we may apply case a) and write

$$f(x+\alpha(x)) = f(\alpha(y)+y) = f(\alpha(y)) + f(y) = f(x) + f(\alpha(x)).$$

The continuity of $f$ follows from (H) and the continuity of $f_0$. $\Diamond$

Now we are going to prove that the extension of $f_0$ to $f$, given by (5), is unique.

**Theorem 2.** *Let* (H) *be satisfied and let* $f_0 \colon (0, 2c) \to \mathbb{R}$ *be any function. There exists the unique solution* $f$ *of equation* (2) *which coincides with* $f_0$ *on* $(0, 2c)$.

**Proof.** Suppose we are given two solutions: $f$ and $f^*$ of (2) such that

$$f(x) = f^*(x) \quad \text{for} \quad x \in (0, 2c) \tag{6}$$

and that there is a $t \in (0, 2c)$ such that $f(t) \neq f^*(t)$. Because of (4), $x \in [x_i, x_{i+1})$ for some $i \in \mathbb{N}$, thus $h(t) \in [x_{i-1}, x_i)$ and

$$t = g_i \circ h_i(t) = h_i(t) + \alpha \circ h_i(t) \tag{7}$$

and $\alpha \circ h_i(t) \in (0, c)$. We now use (2) and (7) to get

$$f(t) = f \circ h_i(t) + f \circ \alpha \circ h_i(t) \neq f^*(t) = f^* \circ h_i(t) + f^* \circ \alpha \circ h_i(t).$$

But the second terms here are equal, because of (5), so we end up with $f \circ h_i(t) \neq f^* \circ h_i(t)$. Now, by the same argument with $h_i(t)$ in place of $t$ we arrive at $f \circ h_{i-1} \circ h_i(t) \neq f^* \circ h_{i-1} \circ h_i(t)$ and eventually at $f \circ h_1 \circ \ldots \circ h_i(t) \neq f^* \circ h_1 \circ \ldots \circ h_i(t)$. Since $h_1 \circ \ldots \circ h_i(t)$ here belongs to $[x_0, x_1] = [c, 2c) \subset (0, 2c)$, we get a contradiction with (5). $\Diamond$

As a consequence of this theorem we get the following

**Corollary**. *If* (H) *holds then every solution of equation* (2) *is given by the construction* (4).

**Proof.** Indeed, let $f^*$ be a solution of (2). Take the solution $f$ of (2) given by (4) with $f_0 = f^* \mid_{(0,2c)}$. According to Th. 2, since $f$ and $f^*$ coincide on $(0, 2c)$, there is $f = f^*$. $\Diamond$

**Remarks**. (1) The interval $(0, 2c)$ is the maximal set on which one may arbitrarily prescribe a solution to (2). Indeed, given any set $M \subset (0, 2c)$ let us take an $x_0 \in (0, 2c) \setminus M$ such that $f(x_0)$ can be determined with the use of the values of $f$ given on $M$. Since $f$ satisfies (2), the following may happen: either there is an $x$ such that $x_0 = x + \alpha(x) = g(x) < < 2c$, contrary to $g(x) \geq 2c$; or (2) is satisfied either with $x_0$, then $x_0 + \alpha(x_0) > 2c$ or for an $y$ such that $x_0 = \alpha(y)$, then $y + \alpha(y) > 2c$. In the latter case both arguments do not belong to $M$, so that the value of $f$ is not defined, contrary to the hypothesis.

(2) Since the function $\alpha(x) = 1/x$, $x \in \mathbb{R}^+$, satisfies hypothesis (H), Th. 1 determines also the general solution of equation (1).

(3) The solutions of (2) defined on $(-\infty, 0)$, respectively on $\mathbb{R}\setminus\{0\}$, can be described in a similar way as those defined on $\mathbb{R}^+$.

**3.** The following result by J. Matkowski and M. Sablik ([4], Th. 4) yields another construction of the general solution to (2) under more general assumptions than (H).

**Proposition.** *Let $\alpha\colon \mathbb{R}^+ \to \mathbb{R}^+$ be an involution satisfying $\alpha(c) = c$ and $\alpha((0,c)) \subset (c, +\infty)$ and $\alpha((c,+\infty)) \subseteq (0,c)$. Then every function $f_0\colon (c,+\infty) \to \mathbb{R}$ such that*

$$f_0(2c) = 2f_0(c) \tag{7}$$

*can uniquely be extended to a solution $f\colon \mathbb{R}^+ \to \mathbb{R}$ of equation* (2). *Moreover, if $\alpha$ and $f_0$ are continuous then so is $f$.*

An analogous result can be obtained for equation (3), i.e., for the Cauchy equation on the graph of a parametrically given curve $(\phi(x), \psi(x))$.

**Theorem 3.** *Let $\phi, \psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ satisfy $\phi(c) = c$, $\phi((0,c)) \subset (c,+\infty)$, $\phi((c,+\infty)) \subset (0,c)$ and*

(8) *$\phi \circ \psi(x) = \phi(x)$ for $x \in (0,c]$ and $\psi \circ \psi(x) = \phi(x)$ for $x \in (c,+\infty)$.*

*Then every function $f_0\colon [c,+\infty) \to \mathbb{R}$ satisfying* (7) *can uniquely be extended to a solution $f\colon \mathbb{R}^+ \to \mathbb{R}$ of equation* (3). *Moreover, if $\phi$, $\psi$ and $f_0$ are continuous then so is $f$.*

**Proof.** Given an $f_0$ as claimed we define $f$ as follows

$$f(x) = \begin{cases} f_0(x + \phi(x)) - f_0(\phi(x)) & \text{if } x \in (0,c), \\ f_0(x) & \text{if } x \in [c,+\infty) \end{cases} \tag{9}$$

The function $f$ is well defined since $x \in (0,c)$ implies $\phi(x) > c$ and $x + \phi(x) > c$. If $x \in (0,c)$, then from (8) we get $\psi(x) \in (0,c)$ so that $\phi(x) + \psi(x) \in (c,+\infty)$ and

$$f \circ \phi(x) + f \circ \psi(x) = f_0 \circ \phi(x) + f_0(\psi(x) + \phi \circ \psi(x)) - f_0 \circ \phi \circ \psi(x) =$$
$$= f_0(\phi(x) + \psi(x)) = f(\phi(x) + \psi(x)).$$

If $x \in (c, +\infty)$, then $\phi(x) \in (0, c)$, $\phi \circ \phi(x) \in (c, +\infty)$, whence $\psi(x) \in$ $\in (c, +\infty)$, $\phi(x) + \psi(x) \in (c, +\infty)$. Therefore

$$f \circ \phi(x) + f \circ \psi(x) = f_0\phi(x) + \phi \circ \phi(x)) - f_0 \circ \phi \circ \phi(x) + f_0 \circ \psi(x) =$$
$$= f_0(\phi(x) + \psi(x)) = f(\phi(x) + \psi(x)).$$

For $x = c$, (3) results from (7).

If the functions involved in its definition are continuous, then the continuity of $f$ given by (9) is obvious for $x \neq c$, whereas for $x = c$ it results from (7) and (9). $\Diamond$

**Concluding remark.** The question (cf. [1]) under what conditions equation (2) has linear solutions only (or a finite-parameter family of solutions) remains unanswered. In this connection, during the XXXI International Symposium on Functional Equations (August 1993, Debrecen), J. Matkowski proposed the following, more adequate, problem:

Consider the system of functional equations (2) with two given involutions and establish conditions under which the only solution to the system is the identity function.

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