Mathematica Pannonica
5/2 (1994), 177 – 196

SUBSTRUCTURES AND RADICALS OF MORITA CONTEXTS FOR NEAR-RINGS AND MORITA NEAR-RINGS

Stefan **VELDSMAN**

Department of Mathematics, University of Port Elizabeth, PO Box 1600, 6000 Port Elizabeth, South Africa

Received February 1994

AMS Subject Classification: 16 Y 30

Keywords: Near-ring, morita context, radical.

Abstract: The relationships between certain substructures of a morita context for near-rings and the associated morita near-ring is determined. This is then used to determine the relationship between the radicals of the two near-rings in the morita context and the radical of the associated morita near-ring.

1. Introduction and preliminaries

Morita contexts have proved to be a useful tool in ring theory in determining the transfer of structural properties between two rings, especially as far as the radicals are concerned, see for example Amitsur [1] and Sands [6]. In [3] we have defined morita contexts for near-rings and in [4] we showed that two much studied cases from the theory of near-rings can be accommodated in this setting and how the tools provided by the morita context facilitates their investigation. The two cases referred to are, firstly, the transfer of structural properties between a (right) ring module and the associated near-ring of homogeneous maps on the group and secondly, that of a near-ring and the associated matrix near-ring. Here, in Section 3, we study the relationships between the radicals of the two near-rings L and R from a morita context for near-rings $\Gamma = (L, G, H, R)$ and the radical of the associated morita near-ring

$M_2(\Gamma)$. We give explicit conditions which ensures that the radical of $M_2(\Gamma)$ can be expressed in terms of the radicals L and R – initially having to first determine the relationship between the radicals of L and R. In doing this, we had to establish various relationships between some substructures of the morita context and the morita near-ring (Section 2). But firstly we have to recall some relevant definitions and earlier results.

All near-rings will be right distributive and 0-symmetric. Let R and L be near-rings and let G be a group. G is a *L-R-bimodule* if there are functions

$$L \times G \to G, (x, g) \longmapsto xy \quad \text{ and } \quad G \times R \to G,\ (g, r) \longmapsto gr$$

such that $(x_1 + x_2)g = x_1g + x_2g$, $(g_1 + g_2)r = g_1r + g_2r$, $(x_1x_2)g = = x_1(x_2g)$, $(gr_1)r_2 = g(r_1r_2)$ and $(xg)r = x(gr)$ for all $x, x_1, x_2 \in L$, $g, g_1, g_2 \in G$, $r, r_1, r_2 \in R$. (Strictly speaking we should call G a near-ring L-R-bimodule, for even if both L and R are rings, G need not be a ring bimodule.) A normal subgroup K of G, G a L-R-bimodule, is an *ideal* of G if

$$KR := \{kr \mid k \in K, r \in R\} \subseteq K \quad \text{ and }$$
$$L * K := \{x(g + k) - xg \mid x \in L, g \in G, k \in K\} \subseteq K.$$

Let $N_2 := \{1, 2\}$. For $i \in N_2$, we use i_c to denote the complement of i in N_2. For each $i, j \in N_2$, let Γ_{ij} be a group. The quadruple $\Gamma = (\Gamma_{11}, \Gamma_{12}, \Gamma_{21}, \Gamma_{22})$ is a *morita context* (for near-rings) if for every $i, j, k \in N_2$ there is a function

$$\Gamma_{jk} \times \Gamma_{ki} \to \Gamma_{ji},\ \ (x, y) \longmapsto xy$$

which satisfies $(a + b)c = ac + bc$ and $(db)e = d(be)$ for all $a, b \in \Gamma_{jk}$, $c \in \Gamma_{ki}$, $d \in \Gamma_{ij}$ and $e \in \Gamma_{km}$. It is clear that for each $i, j \in N_2$, Γ_{ij} is a Γ_{ii}-Γ_{jj}-bimodule and, in particular, for $i = j$, Γ_{ii} is a near-ring. As agreed earlier on, we only consider 0-symmetric near-rings. Hence we should add the requirement that $a0 = 0$ for each $a \in \Gamma_{ii}$, $i = 1, 2$. Then $x0_{jk} = 0_{ik}$ for all $x \in \Gamma_{ij}$, $i, j, k \in N_2$. We will usually not write the subscripts in 0_{ij}. It is clear that if $\Gamma = (\Gamma_{11}, \Gamma_{12}, \Gamma_{21}, \Gamma_{22})$ is a morita context, then so is $(\Gamma_{22}, \Gamma_{21}, \Gamma_{12}, \Gamma_{11})$, the one being called the dual of the other. Often, for a fixed $i \in N_2$, we will thus talk about the morita context $(\Gamma_{ii}, \Gamma_{ii_c}, \Gamma_{i_ci}, \Gamma_{i_ci_c})$. For each $i, j \in N_2$, let $\Delta_{ij} \subseteq \Gamma_{ij}$. The quadruple $\Delta = (\Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{22})$ is an *ideal of the morita context* $\Gamma = (\Gamma_{11}, \Gamma_{12}, \Gamma_{21}, \Gamma_{22})$ if each Δ_{ij} is a normal subgroup of Γ_{ij}, $\Delta_{ij}\Gamma_{jk} \subseteq \Delta_{ik}$ and $\Gamma_{ki} * \Delta_{ij} := \{a(b+c)-ab \mid a \in \Gamma_{ki}, b \in \Gamma_{ij}, c \in$ $\in \Delta_{ij}\} \subseteq \Delta_{kj}$ for all $i,j,k \in N_2$. In this case we get the quotient morita context

$$\frac{\Gamma}{\Delta} := \left[\frac{\Gamma_{11}}{\Delta_{11}}, \frac{\Gamma_{12}}{\Delta_{12}}, \frac{\Gamma_{21}}{\Delta_{21}}, \frac{\Gamma_{22}}{\Delta_{22}}\right]$$

where the relevant maps

$$\frac{\Gamma_{ij}}{\Delta_{ij}} \times \frac{\Gamma_{jk}}{\Delta_{jk}} \to \frac{\Gamma_{ik}}{\Delta_{ik}}$$

are defined by

$$(x + \Delta_{ij}, y + \Delta_{jk}) \longmapsto xy + \Delta_{ik}.$$

For the morita context $\Gamma = (\Gamma_{11}, \Gamma_{12}, \Gamma_{21}, \Gamma_{22})$, let $\Gamma^+ = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix}$ be the associated matrix group. Let $\pi_{ij} \colon \Gamma^+ \to \Gamma_{ij}$ and $\tau_{ij} \colon \Gamma_{ij} \to \Gamma^+$ be the (i,j)-th projection and (i,j)-th injection respectively. For $i \in N_2$, let $\Gamma_{i1} \oplus \Gamma_{i2}$ be the direct sum of the two groups Γ_{i1} and Γ_{i2} and let $\pi_i \colon \Gamma^+ \to \Gamma_{i1} \oplus \Gamma_{i2}$ and $\tau_i \colon \Gamma_{i1} \oplus \Gamma_{i2} \to \Gamma^+$ be the obvious projection and injection respectively. Let $u_{ij} \colon \Gamma_{ij} \to \mathrm{Map}\,(\Gamma_{j1} \oplus \Gamma_{j2}, \Gamma_{i1} \oplus \Gamma_{i2})$ be defined by

$$u_{ij}(x) = u^x_{ij} \colon \Gamma_{ji} \oplus \Gamma_{j2} \to \Gamma_{i1} \oplus \Gamma_{i2}, \quad u^x_{ij}(a_1, a_2) := (xa_1, xa_2).$$

Finally, for each $i,j \in N_2$, $x \in \Gamma_{ij}$, let $s^x_{ij} := \tau_i \circ u^x_{ij} \circ \pi_j$.

The *morita near-ring determined by* Γ, denoted by $M_2(\Gamma)$, is the subnear-ring of

$$\mathrm{Map}\,(\Gamma^+, \Gamma^+) := \{f \colon \Gamma^+ \to \Gamma^+ \mid f \text{ a function with } f(0) = 0\}$$

generated by $\{s^x_{ij} \mid x \in \Gamma_{ij},\, i,j \in N_2\}$. $M_2(\Gamma)$ is a 0-symmetric near-ring which has an identity $s^1_{11} + s^1_{22}$ if both Γ_{11} and Γ_{22} have identities (here 1 denotes both the identity of Γ_{11} and Γ_{22}). A proof technique which is quite useful when dealing with elements of $M_2(\Gamma)$ is "induction on the weight $w(u)$ of $U \in M_2(\Gamma)$". The weight of $U \in M_2(\Gamma)$, written as $w(u)$, is the smallest number of s^x_{ij} needed to represent U. If Γ_d denotes the dual of the morita context Γ, then $M_2(\Gamma) \cong M_2(\Gamma_d)$. Some useful facilities for doing calculations in $M_2(\Gamma)$ are (cf. [3]):

1.1 $$s^x_{ij} + s^y_{ij} = s^{x+y}_{ij};$$

1.2 $$s^x_{ij} + s^y_{km} = s^y_{km} + s^x_{ij} \quad \text{if} \quad i \neq k;$$

1.3 $$s_{ij}^x s_{km}^y = \begin{cases} s_{im}^{xy} & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases};$$

1.4 $$s_{ij}^x (s_{1k_1}^{y_1} + s_{2k_2}^{y_2}) = s_{ik_j}^{xy_j};$$

1.5 $$\text{for any } U \in M_2(\Gamma),\ U \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = U \begin{bmatrix} a_{11} & 0 \\ a_{21} & 0 \end{bmatrix} + U \begin{bmatrix} 0 & a_{12} \\ 0 & a_{22} \end{bmatrix};$$

1.6 $$\text{for any } U, V \in M_2(\Gamma),\ U \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} + V \begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix} = V \begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix} + V \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix};$$

1.7 $$\text{for } k \in N_2,\ C_k := \{ s_{1k}^{x_1} + s_{2k}^{x_2} \mid x_1 \in \Gamma_{1k}, x_2 \in \Gamma_{2k} \}$$ is a left invariant subgroup of $M_2(\Gamma)$;

1.8 $$\text{for } U \in M_2(\Gamma), U \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$ if and only if $U(s_{1i}^{a_{1i}} + s_{2i}^{a_{2i}}) = s_{1i}^{b_{1i}} + s_{2i}^{b_{2i}}$ for all $i \in N_2$.

All our consideration to follow, will be in what we call a standard morita context. A morita context $\Gamma = (\Gamma_{11}, \Gamma_{12}, \Gamma_{21}, \Gamma_{22})$ is a *standard morita context* if both Γ_{11} and Γ_{22} have identities, all bimodules in Γ are unital (both left and right) and $\overline{\Gamma_{jj_c}\Gamma_{j_cj}} = \Gamma_{jj}$ for all $j \in N_2$ where $\overline{\Gamma_{jj_c}\Gamma_{j_cj}}$ denotes the subgroup of Γ_{jj} generated by $\Gamma_{jj_c}\Gamma_{j_cj}$. Some useful consequences are:

1.9 For all $j, k \in N_2$ and $x \in \Gamma_{jk}$, $x \in (\overline{\Gamma_{jj_c}\Gamma_{j_cj}})x$;

1.10 $\Gamma_{jn} = \overline{\Gamma_{jk}\Gamma_{kn}}$ for all $j, k, n \in N_2$;

1.11 if $x \in \Gamma_{jk}$ and A is a subgroup of Γ_{jk}, then $\Gamma_{jj_c}\Gamma_{j_cj}x \subseteq A$ implies $x \in A$;

1.12 for $x \in \Gamma_{jk}$, if $\Gamma_{nj}x \subseteq \Delta_{nk}$, then $x \in \Delta_{nk}$ where $(\Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{22})$ is an ideal of Γ.

Finally, it is clear that if Γ is a standard morita context, then so is its dual as well as $\frac{\Gamma}{\Delta}$ for any ideal Δ of Γ.

In all that follows, we assume that the morita contexts under discussion are all standard morita contexts.

2. Substructures of morita contexts and morita near-rings

Let $\Delta = (\Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{22})$ be an ideal of the morita context $\Gamma = (\Gamma_{11}, \Gamma_{12}, \Gamma_{21}, \Gamma_{22})$. Then $\Delta^* := \{U \in M_2(\Gamma) \mid U\Gamma^+ \subseteq \Delta^+ +\}$ is an ideal of the near-ring $M_2(\Gamma)$ where Δ^+ is the matrix group $\Delta^+ = \begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{bmatrix}$. Let $\mathcal{T}$ be an ideal of the morita near-ring $M_2(\Gamma)$. For each $i, j \in N_2$, let $\mathcal{T}_{ij} = \{x \in \Gamma_{ij} \mid S_{ij}^x \in \mathcal{T}\}$. Then $\mathcal{T}_* := := (\mathcal{T}_{11}, \mathcal{T}_{12}, \mathcal{T}_{21}, \mathcal{T}_{22})$ is an ideal of the morita context Γ. If $U \in \mathcal{T}$, and $U \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, then $s_{ij}^{b_{ij}} \in \mathcal{T}$ for all $i, j \in N_2$. It is clear that if $\mathcal{T}^1$ and $\mathcal{T}^2$ are ideals of $M_2(\Gamma)$ with $\mathcal{T}^1 \subseteq \mathcal{T}^2$, then $(\mathcal{T}^1)_* \subseteq (\mathcal{T}^2)_*$. Moreover, if $\mathcal{T}^\alpha$ is an ideal of $M_2(\Gamma)$ where each α is from some index set, then $(\bigcap_\alpha \mathcal{T}^\alpha)_* = \bigcap_\alpha ((\mathcal{T}^\alpha)_*)$. For any ideal Δ of Γ, $(\Delta^*)_* = \Delta$ and if $\mathcal{T}$ is an ideal of $M_2(\Gamma)$, then (in general) only $\mathcal{T} \subseteq (\mathcal{T}_*)^*$ holds. If $\mathcal{T} = (\mathcal{T}_*)^*$, then $\mathcal{T}$ is called a *full ideal* of $M_2(\Gamma)$. From the above and [3], we have

2.1 Proposition.

(1) *An ideal $\mathcal{T}$ of $M_2(\Gamma)$ is full if and only if it satisfies:*

$$UM_2(\Gamma)s_{kk}^1 \subseteq \mathcal{T} \text{ for all } k \in N_2 \text{ implies } U \in \mathcal{T}.$$

(2) *There is a one-to-one correspondence, which preserves inclusions and intersections, between the ideals of the morita context Γ and all the full ideals of the associated morita near-ring $M_2(\Gamma)$ given by $\Delta \longmapsto \Delta \longmapsto \Delta^* \longmapsto (\Delta^*)_* = \Delta$.* $\Diamond$

Let Δ_{ij} be an ideal of the Γ_{ii}-Γ_{jj}-bimodule Γ_{ij}. For $k \in N_2$, let $\Delta_{ij}\Gamma_{kj}^{-1} := \{x \in \Gamma_{ik} \mid x\Gamma_{kj} \subseteq \Delta_{ij}\}$; it is an ideal of the Γ_{ii}-Γ_{kk}-bimodule Γ_{ik}. Let $\Gamma_{ik}^{-1}\Delta_{ij}$ be the ideal of the Γ_{kk}-Γ_{jj}-bimodule Γ_{kj} generated by $\Gamma_{ki} * \Delta_{ij}$. Part of the next result follows from [4]:

2.2 Proposition. *Let $i \in N_2$ be fixed. Let $\Gamma = (\Gamma_{ii}, \Gamma_{ii_c}, \Gamma_{i_ci}, \Gamma_{i_ci_c})$ be a* (as usual) *standard morita context. For each $j \in N_2$, let Δ_{jj} be an ideal of the near-ring Γ_{jj}. Let*

$$\Delta_i = (\Delta_{ii}, \Delta_{ii}\Gamma_{i_ci}^{-1}, \Gamma_{ii_c}^{-1}\Delta_{ii}, (\Gamma_{ii_c}^{-1}\Delta_{ii})\Gamma_{i_ci}^{-1})$$

and let

$$\Delta_{i_c} = ((\Gamma_{i_c i}^{-1}\Delta_{i_c i_c})\Gamma_{ii_c}^{-1}, \Gamma_{i_c i}^{-1}\Delta_{i_c i_c}, \Delta_{i_c i_c}\Gamma_{ii_c}^{-1}, \Delta_{i_c i_c}).$$

Then:

(1) *For every $j \in N_2$, Δ_j is an ideal of Γ if and only if $\Gamma_{jj_c} * \Gamma_{jj_c}^{-1}\Delta_{jj} \subseteq \subseteq \Delta_{jj}$. In case $\Gamma_{jj_c} * \Gamma_{jj_c}^{-1}\Delta_{jj} \subseteq \Delta_{jj}$, there is no need to insert brackets in $(\Gamma_{jj_c}^{-1}\Delta_{jj})\Gamma_{j_c j}^{-1}$, since $(\Gamma_{jj_c}^{-1}\Delta_{jj})\Gamma_{j_c j}^{-1} = \Gamma_{jj_c}^{-1}(\Delta_{jj}\Gamma_{j_c j}^{-1})$.*

(2) *If $\Delta_i = \Delta_{i_c}$, then Δ_i is an ideal of Γ.*

Proof. (1) follows from [4].

(2): From (1) above, we need $\Gamma_{ii_c} * \Gamma_{ii_c}^{-1}\Delta_{ii} \subseteq \Delta_{ii}$. But our assumption reduces our need to $\Gamma_{ii_c} * \Delta_{i_c i_c}\Gamma_{ii_c}^{-1} \subseteq (\Gamma_{i_c i}^{-1}\Delta_{i_c i_c})\Gamma_{ii_c}^{-1}$ i.e., we need

$$(\Gamma_{ii_c} * \Delta_{i_c i_c}\Gamma_{ii_c}^{-1})\Gamma_{ii_c} \subseteq \Gamma_{i_c i}^{-1}\Delta_{i_c i_c}$$

where the latter is the ideal of the Γ_{ii}-$\Gamma_{i_c i_c}$-bimodule generated by $\Gamma_{ii_c} * \Delta_{i_c i_c}$. Now $(\Gamma_{ii_c} * \Delta_{i_c i_c}\Gamma_{ii_c}^{-1})\Gamma_{ii_c} \subseteq \Gamma_{ii_c} * ((\Delta_{i_c i_c}\Gamma_{ii_c}^{-1})\Gamma_{ii_c})\Gamma_{ii_c}) \subseteq \subseteq \Gamma_{ii_c} * \Delta_{i_c i_c} \subseteq \Gamma_{i_c i}^{-1}\Delta_{i_c i_c}$. ◊

For any $j \in N_2$, let $\mathcal{S}^*(\Gamma_{jj}) := \{\Delta_{jj} \subseteq \Gamma_{jj} \mid \Delta_{jj}$ is an ideal of Γ_{jj} for which $\Gamma_{jj_c} * \Gamma_{jj_c}^{-1}\Delta_{jj} \subseteq \Delta_{jj}$ and for $x \in \Gamma_{jj}$, $x\Gamma_{jj_c}\Gamma_{j_c j} \subseteq \Delta_{jj}$ implies $x \in \Delta_{jj}\}$. In [4] it was shown that this class of ideals is closed under intersections and

2.3 Proposition [4]. *There is a one-to-one correspondence, which preserves inclusions and intersections, between $\mathcal{S}^*(\Gamma_{ii})$ and $\mathcal{S}^*(\Gamma_{i_c i_c})$ given by*

$$\Delta_{ii} \longmapsto \Gamma_{ii_c}^{-1}\Delta_{ii}\Gamma_{i_c i}^{-1} \longmapsto \Gamma_{i_c i}^{-1}(\Gamma_{ii_c}^{-1}\Delta_{ii}\Gamma_{i_c i}^{-1})\Gamma_{ii_c}^{-1} = \Delta_{ii}.$$

Recall, an ideal I of a near-ring N is a 2-*semiprime ideal* if for any left invariant subgroup A of N, $A^2 \subseteq I$ implies $A \subseteq I$. I is a 3-*semiprime ideal* if $xNx \subseteq I$ implies $x \in I$. The near-ring N is 2-semiprime (resp. 3-semiprime) if 0 is a 2-semiprime (resp. 3-semiprime) ideal of N. Any intersection of 2-semiprime (3-semiprime) ideals is 2-semiprime (3-semiprime). It is clear that any 3-semiprime near-ring is 2-semiprime; our interest here in 2-semiprime near-rings is mainly because of the following which is easy to verify:

2.4 Proposition. *Let N be a near-ring with identity. An ideal I of N is 3-semiprime if and only if it is 2-semiprime.* ◊

2.5 Proposition. *Any 3-semiprime (= 2-semiprime) ideal $\mathcal{T}$ of $M_2(\Gamma)$ is full.*

Proof. We use 2.1(1) above. Let $U \in M_2(\Gamma)$ and suppose $UM_2(\Gamma)s_{kk}^1 \subseteq \mathcal{T}$ for all $k \in N_2$. Then $(s_{kk}^1 U)M_2(\Gamma)(s_{kk}^1 U) \subseteq \mathcal{T}$ and hence $s_{kk}^1 U \in \mathcal{T}$ for all $k \in N_2$. Then $U = (s_{11}^1 + s_{22}^1)U = s_{11}^1 U + s_{22}^1 U \in \mathcal{T}$. ◊

Let $\mathcal{S}_p^*(\Gamma_{jj}) := \{\Delta_{jj} \subseteq \Gamma_{jj} \mid \Delta_{jj}$ is a 3-semiprime ideal of Γ_{jj} which satisfies $\Gamma_{jj_c} * \Gamma_{jj_c}^{-1}\Delta_{jj} \subseteq \Delta_{jj}\}$. From [4] we need two more results:

2.6 Proposition [4]. *Let Δ_{jj} be a 3-semiprime ideal of Γ_{jj}. Then*

(1) $$x\Gamma_{jj_c}\Gamma_{j_cj} \subseteq \Delta_{jj} \text{ implies } x \in \Delta_{jj};$$

(2) $$(\Gamma_{jj_c}^{-1}\Delta_{jj})\Gamma_{j_cj}^{-1} \text{ is a 3-semiprime ideal of } \Gamma_{j_cj_c}.$$

2.7 Proposition [4]. *There is a one-to-one correspondence, which preserves inclusions and intersections between $\mathcal{S}_p^*(\Gamma_{ii})$ and $\mathcal{S}_p^*(\Gamma_{i_ci_c})$* (given by the same map as in 2.3 above).

2.8 Proposition. *Let $i \in N_2$ be fixed. Let $\Delta = (\Delta_{ii}, \Delta_{ii_c}, \Delta_{i_ci}, \Delta_{i_ci_c})$ be an ideal of $\Gamma = (\Gamma_{ii}, \Gamma_{ii_c}, \Gamma_{i_ci}, \Gamma_{i_ci_c})$. Then*

(1) $$\Delta_{ii_c} \subseteq \Delta_{ii}\Gamma_{i_ci}^{-1} = \{x \in \Gamma_{ii_c} \mid x\Gamma_{i_ci} \subseteq \Delta_{ii}\};$$

(2) $$\Delta_{i_ci} = \Gamma_{ii_c}^{-1}\Delta_{ii} = \{x \in \Gamma_{i_ci} \mid \Gamma_{ii_c}x \subseteq \Delta_{ii}\} \text{ and } \Gamma_{ii_c} * \Gamma_{ii_c}^{-1}\Delta_{ii} \subseteq \Delta_{ii};$$

(3) $$\Delta_{i_ci_c} \subseteq \Gamma_{ii_c}^{-1}\Delta_{ii}\Gamma_{i_ci}^{-1} = \{x \in \Gamma_{i_ci_c} \mid \Gamma_{ii_c}x\Gamma_{i_ci} \subseteq \Delta_{ii}\};$$

(4) $$(\Delta_{ii}, \Delta_{ii_c}, \Delta_{i_ci}, \Delta_{i_ci_c}) = (\Delta_{ii}, \Delta_{ii}\Gamma_{i_ci}^{-1}, \Gamma_{ii_c}^{-1}\Delta_{ii}, \Gamma_{ii_c}^{-1}\Delta_{ii}\Gamma_{i_ci}^{-1})$$ *if and only if for $x \in \Gamma_{ii_c}$, $x\Gamma_{i_ci} \subseteq \Delta_{ii}$ implies $x \in \Delta_{ii_c}$.*

Proof. (1): Let $x \in \Delta_{ii_c}$. Then $x\Gamma_{i_ci} \subseteq \Delta_{ii_c}\Gamma_{i_ci} \subseteq \Delta_{ii}$; hence $x \in \Delta_{ii}\Gamma_{i_ci}^{-1} = \{x \in \Gamma_{ii_c} \mid x\Gamma_{i_ci} \subseteq \Delta_{ii}\}$ (by definition).

(2): Let $x \in \Delta_{i_ci}$. Then $\Gamma_{i_ci}\Gamma_{ii_c}x \subseteq \Gamma_{i_ci}(\Gamma_{ii_c}\Delta_{i_ci}) \subseteq \Gamma_{i_ci}\Delta_{ii} \subseteq \Gamma_{i_ci} * \Delta_{ii} \subseteq \Gamma_{i_ci}^{-1}\Delta_{ii}$ by the definition of the latter. By 1.11 we have $x \in \Gamma_{ii_c}^{-1}\Delta_{ii}$. Since $\Gamma_{i_ci} * \Delta_{ii} \subseteq \Delta_{i_ci}$ and Δ_{i_ci} is an ideal of the $\Gamma_{i_ci_c}$-Γ_{ii}-bimodule Γ_{i_ci}, we get $\Gamma_{i_ci}^{-1}\Delta_{ii} \subseteq \Delta_{i_ci}$. Hence $\Delta_{i_ci} = \Gamma_{ii_c}^{-1}\Delta_{ii}$ and so $\Gamma_{ii_c} * \Gamma_{ii_c}^{-1}\Delta_{ii} = \Gamma_{ii_c} * \Delta_{i_ci} \subseteq \Delta_{ii}$. For the second equality, let $x \in \Gamma_{i_ci}$ such that $\Gamma_{ii_c}x \subseteq \Delta_{ii}$. By 1.12, $x \in \Delta_{i_ci} = \Gamma_{ii_c}^{-1}\Delta_{ii}$ follows. Conversely, if $x \in \Delta_{i_ci}$, then $\Gamma_{ii_c}x \subseteq \Gamma_{ii_c}\Delta_{i_ci} \subseteq \Delta_{ii}$.

(3): Let $x \in \Delta_{i_ci_c}$. Then $x\Gamma_{i_ci} \subseteq \Delta_{i_ci_c}\Gamma_{i_ci} \subseteq \Delta_{i_ci}$; hence $x \in \Delta_{i_ci}\Gamma_{i_ci}^{-1} = \Gamma_{ii_c}^{-1}\Delta_{ii}\Gamma_{i_ci}^{-1}$ from (2) above. The equality $\Gamma_{ii_c}^{-1}\Delta_{ii}\Gamma_{i_ci}^{-1} = \{x \in \Gamma_{i_ci_c} \mid \Gamma_{ii_c}x\Gamma_{i_ci} \subseteq \Delta_{ii}\}$ is obvious from (1) and (2) above.

(4): If the equality holds, then clearly $x \in \Gamma_{ii_c}$ with $x\Gamma_{i_ci} \subseteq \Delta_{ii}$ implies $x \in \Delta_{ii}\Gamma_{i_ci}^{-1} = \Delta_{ii_c}$. Conversely, let $x \in \Delta_{ii}\Gamma_{i_ci}^{-1}$. Then $x\Gamma_{i_ci} \subseteq \Delta_{ii}$ and by the assumption, $x \in \Delta_{ii_c}$. Thus $\Delta_{ii_c} = \Delta_{ii}\Gamma_{i_ci}^{-1}$. Let

$y \in \Gamma_{ii_c}^{-1}\Delta_{ii}\Gamma_{i_ci}^{-1} = \{a \in \Gamma_{i_ci_c} \mid \Gamma_{ii_c}a\Gamma_{i_ci} \subseteq \Delta_{ii}\}$. Then $\Gamma_{ii_c}y\Gamma_{i_ci} \subseteq \Delta_{ii}$ and by the assumption, $\Gamma_{ii_c}y \subseteq \Delta_{ii_c}$. By 1.12 $y \in \Delta_{i_ci_c}$ follows. ◊

2.9 Proposition. *Let $\mathcal{T}$ be a 3-semiprime ideal of $M_2(\Gamma)$. For $i, j, k \in N_2$ and $x \in \Gamma_{ij}$:*

$$x\Gamma_{ji}x \subseteq \mathcal{T}_{ij} \text{ implies } x \in \mathcal{T}_{ij}; \tag{1}$$

in particular, $\mathcal{T}_{ii}$ is a 3-semiprime ideal of Γ_{ii}.

$$x\Gamma_{jk}\Gamma_{kj} \subseteq \mathcal{T}_{ij} \text{ implies } x \in \mathcal{T}_{ij}. \tag{2}$$

Proof. (1): For any $U \in M_2(\Gamma)$, there is some $a_k \in \Gamma_{ki}$ $(k = i, i_c)$ by 1.7, such that $s_{ij}^xUs_{ij}^x = s_{ij}^xUs_{ii}^1s_{ij}^x = s_{ij}^xU(s_{ii}^1 + s_{i_ci_c}^0)s_{ij}^x = s_{ij}^x(s_{ii}^{a_i} + s_{i_ci_c}^{a_{i_c}})s_{ij}^x = s_{ii}^{xa_jx} \in \mathcal{T}$ since $xa_jx \in x\Gamma_{ji}x \subseteq \mathcal{T}_{ij}$. Since $\mathcal{T}$ is 3-semiprime, $s_{ij}^x \in \mathcal{T}$ and hence $x \in \mathcal{T}_{ij}$ follows.

(2): Suppose $x\Gamma_{jk}\Gamma_{kj} \subseteq \mathcal{T}_{ij}$ but $x \notin \mathcal{T}_{ij}$. By (1) above, there is a $y \in \Gamma_{ji} = \overline{\Gamma_{jk}\Gamma_{ki}}$ (cf. 1.10) such that $xyx \notin \mathcal{T}_{ij}$. Assume $y = \sum_{r=1}^{n} \sigma_r a_r b_r$ where $\sigma_r \in \{+, -\}$, $a_r \in \Gamma_{jk}$ and $b_r \in \Gamma_{ki}$. Then $a_{r_0}b_{r_0}x \notin \mathcal{T}_{jj}$ for some $r_0 \in \{1, 2, \ldots, n\}$. Once again, by (1) above (with $i = j$), there is a $u \in \Gamma_{jj}$ such that $a_{r_0}b_{r_0}x\,u\,a_{r_0}b_{r_0}x \notin \mathcal{T}_{jj}$. But $a_{r_0}b_{r_0}x\,u\,a_{r_0}b_{r_0}x = (a_{r_0}b_{r_0})x(ua_{r_0})(b_{r_0}x) \in \Gamma_{ji}x\Gamma_{jk}\Gamma_{kj} \subseteq \Gamma_{ji}\mathcal{T}_{ij} \subseteq \mathcal{T}_{jj}$; a contradiction. ◊

2.10 Proposition. *Let $i \in N_2$ be fixed and let $\Delta = (\Delta_{ii}, \Delta_{ii_c}, \Delta_{i_ci}, \Delta_{i_ci_c})$ be an ideal of $\Gamma = (\Gamma_{ii}, \Gamma_{ii_c}, \Gamma_{i_ci}, \Gamma_{i_ci_c})$. Then Δ^* is a 3-semiprime ideal of $M_2(\Gamma)$ if and only if Δ satisfies:*

$$\Delta_{ii} \text{ is a 3-semiprime ideal of } \Gamma_{ii} \text{ and} \tag{1}$$

$$x\Gamma_{i_ci} \subseteq \Delta_{ii} (x \in \Gamma_{ii_c}) \text{ implies } x \in \Delta_{ii_c}. \tag{2}$$

Proof. If $\mathcal{T} := \Delta^*$ is a 3-semiprime ideal of $M_2(\Gamma)$, then $\Delta_{ii} = \mathcal{T}_{ii}$ a 3-semiprime ideal of Γ_{ii} follows from Prop. 2.10(1). If $x \in \Gamma_{ii_c}$ such that $x\Gamma_{i_ci} \subseteq \Delta_{ii}$, then $x\Gamma_{i_ci}\Gamma_{ii_c} \subseteq \Delta_{ii}\Gamma_{ii_c} \subseteq \Delta_{ii_c} = \mathcal{T}_{ii_c}$ and by Prop. 2.10(2) we have $x \in \mathcal{T}_{ii_c} = \Delta_{ii_c}$.

Conversely, suppose (1) and (2) are satisfied. Let $U \in M_2(\Gamma)$ such that $UM_2(\Gamma)U \subseteq \Delta^*$. Suppose $U \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$. By 1.8 $U(s_{1k}^{a_{1k}} + s_{2k}^{a_{2k}}) = s_{tk}^{b_{1k}} + s_{2k}^{b_{2k}}$ for $k \in N_2$ and

$V := s_{kj}^yU(s_{1k}^{a_{1k}} + s_{2k}^{a_{2k}})s_{kj}^xU(s_{1k}^{a_{1k}} + s_{2k}^{a_{2k}}) \in \Delta^*$ for all $k, j \in N_2$, $x, y \in \Gamma_{kj}$.

But $V = s_{kj}^y(s_{1k}^{b_{1k}} + s_{2k}^{b_{2k}})s_{kj}^x(s_{1k}^{b_{1k}} + s_{2k}^{b_{2k}}) = s_{kk}^{yb_{jk}}s_{kk}^{xb_{jk}} = s_{kk}^{yb_{jk}xb_{jk}} \in \Delta^*$

and so $yb_{jk}xb_{jk} \in \Delta_{kk}$. Thus $\Gamma_{kj}b_{jk}\Gamma_{kj}b_{jk} \subseteq \Delta_{kk}$ for all $j,k \in N_2$. By (1), Δ_{ii} is a 3-semiprime ideal of Γ_{ii} and by Props. 2.6 and 2.10, also $\Delta_{i_ci_c} = \Gamma_{ii_c}^{-1}\Delta_{ii}\Gamma_{i_ci}^{-1}$ is a 3-semiprime ideal of $\Gamma_{i_ci_c}$. This means, for any $k \in N_2$, $(\Gamma_{kj}b_{jk})\Gamma_{kk}(\Gamma_{kj}b_{jk}) \subseteq \Delta_{kk}$ and since Δ_{kk} is a 3-semiprime ideal of $\Gamma_{kk}, \Gamma_{kj}b_{jk} \subseteq \Delta_{kk}$. By 1.12 we have $b_{jk} \in \Delta_{jk}$ for all $j,k \in N_2$. Hence $U\Gamma^+ \subseteq \Delta^+$; so $U \in \Delta^*$. ◊

2.11 Corollary. *Let $i \in N_2$ be fixed and let $\Gamma = (\Gamma_{ii}, \Gamma_{ii_c}, \Gamma_{i_ci}, \Gamma_{i_ci_c})$ be a morita context. Let $\mathcal{T}$ be a 3-semiprime ideal of $M_2(\Gamma)$. Then:*

$$\begin{aligned} \mathcal{T}_{ii_c} &= \mathcal{T}_{ii}\Gamma_{i_ci}^{-1} = \{x \in \Gamma_{ii_c} \mid x\Gamma_{i_ci} \subseteq \mathcal{T}_{ii}\}, \\ \mathcal{T}_{i_ci} &= \Gamma_{ii_c}^{-1}\mathcal{T}_{ii} = \{x \in \Gamma_{i_ci} \mid \Gamma_{ii_c}x \subseteq \mathcal{T}_{ii}\} \ \textit{and} \\ \mathcal{T}_{i_ci_c} &= \Gamma_{ii_c}^{-1}\mathcal{T}_{ii}\Gamma_{i_ci}^{-1} = \{x \in \Gamma_{i_ci_c} \mid \Gamma_{ii_c}x\Gamma_{i_ci} \subseteq \mathcal{T}_{ii}\}; \end{aligned} \tag{1}$$

$$\begin{aligned} \mathcal{T}_* &= (\mathcal{T}_{ii}, \mathcal{T}_{ii_c}, \mathcal{T}_{i_ci}, \mathcal{T}_{i_ci_c}) = (\mathcal{T}_{ii}, \mathcal{T}_{ii}\Gamma_{i_ci}^{-1}, \Gamma_{ii_c}^{-1}\mathcal{T}_{ii}, \Gamma_{ii_c}^{-1}\mathcal{T}_{ii}\Gamma_{i_ci}^{-1}) = \\ &= (\Gamma_{i_ci}^{-1}\mathcal{T}_{i_ci_c}\Gamma_{ii_c}^{-1}, \Gamma_{i_ci}^{-1}\mathcal{T}_{i_ci_c}, \mathcal{T}_{i_ci_c}\Gamma_{ii_c}^{-1}, \mathcal{T}_{i_ci_c}). \end{aligned} \tag{2}$$

Proof. (1): Let $\Delta := \mathcal{T}_* = (\mathcal{T}_{ii}, \mathcal{T}_{ii_c}, \mathcal{T}_{i_ci}, \mathcal{T}_{i_ci_c})$. By Prop. 2.5, $\Delta^* = (\mathcal{T}_*)^* = \mathcal{T}$. The result then follows from Props. 2.10 and 2.8. (2) follows by using (1) above twice; once for i and then for i_c. ◊

2.12 Corollary. *Let $i \in N_2$ be fixed. Let Δ_{ii} be an ideal of Γ_{ii} such that $\Gamma_{ii_c} * \Gamma_{ii_c}^{-1}\Delta_{ii} \subseteq \Delta_{ii}$. Let Δ be the ideal $\Delta = (\Delta_{ii}, \Delta_{ii}\Gamma_{i_ci}^{-1}, \Gamma_{ii_c}^{-1}\Delta_{ii}, \Gamma_{ii_c}^{-1}\Delta_{ii}\Gamma_{i_ci}^{-1})$ of the morita context Γ* (cf. Prop. 2.2). *Then Δ^* is a 3-semiprime ideal of $M_2(\Gamma)$ if and only if Δ_{ii} is a 3-semiprime ideal of Γ_{ii}. If any one of these two conditions holds, then $\Delta_{nj}\Gamma_{kj}^{-1} = \Delta_{nk} = \Gamma_{jn}^{-1}\Delta_{jk}$ for all $j,k,n \in N_2$.*

Proof. The sufficiency is clear from Prop. 2.10. Conversely, since $x\Gamma_{i_ci} \subseteq \Delta_{ii}$ implies $x \in \Delta_{ii}\Gamma_{i_ci}^{-1}$, once again Prop. 2.10 yields the result. For $x \in \Delta_{nj}\Gamma_{kj}^{-1}$, $x\Gamma_{kj} \subseteq \Delta_{nj}$ and so $x\Gamma_{kj}\Gamma_{jk} \subseteq \Delta_{nj}\Gamma_{jk} \subseteq \Delta_{nk}$. By the assumption and Prop. 2.9 we get $x \in \Delta_{nk}$. Conversely, $x \in \Delta_{nk}$ implies $x\Gamma_{kj} \subseteq \Delta_{nj}$ and so $x \in \Delta_{nj}\Gamma_{kj}^{-1}$. Since $\Gamma_{nj} * \Delta_{jk} \subseteq \Delta_{nk}$ and Δ_{nk} is an ideal of the Γ_{nn}-Γ_{kk}-bimodule Γ_{nk}, we have $\Gamma_{jn}^{-1}\Delta_{jk} \subseteq \Delta_{nk}$. Conversely, for $x \in \Delta_{nk}$, we have $\Gamma_{nj}\Gamma_{jn}x \subseteq \Gamma_{nj}\Delta_{jk} \subseteq \Gamma_{nj} * \Delta_{jk} \subseteq \Gamma_{jn}^{-1}\Delta_{jk}$. For $j = n$, $x \in \Gamma_{nn}x = \Gamma_{nj}\Gamma_{jn}x \subseteq \Gamma_{jn}^{-1}\Delta_{jk}$, and for $j = n_c$, by 1.11 we get $x \in \Gamma_{jn}^{-1}\Delta_{jk}$. Thus $\Gamma_{jn}^{-1}\Delta_{jk} = \Delta_{nk}$. ◊

2.13 Proposition. *Let $\Delta = (\Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{22})$ be an ideal of the morita context $\Gamma = (\Gamma_{11}, \Gamma_{12}, \Gamma_{21}, \Gamma_{22})$. Then*

$$M_2\Big(\frac{\Gamma}{\Delta}\Big) \cong \frac{M_2(\Gamma)}{\Delta^*}.$$

Before proceeding with the proof, we need:

2.14 Lemma. *For each $U \in M_2(\Gamma)$, there is a $U_q \in M_2(\frac{\Gamma}{\Delta})$ with the property:*

$$\text{If } U \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},$$

$$\text{then } U_q \begin{bmatrix} a_{11}+\Delta_{11} & a_{12}+\Delta_{12} \\ a_{21}+\Delta_{21} & a_{22}+\Delta_{22} \end{bmatrix} = \begin{bmatrix} b_{11}+\Delta_{11} & b_{12}+\Delta_{12} \\ b_{21}+\Delta_{21} & b_{22}+\Delta_{22} \end{bmatrix}.$$

Proof (by induction on $w(U)$). If $U \in M_2(\Gamma)$ with $w(U) = 1$, then $U = s^x_{kn}$ for some $k, n \in N_2$, $x \in \Gamma_{kn}$. Let $U_q := s^y_{kn}$ where $y = x + + \Delta_{kn}$. Then $U_q \in M_2(\frac{\Gamma}{\Delta})$ and it has the required property. Suppose for all $V \in M_2(\Gamma)$ with $w(V) < m$, $m \geq 2$, such a $V_q \in M_2(\frac{\Gamma}{\Delta})$ has been found. Let $U \in M_2(\Gamma)$ with $w(U) = m$. Then $U = U_1 + U_2$ or $U = U_1U_2$ where $U_1, U_2 \in M_2(\Gamma)$ with $w(U_1) < m$ and $w(U_2) < m$. If $U = U_1 + U_2$, let $U_q = (U_1)_q + (U_2)_q$ and if $U = U_1U_2$, let $U_q = = (U_1)_q(U_2)_q$. It follows readily that for both possibilities, U_q has the desired property. We also remark that, even if $U_1 + U_2 = U = U'_1 + U'_2$ or $U_1+U_2 = U = U'_1U'_2$ respectively, where $w(U'_1) < m$ and $w(U'_2) < m$, then U_q is well-defined since for each case, $(U_1)_q+(U_2)_q = (U'_1)_q+(U'_2)_q$ or $(U_1)_q + (U_2)_q = (U'_1)_q(U'_2)_q$ respectively. ◊

Proof (of Prop. 2.13). For each $U \in M_2(\Gamma)$, the $U_q \in M_2(\frac{\Gamma}{\Delta})$ given by the Lemma is obviously uniquely determined by U; hence we have a well-defined function

$$\varphi\colon M_2(\Gamma) \to M_2\Big(\frac{\Gamma}{\Delta}\Big), \quad \text{given by} \quad \varphi(U) = U_q.$$

Since $(U_1+U_2)_q = (U_1)_q+(U_2)_q$ and $(U_1U_2)_q = (U_1)_q(U_2)_q$, it is a near-ring homomorphism. Let us abbreviate the elements of Γ^+ and $\big(\frac{\Gamma}{\Delta}\big)^+$ by $[a_{ij}]$ and $[a_{ij} + \Delta_{ij}]$ respectively $\Big($meaning of course, for example for $[a_{ij}], [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\Big)$. Note that $U_q[a_{ij} + \Delta_{ij}] = [b_{ij} + \Delta_{ij}]$ if and only if $U[a_{ij}] = [b_{ij} + d_{ij}]$ for some $d_{kn} \in \Delta_{kn}$, $k, n = 1, 2$. Hence we get $\ker\varphi = \{U \in M_2(\Gamma) | U_q = 0\} = \{U \in M_2(\Gamma) | U\Gamma^+ \subseteq \Delta^+\} = = \Delta^*$. Finally we show φ is surjective from which $\frac{M_2(\Gamma)}{\Delta^*} \cong M_2\big(\frac{\Gamma}{\Delta}\big)$ will follow. Let $V \in M_2\big(\frac{\Gamma}{\Delta}\big)$ and let U be an element of $M_2(\Gamma)$ which is obtained from V by replacing each $s_{ij}^{x+\Delta_{ij}}$ present in V by s_{ij}^{x}. Of course there may be many different such U's (x in s_{ij}^{x} can be replaced by other representative from $x+\Delta_{ij}$); for our purposes any one such U will do. A straightforward induction on $w(V)$ will show that $V=U_q$; hence φ is surjective. $\Diamond$

Let $i \in N_2$ be fixed. For $k=1,2$, let Δ_{ik} be a subgroup of Γ_{ik}. Then $\tau_i(\Delta_{i1},\Delta_{i2}) := \{\tau_i(a,b) \mid a \in \Delta_{i1}, b \in \Delta_{i2}\}$ is a subgroup of Γ^+ (which is normal if each Δ_{ik} is normal in Γ_{ik}). Let $\mathcal{R}_i(\Delta_{i1},\Delta_{i2}) := \{U \in M_2(\Gamma) \mid U\Gamma^+ \subseteq \tau_i(\Delta_{i1},\Delta_{i2})\}$.

2.15 Proposition. *$\mathcal{R}_i(\Delta_{i1},\Delta_{i2})$ is a right invariant subgroup of $M_2(\Gamma)$. It is a right ideal of $M_2(\Gamma)$ if Δ_{ik} is normal in Γ_{ik} for $k=1,2$. If Δ_{ii} is a subnear-ring of Γ_{ii}, then*

$$\varphi_i\colon \mathcal{R}_i(\Delta_{i1},\Delta_{i2}) \to \Delta_{ii}, \text{ defined by } \varphi_i(U) := \pi_{ii}(U(\pi_{ii}(1)))$$

is a near-ring homomorphism. It is surjective if $\Delta_{ii}\Gamma_{ij} \subseteq \Delta_{ij}$ for all $j=1,2$. Moreover, $\ker\varphi_i \subseteq \{U \in \mathcal{R}_i(\Delta_{i1},\Delta_{i2}) \mid U\mathcal{R}_i(\Delta_{i1},\Delta_{i2})=0\}$; in particular, $(\ker\varphi_i)^2=0$.

Proof. It is straightforward to see that $\mathcal{R}_i(\Delta_{i1},\Delta_{i2})$ is a right invariant subgroup, which is a right ideal of $M_2(\Gamma)$ if the Δ_{ik}'s are normal. We show that φ_i is well defined, i.e. $\varphi_i(U) \in \Delta_{ii}$. Firstly note that for any $U \in \mathcal{R}_i(\Delta_{i1},\Delta_{i2})$, $Us_{ii}^1 = s_{ii}^u$ for some unique $u \in \Delta_{ii}$. Thus

$$\varphi_i(U) = \pi_{ii}(U(\tau_{ii}(1))) = \pi_{ii}(Us_{ii}^1(\tau_{ii}(1))) = \pi_{ii}(s_{ii}^u(\tau_{ii}(1))) = u \in \Delta_{ii}.$$

Let $U_1,U_2 \in \mathcal{R}_i(\Delta_{i1},\Delta_{i2})$ with $U_1s_{ii}^1 = s_{ii}^{u_1}$ and $U_2s_{ii}^1 = s_{ii}^{u_2}$. Then $(U_1+U_2)s_{ii}^1 = s_{ii}^{u_1+u_2}$ and so $\varphi_i(U_1+U_2) = u_1+u_2 = \varphi(U_1)+\varphi(U_2)$. Furthermore, $\varphi_i(U_1U_2) = \pi_{ii}(U_1(U_2s_{ii}^1\tau_{ii}(1))) = \pi_{ii}(U_1s_{ii}^1s_{ii}^{u_2}\tau_{ii}(1)) = \pi_{ii}(s_{ii}^{u_1}s_{ii}^{u_2}\tau_{ii}(1)) = \pi_{ii}(s_{ii}^{u_1u_2}\tau_{ii}(1)) = u_1u_2 = \varphi(U_1)\varphi(U_2)$. Let $K \in \ker\varphi_i$ and $U \in \mathcal{R}_i(\Delta_{i1},\Delta_{i2})$. For any $a_{jk} \in \Gamma_{jk}$, $j,k \in N_2$, and for some $b_{i1} \in \Delta_{ii}$ and $b_{i2} \in \Delta_{i2}$, $KU\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix} = K(\tau_i(b_{i1},b_{i2})) = Ks_{ii}^1\tau_i(b_{i1},b_{i2}) = s_{ii}^k\tau_i(b_{i1},b_{i2}) = 0$ since $K \in \ker\varphi_i$ implies $k = \varphi_i(K) = 0$ where $Ks_{ii}^1 = s_{ii}^k$. Thus $(\ker\varphi_i)\cdot\mathcal{R}_i(\Delta_{i1},\Delta_{i2}) = 0$. Finally, suppose $\Delta_{ii}\Gamma_{ij} \subseteq \Delta_{ij}$ for $j=1,2$. Let $d \in \Delta_{ii}$. Then $s_{ii}^d \in \mathcal{R}_i(\Delta_{i1},\Delta_{i2})$ and $\varphi_i(s_{ii}^d) = d$. Thus φ_i is surjective. $\Diamond$

Once again, let $i \in N_2$ be fixed and for each $k=1,2$, let Δ_{ki} be a subgroup of Γ_{ki}. Let $\mathcal{C}_i(\Delta_{1i},\Delta_{2i}) := \{s_{1i}^{x_1} + s_{2i}^{x_2} \mid x_k \in \Delta_{ki},\ k=1,2\}$. Note that if Δ_{ki} for $k=1,2$, then $\mathcal{C}_i(\Delta_{1i},\Delta_{2i}) = \mathcal{C}_i(\Gamma_{1i},\Gamma_{2i}) = C_i$ (cf. 1.7).

2.16 Proposition. *Suppose $\Gamma_{jk}\Delta_{ki} \subseteq \Delta_{ji}$ for all $k,j \in N_2$. Then*

$\mathcal{C}_i(\Delta_{1i},\Delta_{2i})$ is a left invariant subgroup of $M_2(\Gamma)$ and if $\gamma_i : \mathcal{C}_i(\Delta_{1i},\Delta_{2i}) \to \Delta_{ii}$ is defined by $\gamma_i(s_{1i}^{x_1}+s_{2i}^{x_2}) := x_i$, then γ_i is a surjective near-ring homomorphism with $(\ker\gamma_i)^2 = 0$.

Proof. $\mathcal{C}_i(\Delta_{1i},\Delta_{2i})$ is clearly a subgroup of $M_2(\Gamma)$. Let $U \in M_2(\Gamma)$ and let $s_{1i}^{x_1}+s_{2i}^{x_2} \in \mathcal{C}_i(\Delta_{1i},\Delta_{2i})$. We show that $U(s_{1i}^{x_1}+s_{2i}^{x_2}) \in \mathcal{C}_i(\Delta_{1i},\Delta_{2i})$ by induction on $w(U)$. If $w(U)=1$, then $U = s_{kj}^{y}$ for some $k,j \in N_2$, $y \in \Gamma_{kj}$. Then

$$s_{kj}^{y}(s_{1i}^{x_1}+s_{2i}^{x_2}) = s_{ki}^{yx_j} = s_{ki}^{yx_j} + s_{k_ci}^{0} \in \mathcal{C}_i(\Delta_{1i},\Delta_{2i})$$

by 1.2 and since $yx_j \in \Gamma_{kj}\Delta_{ji} \subseteq \Delta_{ki}$.

Suppose $V(s_{1i}^{a}+s_{2i}^{b}) \in \mathcal{C}_i(\Delta_{1i},\Delta_{2i})$ for all $V \in M_2(\Gamma)$ with $w(V) < m$ $(m \geq 2)$ and $s_{1i}^{a}+s_{2i}^{b} \in \mathcal{C}_i(\Delta_{1i},\Delta_{2i})$. Let $U \in M_2(\Gamma)$ with $w(U)=m$. Then $U = U_1+U_2$ or $U = U_1U_2$ for some $U_j \in M_2(\Gamma)$ with $w(U_j) < m$, $j = 1,2$. Then $U(s_{1i}^{x_1}+s_{2i}^{x_2}) = U_1(s_{1i}^{x_1}+s_{2i}^{x_2}) + {}+U_2(s_{1i}^{x_1}+s_{2i}^{x_2}) \in \mathcal{C}_i(\Delta_{1i},\Delta_{2i})$ by the induction assumption, or, by 1.7, $U(s_{1i}^{x_1}+s_{2i}^{x_2}) = U_1(U_2(s_{1i}^{x_1}+s_{2i}^{x_2})) = U_1(s_{1i}^{y_1}+s_{2i}^{y_2}) \in \mathcal{C}_i(\Delta_{1i},\Delta_{2i})$, once again by the induction assumption.

Note that by the assumptions on Δ_{1i} and Δ_{2i}, Δ_{ii} is a subnear-ring of Γ_{ii}. Clearly γ_i is a group homomorphism and $\gamma_i((s_{1i}^{x_1}+s_{2i}^{x_2})(s_{1i}^{y_1}+ {}+ s_{2i}^{y_2})) = \gamma_i(s_{1i}^{x_1y_i}+s_{2i}^{x_2y_i}) = x_iy_i = \gamma_i(s_{1i}^{x_1}+s_{2i}^{x_2})\cdot\gamma_i(s_{1i}^{y_1}+s_{2i}^{y_2})$. It is clear that γ_i is surjective and $\ker\gamma_i = \{s_{1i}^{x_1}+s_{2i}^{x_2} \in \mathcal{C}_i(\Delta_{1i},\Delta_{2i}) \mid x_i = {}= 0\} = \{s_{i_ci}^{x_{i_c}} \mid x_{i_c} \in \Delta_{i_ci}\}$. Thus, $(\ker(\gamma_i)^2 = 0$. $\Diamond$

3. Radical theory

Here we investigate the relationship between the radical of the near-ring $M_2(\Gamma)$ and the radicals of the near-rings Γ_{11} and Γ_{22}. We once again stress our assumption that $\Gamma = (\Gamma_{ii},\Gamma_{ii_c},\Gamma_{i_ci},\Gamma_{i_ci_c})$ is always a standard morita context for near-rings. We shall give two approaches to establish this relationship. The first is by placing additional conditions on a Kurosh–Amitsur radical, and the second will be by considering conditions on a class of near-rings such that the corresponding Hoehnke radical has the desired properties. Throughout this section $\mathcal{M}$ is a class of 2-semiprime near-rings. Let ϱ be the corresponding Hoehnke radical, i.e. $\varrho N = \cap(I$ an ideal of $N \mid N/I \in \mathcal{M})$ for all near-rings N.

Conditions on ϱ. Here we suppose that ϱ is a Kurosh–Amitsur radical. For more information on the relevant requirements for this to hold, [5] can be consulted. Of importance here, are the following:

Let $\mathcal{R}_\varrho := \{\text{near-rings } N \mid \varrho N = N\}$ be the radical class determined by ϱ. Then

3.1.1 $\mathcal{R}_\varrho$ is homomorphically closed;

3.1.2 $\varrho N \in \mathcal{R}_\varrho$ for all N;

3.1.3 $\mathcal{R}_\varrho$ is closed under extensions, i.e. if I is an ideal of N such that both I and N/I are in $\mathcal{R}_\varrho$, then $N \in \mathcal{R}_\varrho$;

3.1.4 if I is an ideal of N with $I \in \mathcal{R}_\varrho$, then $I \subseteq \varrho N$;

3.1.5 since $\mathcal{M}$ consists of 2-semiprime near-rings, $\mathcal{R}_\varrho$ contains all the near-rings with zero multiplication.

Examples of such radicals are the Jacobson radicals J_2 and J_3, the Brown-McCoy radical $\mathcal{G}$ and the equiprime radical e determined by the classes of 2-primitive, 3-primitive, simple near-rings with identity and the equiprime near-rings respectively. If a 2-primitive near-ring N has an identity, then it is 3-primitive and consequently the J_2 and J_3 radicals of any near-ring with identity coincide. Additional properties that the radical ϱ may satisfy are:

ϱ is *right strong* (resp. *left invariantly strong*) if whenever I is a right ideal (resp. left invariant subgroup) of N with $\varrho I = I$, then $I \subseteq \varrho N$.

ϱ is *hereditary on right ideals* (resp. *hereditary on left invariant subgroups*) if whenever I is a right ideal (resp. left invariant subgroup) of $N \in \mathcal{R}_\varrho$, then $I \in \mathcal{R}_\varrho$.

In the sequel, we let $\mathcal{T} := \varrho(M_2(\Gamma))$. Since $\mathcal{T}$ is an intersection of 3-semiprime ideals (= intersection of 2-semiprime ideals since $M_2(\Gamma)$ has an identity), $\mathcal{T}$ itself is 3-semiprime and thus full (cf. 2.5).

3.1.6 Proposition. *Suppose ϱ is right strong. Then $\varrho(\Gamma_{ii}) \subseteq \mathcal{T}_{ii}$ for all $i = 1, 2$.*

Proof. Let $i \in N_2$ be fixed. Let $\Delta_{ii} = \varrho(\Gamma_{ii})$ and let $\Delta_{ii_c} = \Gamma_{ii_c}$. Then Δ_{ik} is a normal subgroup of Γ_{ik} and $\Delta_{ii}\Gamma_{ik} \subseteq \Delta_{ik}$ for each $k = 1, 2$. By Prop. 2.15, $\varphi_i \colon \mathcal{R}_i(\Delta_{ii}, \Delta_{ii_c}) \to \Delta_{ii}$ is a surjective near-ring homomorphism with $K^2 = 0$ where $K = \ker \varphi_i$. (Note the inconsistency here, as well as in a few other places in the sequel of our notation; strictly speaking for $i = 2$, we should write $\mathcal{R}_i(\Delta_{ii_c}, \Delta_{ii})$ instead of $\mathcal{R}_i(\Delta_{ii}, \Delta_{ii_c})$.) By 3.1.5, 3.1.2 and 3.1.3 we get $\mathcal{R}_i(\Delta_{ii}, \Delta_{ii_c}) \in \mathcal{R}_\varrho$. Since $\mathcal{R}_i(\Delta_{ii}, \Delta_{ii_c})$ is a right ideal of $M_2(\Gamma)$ and ϱ is right strong, we get $\mathcal{R}_i(\Delta_{ii}, \Delta_{ii_c}) \subseteq \subseteq \varrho(M_2(\Gamma)) = \mathcal{T}$. Let $x \in \varrho(\Gamma_{ii}) = \Delta_{ii}$. Then $s^x_{ii} \in \mathcal{R}_i(\Delta_{ii}, \Delta_{ii_c}) \subseteq \mathcal{T}$; hence $x \in \mathcal{T}_{ii}$. $\Diamond$

3.1.7 Proposition. *Suppose ϱ is hereditary on right ideals. Then $\mathcal{T}_{ii} \subseteq \varrho(\Gamma_{ii})$ for all $i = 1, 2$.*

Proof. Let $i \in N_2$ be fixed. For each $k \in N_2$, let $\Delta_{ik} = \mathcal{T}_{ik}$. Then Δ_{ik} is a normal subgroup of Γ_{ik} and $\Delta_{ii}\Gamma_{ik} \subseteq \Delta_{ik}$ for all $k = 1, 2$. By Prop. 2.15, $\mathcal{R}_i(\Delta_{ii}, \Delta_{ii_c})$ is a right ideal of $M_2(\Gamma)$ and $\varphi_i \colon \mathcal{R}_i(\Delta_{ii}, \Delta_{ii_c}) \to \to \Delta_{ii}$ is a surjective near-ring homomorphism. Let $U \in \mathcal{R}_i(\Delta_{ii}, \Delta_{ii_c})$. Then $U\Gamma^+ \subseteq \tau_i(\Delta_{ii}, \Delta_{ii_c}) \subseteq (\mathcal{T}_*)^+$. Thus $U \in (\mathcal{T}_*)^* = \mathcal{T}$; hence $\mathcal{R}_i(\Delta_{ii}, \Delta_{ii_c}) \subseteq \mathcal{T} = \varrho(M_2(\Gamma)) \in \mathcal{R}_\varrho$. By our assumption on ϱ, $\mathcal{R}_i(\Delta_{ii}, \Delta_{ii_c}) \in \mathcal{R}_\varrho$ and thus also $\mathcal{T}_{ii} = \Delta_{ii} \in \mathcal{R}_\varrho$ (by 3.1.1). Since $\mathcal{T}_{ii}$ is an ideal of Γ_{ii}, we have $\mathcal{T}_{ii} \subseteq \varrho(\Gamma_{ii})$ by 3.1.4. $\Diamond$

Since $\mathcal{T} = \varrho(M_2(\Gamma))$ is a 3-semiprime ideal of $M_2(\Gamma)$, we have by Cor. 2.11(2)

$$\begin{aligned}\mathcal{T}_* &= (\mathcal{T}_{ii}, \mathcal{T}_{ii_c}, \mathcal{T}_{i_ci}, \mathcal{T}_{i_ci_c}) = \\ &= (\mathcal{T}_{ii}, \mathcal{T}_{ii}\Gamma_{i_ci}^{-1}, \Gamma_{ii_c}^{-1}\mathcal{T}_{ii}, \mathcal{T}_{i_ci_c}) = \\ &= (\mathcal{T}_{ii}, \Gamma_{i_ci}^{-1}\mathcal{T}_{i_ci_c}, \mathcal{T}_{i_ci_c}\Gamma_{ii_c}^{-1}, \mathcal{T}_{i_ci_c}).\end{aligned}$$

Let $\varrho_1(\Gamma) := (\varrho(\Gamma_{11}), \varrho(\Gamma_{11})\Gamma_{21}^{-1}, \Gamma_{12}^{-1}\varrho(\Gamma_{11}), (\Gamma_{12}^{-1}\varrho(\Gamma_{11}))\Gamma_{21}^{-1})$ and let

$$\varrho_2(\Gamma) := ((\Gamma_{21}^{-1}\varrho(\Gamma_{22}))\Gamma_{12}^{-1}, \Gamma_{21}^{-1}\varrho(\Gamma_{22}), \varrho(\Gamma_{22})\Gamma_{12}^{-1}, \varrho(\Gamma_{22})).$$

In general, neither of these need to be an ideal of the morita context Γ and they need not be equal. However, if $\varrho_1(\Gamma) = \varrho_2(\Gamma)$, then $\varrho_1(\Gamma)$ is an ideal of Γ (cf. 2.2) and in this case we say the radical of Γ exists and call it the *radical of the morita context* Γ. We denote it by $\varrho(\Gamma)$.

3.1.8 Corollary. *Suppose ϱ is right strong and hereditary on right ideals. Then $\varrho(\Gamma_{jj}) = \mathcal{T}_{jj}$ for all $j \in N_2$ and $\varrho(\Gamma) = \varrho_1(\Gamma) = \varrho_2(\Gamma)$; hence $\varrho(M_2(\Gamma)) = \varrho(\Gamma)^*$.*

Proof. By 3.1.6 and 3.1.7 we have $\varrho(\Gamma_{jj}) = \mathcal{T}_{jj}$ and by the discussion preceding the corollary, we get $\varrho_1(\Gamma) = \mathcal{T}_* = \varrho_2(\Gamma)$. Hence $\varrho(\Gamma)$ exists and $\varrho(\Gamma) = \mathcal{T}_*$. Thus $\varrho(M_2(\Gamma)) = \mathcal{T} = (\mathcal{T}_*)^* = (\varrho(\Gamma))^*$. $\Diamond$

3.1.9 Proposition. *Suppose ϱ is hereditary on left invariant subgroups. Then $\mathcal{T}_{ii} \subseteq \varrho(\Gamma_{ii})$ for all $i \in N_2$.*

Proof. Let $i \in N_2$ be fixed and for each $k \in N_2$, let $\Delta_{ki} = \mathcal{T}_{ki}$. Then Δ_{ki} is a subgroup of Γ_{ki} and $\Gamma_{jk}\Delta_{ki} \subseteq \Delta_{ji}$ for all $k, j \in N_2$. By Prop. 2.16, $\mathcal{C}_i(\Delta_{ii}, \Delta_{i_ci})$ is a left invariant subgroup of $M_2(\Gamma)$ and $\gamma_i \colon \mathcal{C}_i(\Delta_{ii}, \Delta_{i_ci}) \to \Delta_{ii}$ is a surjective near-ring homomorphism. Now $\mathcal{C}_i(\Delta_{ii}, \Delta_{i_ci}) \subseteq \mathcal{T} = \varrho(M_2(\Gamma))$: Indeed, for all $n, m \in N_2$, let $a_{nm} \in \in \Gamma_{nm}$. For $x_i \in \Delta_{ii}$ and $x_{i_c} \in \Delta_{i_ci}$, we have

$$\left(s_{ii}^{x_i} + s_{i_ci}^{x_{i_c}}\right) \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \tau_i(x_ia_{i1}, x_ia_{i2}) + \tau_{i_c}(x_{i_c}a_{i1}, x_{i_c}a_{i2}) \in (\mathcal{T}_*)^+$$

since $x_ka_{ij} \in \Delta_{ki}\Gamma_{ij} = \mathcal{T}_{ki}\Gamma_{ij} \subseteq \mathcal{T}_{kj}$ for all $k, j \in N_2$. Thus $s_{ii}^{x_i} + s_{i_ci}^{x_{i_c}} \in$

$\in (\mathcal{T}_*)^* = \mathcal{T}$. Since ϱ is hereditary on left invariant subgroups, we get $\mathcal{C}_i(\Delta_{ii}, \Delta_{i_c i})$ and thus also $\Delta_{ii} = \mathcal{T}_{ii}$ in $\mathcal{R}_\varrho$ (by 3.1.1). Thus $\mathcal{T}_{ii} \subseteq \varrho(\Gamma_{ii})$ by 3.1.4. $\Diamond$

3.1.10 Proposition. *Suppose ϱ is left invariantly strong. For $i \in N_2$ fixed, let $\Delta_{ii} = \varrho(\Gamma_{ii})$ and let $\Delta_{i_c i}$ be the subgroup of $\Gamma_{i_c i}$ generated by $\Gamma_{i_c i}\Delta_{ii}$. If $\Gamma_{ii_c}\Delta_{i_c i} \subseteq \Delta_{ii}$, then $\varrho(\Gamma_{ii}) \subseteq \mathcal{T}_{ii}$.*

Proof. Firstly note that $\Gamma_{jk}\Delta_{ki} \subseteq \Delta_{ji}$ for all $j, k \in N_2$ by Proposition 2.16, $\mathcal{C}_i(\Delta_{ii}, \Delta_{i_c i})$ is a left invariant subgroup of $M_2(\Gamma)$ and $\gamma_i \colon \mathcal{C}_i(\Delta_{ii}, \Delta_{i_c i}) \to \Delta_{ii}$ is a surjective near-ring homomorphism with $K^2 = 0$ where $K = \ker\theta$. By 3.1.5, 3.1.2 and 3.1.3 we get $\mathcal{C}_i(\Delta_{ii}, \Delta_{i_c i}) \in \in \mathcal{R}_\varrho$. By the assumption on ϱ, we get $\mathcal{C}_i(\Delta_{ii}, \Delta_{i_c i}) \subseteq \varrho(M_2(\Gamma)) = \mathcal{T}$. For $x \in \varrho(\Gamma_{ii}) = \Delta_{ii}$, $s^x_{ii} \in \mathcal{C}_i(\Delta_{ii}, \Delta_{i_c i}) \subseteq \mathcal{T}$; hence $x \in \mathcal{T}_{ii}$. $\Diamond$

As in 3.1.8, we get:

3.1.11 Corollaries. *If*

(i) *ϱ is right strong and hereditary on left invariant subgroups*

or if

(ii) *ϱ is hereditary on left invariant subgroups and left invariantly strong such that $\Gamma_{jj_c}\Delta_{j_c j} \subseteq \varrho(\Gamma_{jj})$ where $\Delta_{j_c j}$ is the subgroup of $\Gamma_{j_c j}$ generated by $\Gamma_{j_c j}\varrho(\Gamma_{jj})$ for all $j = 1, 2$*

then

$$\varrho(\Gamma_{ii}) = \mathcal{T}_{ii} \text{ for all } i \in N_2,\ \varrho(\Gamma) \text{ exists and } \varrho(M_2(\Gamma)) = (\varrho(\Gamma))^*. \ \Diamond$$

3.2 Conditions on $\mathcal{M}$. Throughout this section, $\Gamma = (\Gamma_{11}, \Gamma_{12}, \Gamma_{21}, \Gamma_{22})$ is a fixed standard morita context and $\mathcal{M}$ is a class of 2-semiprime near-rings with ϱ the corresponding Hoehnke radical. We write $\mathcal{T}$ for $\varrho(M_2(\Gamma))$.

3.2.1 Proposition. *Suppose $\mathcal{M}$ satisfies:*

(I) *If A is an ideal of $M_2(\Gamma)$ with $M_2(\Gamma)/\mathcal{A} \in \mathcal{M}$, then $\Gamma_{ii}/\mathcal{A}_{ii} \in \mathcal{M}$ for $i \in N_2$.*

Then $\varrho(\Gamma_{ii}) \subseteq \mathcal{T}_{ii}$.

Proof. Let $x \in \varrho(\Gamma_{ii})$ and let $\mathcal{A}$ be an ideal of $M_2(\Gamma)$ with $M_2(\Gamma)/\mathcal{A} \in \in \mathcal{M}$. By condition (I) $x \in \varrho(\Gamma_{ii}) \subseteq \mathcal{A}_{ii}$ and so $s^x_{ii} \in \mathcal{A}$. Since this holds for all such ideals $\mathcal{A}$ of $M_2(\Gamma)$, $s^x_{ii} \in \varrho(M_2(\Gamma)) = \mathcal{T}$. Thus $x \in \mathcal{T}_{ii}$. $\Diamond$

3.2.2 Proposition. *Suppose $\mathcal{M}$ and Γ satisfy:*

(II) *For $i \in N_2$, if Δ_{ii} is an ideal of Γ_{ii} with $\Gamma_{ii}/\Delta_{ii} \in \mathcal{M}$, then*

$$\Gamma_{ii_c} * \Gamma_{ii_c}^{-1}\Delta_{ii} \subseteq \Delta_{ii} \text{ and } M_2(\Gamma)/\Delta^* \in \mathcal{M}$$

where $\Delta^* = (\Delta_{ii}, \Delta_{ii}\Gamma_{i_ci}^{-1}, \Gamma_{ii_c}^{-1}\Delta_{ii}, \Gamma_{ii_c}^{-1}\Delta_{ii}\Gamma_{i_ci}^{-1})$.
Then $\mathcal{T}_{ii} \subseteq \varrho(\Gamma_{ii})$.
Proof. Let $x \in \mathcal{T}_{ii}$. Then $s_{ii}^x \in \mathcal{T}$. Let Δ_{ii} be an ideal of Γ_{ii} with $\Gamma_{ii}/\Delta_{ii} \in \mathcal{M}$. By condition (II) we get $s_{ii}^x \in \mathcal{T} = \varrho(M_2(\Gamma)) \subseteq \Delta^*$; hence $x \in (\Delta^*)_{ii} = \Delta_{ii}$. Thus $x \in \varrho(\Gamma_{ii})$. ◊
3.2.3 Theorem. *Suppose* $\mathcal{M}$ *and* Γ *satisfy conditions* (I) *and* (II). *Then* $\varrho(\Gamma_{ii}) = \mathcal{T}_{ii}$ *for* $i \in N_2$, $\varrho(\Gamma)$ *exists and* $\varrho(M_2(\Gamma)) = (\varrho(\Gamma))^*$.
Proof. By the previous two results and Cor. 2.11, we have that $\varrho(\Gamma)$ exists and $\varrho(\Gamma) = \mathcal{T}_*$. Thus $\varrho(M_2(\Gamma)) = \mathcal{T} = (\mathcal{T}_*)^* = (\varrho(\Gamma))^*$. ◊
3.2.4 Proposition. *The conditions* (I) *and* (II) *are equivalent to* (A) *and* (B) *where:*

(A) *For* $i \in N_2$, *if* Δ_{ii} *is an ideal of* Γ_{ii} *with* $\Gamma_{ii}/\Delta_{ii} \in \mathcal{M}$, *then* $\Gamma_{ii_c} * \Gamma_{ii_c}^{-1}\Delta_{ii} \subseteq \Delta_{ii}$ *and* $\Gamma_{i_ci_c}/\Gamma_{ii_c}^{-1}\Delta_{ii}\Gamma_{i_ci}^{-1} \in \mathcal{M}$.

(B) *Let* $\Delta = (\Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{22})$ *be an ideal of* Γ. *Then* $M_2(\Gamma/\Delta) \in$ $\in \mathcal{M}$ *if and only if* $\Gamma_{ii}/\Delta_{ii} \in \mathcal{M}$ *for* $i \in N_2$.

Proof. Suppose (I) and (II) hold. Let $i \in N_2$ and suppose Δ_{ii} is an ideal of Γ_{ii} with $\Gamma_{ii}/\Delta_{ii} \in \mathcal{M}$. By (II), $\Gamma_{ii_c} * \Gamma_{ii_c}^{-1}\Delta_{ii} \subseteq \Delta_{ii}$ and so for $\Delta = (\Delta_{ii}, \Delta_{ii}\Gamma_{i_ci}^{-1}, \Gamma_{ii_c}^{-1}\Delta_{ii}, \Gamma_{ii_c}^{-1}\Delta_{ii}\Gamma_{i_ci}^{-1})$, Δ^* is an ideal of $M_2(\Gamma)$ for which $M_2(\Gamma)/\Delta^* \in \mathcal{M}$. By (I) we then get $\Gamma_{i_ci_c}/\Gamma_{ii_c}^{-1}\Delta_{ii}\Gamma_{i_ci}^{-1} \in \mathcal{M}$ and so (A) holds.

Let $\Delta = (\Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{22})$ be an ideal of Γ. If $M_2(\Gamma/\Delta) \in \mathcal{M}$, then $M_2(\Gamma)/\Delta^* \in \mathcal{M}$ (by Prop. 2.13) and from (I) we get $\Gamma_{ii}/\Delta_{ii} \in$ $\in \mathcal{M}$. If $\Gamma_{ii}/\Delta_{ii} \in \mathcal{M}$, then (II) gives $M_2(\Gamma/\Delta) \cong M_2(\Gamma)/\Delta^* \in \mathcal{M}$ which shows the validity of (B).

Conversely, suppose (A) and (B) hold. Let $\mathcal{A}$ be an ideal of $M_2(\Gamma)$ such that $M_2(\Gamma)/\mathcal{A} \in \mathcal{M}$. Then $M_2(\Gamma/\mathcal{A}_*) \in \mathcal{M}$ and by (B), $\Gamma_{ii}/\mathcal{A}_{ii} =$ $= (\Gamma/\mathcal{A}_*)_{ii} \in \mathcal{M}$. Thus (I) holds. Let Δ_{ii} be an ideal of Γ_{ii} with $\Gamma_{ii}/\Delta_{ii} \in \mathcal{M}$. By (A), $\Gamma_{ii_c} * \Gamma_{ii_c}^{-1}\Delta_{ii} \subseteq \Delta_{ii}$ and $\Gamma_{i_ci_c}/\Gamma_{ii_c}^{-1}\Delta_{ii}\Gamma_{i_ci}^{-1} \in \mathcal{M}$. For the ideal

$$\Delta = (\Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{22}) := (\Delta_{11}, \Delta_{11}\Gamma_{21}^{-1}, \Gamma_{12}^{-1}\Delta_{11}, \Gamma_{12}^{-1}\Delta_{11}\Gamma_{12}^{-1}) =$$
$$= (\Gamma_{21}^{-1}\Delta_{22}\Gamma_{12}^{-1}, \Gamma_{21}^{-1}\Delta_{22}, \Delta_{22}\Gamma_{12}^{-1}, \Delta_{22})$$

we then have $\Gamma_{ii}/\Delta_{ii} \in \mathcal{M}$ for $i \in N_2$. By (B), $M_2(\Gamma)/\Delta^* = M_2(\Gamma/\Delta) \in$ $\mathcal{M}$ which yields (II). ◊

Contrary to the ring case the conditions (I) and (II) for the near-ring case can apparently not be expressed in terms of standard morita context without reference to ideals (e.g. in the case of (II), $\Gamma_{ii} \in \mathcal{M}$ implies $M_2(\Gamma) \in \mathcal{M}$). The reason being that for an ideal Δ_{ii} of Γ_{ii}, $\Delta = (\Delta_{11}, \Delta_{11}\Gamma_{21}^{-1}, \Gamma_{12}^{-1}\Delta_{11}, \Gamma_{12}^{-1}\Delta_{11}\Gamma_{12}^{-1})$ is not necessarily an ideal of Γ (cf. Prop. 2.2).

4. Examples

4.1. Let $\mathcal{M}$ be a class of 2-semiprime near-rings which has the matrix extension property, i.e. if A is a near-ring with identity, then $A \in \mathcal{M}$ if and only if $M_n(A) \in \mathcal{M}$ where $M_n(A)$ is the $n \times n$ matrix near-ring over A. Let N be a 0-symmetric near-ring with identity. Then $\Gamma := (N, N^+, N^+, N)$ is a standard morita context (all multiplications are just the near-ring multiplication). In this case $M_2(\Gamma) \cong M_2(N)$ (cf. [3]) and Γ and $\mathcal{M}$ clearly satisfy the conditions (A) and (B). Hence $\varrho(M_2(\Gamma)) = (\varrho(\Gamma))^*$. But $\varrho(M_2(\Gamma)) \cong \varrho(M_2(N))$ and

$$(\varrho(\Gamma))^* \cong \left\{U \in M_2(N) \mid U \begin{bmatrix} a \\ b \end{bmatrix} \subseteq \begin{bmatrix} \varrho(N) \\ \varrho(N) \end{bmatrix}\right\} = (\varrho(N))^*.$$

Hence $\varrho(M_2(N)) = (\varrho(N))^*$, confirming a well-known result (cf. [7]).

4.2. Let $\mathcal{M}$ be the class of all 2-semiprime near-rings. Let $\Gamma = (\Gamma_{11}, \Gamma_{12}, \Gamma_{21}, \Gamma_{22})$ be a standard morita context. By Prop. 2.9, if $\mathcal{A}$ is a 2-semiprime ideal (= 3-semiprime ideal) of $M_2(\Gamma)$, then $\mathcal{A}_{ii}$ is a 2-semiprime ideal of Γ_{ii} and so condition (I) is satisfied. If the context Γ has the property that for each $i \in N_2$, whenever Δ_{ii} is a 3-semiprime ideal of Γ_{ii}, then $\Gamma_{ii_c} * \Gamma_{ii_c}^{-1}\Delta_{ii} \subseteq \Delta_{ii}$, then also condition (II) is satisfied by Cor. 2.12.

Let us mention that the context $\Gamma = (\Gamma_{11}, \Gamma_{12}, \Gamma_{21}, \Gamma_{22}) := (N, G, H, M_n(N))$ (cf. [4]) has the property that for any ideal Δ_{ii} of Γ_{ii}, $\Gamma_{ii_c} * \Gamma_{ii_c}^{-1}\Delta_{ii} \subseteq \Delta_{ii}$.

4.3. Let $\mathcal{M}$ be the class of 3-primitive near-rings. Then $\varrho = J_3$. Anderson, Kaarli and Wiegandt [2] have shown that J_3 is a right strong radical (a note of caution, they deal with *left near-rings* and consequently show that J_3 is *left strong*). Thus $J_3(\Gamma_{ii}) \subseteq \mathcal{T}_{ii}$ where $\mathcal{T} = \varrho(M_2(\Gamma))$ for any standard morita context Γ. This result also follows from condition (I) which we now verify:

4.3.1 Proposition. *The class of* 3*-primitive near-rings satisfies condition* (I).

Proof. Condition (I) will follow from: $M_2(\Gamma)$ 2-primitive implies Γ_{ii} 2-primitive for any standard morita context $\Gamma = (\Gamma_{11}, \Gamma_{12}, \Gamma_{21}, \Gamma_{22})$

(by using Prop. 2.13 and the fact that the concepts 3-primitivity and 2-primitivity coincide on near-rings with identity). If $M_2(\Gamma)$ is 2-primitive, there is a faithful $M_2(\Gamma)$-group G which has no non-trivial $M_2(\Gamma)$-subgroups. Let $H = s^1_{ii}G$. Then H is a subgroup of G, for if $h_1 = s^1_{ii}g_1$ and $h_2 = s^1_{ii}g_2$ are elements of H, then $h_1 - h_2 = s^1_{ii}(g_1 - g_2) \in H$. Here we have used the fact that for any distributive element $U \in M_2(\Gamma)$, $U(g_1 + g_2) = Ug_1 + Ug_2$. Indeed, if $g_1 = 0$ or $g_2 = 0$ it clearly holds. Suppose thus $g_1 \neq 0$ and $g_2 \neq 0$. Then $M_2(\Gamma)g_1 = G = M_2(\Gamma)g_2$ and so $g_1 = Vg_1$ and $g_2 = Wg_1$ for some $V, W \in M_2(\Gamma)$. Then $U(g_1 + g_2) = = U((V + W)g_1) = (U(V + W))g_1 = (UV)g_1 + (UW)g_1 = Ug_1 + Ug_2$.

We now show that H is a faithful Γ_{ii}-group of type 2. Define $\Gamma_{ii} \times H \to H$ by

$$(a, s^1_{ii}g) \longmapsto s^a_{ii}g \quad \text{for all} \quad a \in \Gamma_{ii},\ g \in G.$$

It is well-defined since $s^a_{ii}g = s^1_{ii}(s^a_{ii}g) \in s^1_{ii}G = H$. Also, $(a+b)(s^1_{ii}g) = = as^1_{ii}g + bs^1_{ii}g$ and $(ab)s^1_{ii}g = s^{ab}_{ii}g = s^a_{ii}(s^1_{ii}(s^b_{ii}g)) = a(b(s^1_{ii}g)) = = a(b(s^1_{ii}g))$. Thus H is a Γ_{ii}-group. Moreover, if $aH = 0$, then $s^a_{ii} \in \in (0 : G)_{M_2(\Gamma)} = 0$. Thus $a = 0$ and so H is faithful. Finally, for $0 \neq s^1_{ii}g \in H$, we show $\Gamma_{ii}(s^1_{ii}g) = H$: Let $0 \neq s^1_{ii}g' \in H$. Then $s^1_{ii}g' \in G = M_2(\Gamma)(s^1_{ii}g)$; say $s^1_{ii}g' = Us^1_{ii}g$ for some $U \in M_2(\Gamma)$. Then, for some $a_j \in \Gamma_{ji}$ $(j = 1, 2)$ we have $s^1_{ii}g' = s^1_{ii}(s^1_{ii}g') = s^1_{ii}(Us^1_{ii}g) = = s^1_{ii}U(s^1_{ii} + s^0_{i_ci})g = s^1_{ii}(s^{a_i}_{ii} + s^{a_{i_c}}_{i_ci})g = s^{a_i}_{ii}g = a_i(s^1_{ii}g) \in \mathcal{G}_{ii}(s^1_{ii}g)$. As the other inclusion is obvious, we have $H = \Gamma_{ii}(s^1_{ii}g)$. Thus Γ_{ii} is a 2-primitive near-ring. $\Diamond$

4.4. Let $\mathcal{M}$ be the class of equiprime near-rings. Recall, a near-ring N is *equiprime* if $anx = any$ for all $n \in N$ implies $a = 0$ or $x = y$. Then $\varrho = e$ is the equiprime radical.

4.4.1 Proposition. *The equiprime radical is right strong.*

Proof. Let I be a right ideal of the near-ring N with $e(I) = I$. Let P be any equiprime ideal of N. We show $K := \{x \in I \mid xI \subseteq P)$ is an equiprime ideal of I. It is clearly an ideal. Let $a, x, y \in I$ such that $aix - aiy \in K$ for all $i \in I$. Then $aixj - aiyj \in P$ for all $i, j \in I$. If $a \notin K$, then $ai_0 \notin P$ for some $i_0 \in I$. Suppose also $x - y \notin K$. Then $xj_0 - yj_0 \notin P$ for some $j_0 \in I$. Since P is an equiprime ideal of N, there is an $n_0 \in N$ such that $(ai_0)n_0(xj_0) - (ai_0)n_0(yj_0) \notin P$. But $(ai_0)n_0(xj_0) - (ai_0)n_0(yj_0) = a(i_0n_0)xj_0 - a(i_0n_0)yj_0 \in P$ since $IN \subseteq \subseteq I$, which is a contradiction. Thus K is an equiprime ideal of I and so $I = e(I) \subseteq K$; i.e. $I^2 \subseteq P$. Let $a \in I$. Then $aNa = (aN)a \subseteq I^2 \subseteq P$ and since P is an equiprime ideal, it is 3-prime and so $a \in P$. Thus

$I \subseteq P$ and we conclude that $I \subseteq e(N)$. $\Diamond$

This result yields $e(\Gamma_{ii}) \subseteq \mathcal{T}_{ii}$ where $\mathcal{T} = e(M_2(\Gamma))$. It also follows from condition (I) which we now verify.

4.4.2 Proposition. *The class of equiprime near-rings satisfy condition* (I).

Proof. Let $\mathcal{A}$ be an equiprime ideal of $M_2(\Gamma)$. We show $\mathcal{A}_{ii}$ is an equiprime ideal of Γ_{ii}. Let $a, b, c \in \Gamma_{ii}$ such that $anb - anc \in \mathcal{A}_{ii}$ for all $n \in \Gamma_{ii}$, i.e. $s_{ii}^{anb-anc} \in \mathcal{A}$ for all $n \in \Gamma_{ii}$. Suppose both a and $b - c$ are not in $\mathcal{A}_{ii}$. Then both s_{ii}^{a} and $s_{ii}^{b} - s_{ii}^{c}$ are not in $\mathcal{A}$ and consequently there is a $U \in M_2(\Gamma)$ such that $s_{ii}^{a}Us_{ii}^{b} - s_{ii}^{a}Us_{ii}^{c} \notin \mathcal{A}$. Now $Us_{ii}^{1} = s_{1i}^{x_1} + s_{2i}^{x_2}$ for some $x_j \in \Gamma_{ji}$, $j = 1, 2$ (cf. 1.7), and so $s_{ii}^{ax_ib-ax_ic} = s_{ii}^{a}(s_{1i}^{x_1}+s_{2i}^{x_2})s_{ii}^{b} - s_{ii}^{a}(s_{1i}^{x_1}+s_{2i}^{x_2})s_{ii}^{c} = s_{ii}^{a}Us_{ii}^{1}s_{ii}^{b} - s_{ii}^{a}Us_{ii}^{1}s_{ii}^{c} = s_{ii}^{a}Us_{ii}^{b} - s_{ii}^{a}Us_{ii}^{c} \notin \mathcal{A}$; a contradiction. $\Diamond$

4.4.3 Proposition. *Let* $\Gamma = (\Gamma_{11}, \Gamma_{12}, \Gamma_{21}, \Gamma_{22})$ *be a standard morita context such that* $\Gamma_{ii_c} * \Gamma_{ii_c}^{-1}\Delta_{ii} \subseteq \Delta_{ii}$ *for any equiprime ideal* Δ_{ii} *of* Γ_{ii}. *Then* Γ *and* $\mathcal{M}$ *satisfy condition* (II).

Proof. Let Δ_{ii} be an equiprime ideal of Γ_{ii}. We show that Δ^* is an equiprime ideal of $M_2(\Gamma)$ where Δ is the ideal of $\Gamma = (\Gamma_{ii}, \Gamma_{ii_c}, \Gamma_{i_ci}, \Gamma_{i_ci_c})$, cf. Prop. 2.2 and our assumption, defined by $\Delta = (\Delta_{ii}, \Delta_{ii_c}, \Delta_{i_ci}, \Delta_{i_ci_c}) := (\Delta_{ii}, \Delta_{ii}\Gamma_{i_ci}^{-1}, \Gamma_{ii_c}^{-1}\Delta_{ii}, \Gamma_{ii_c}^{-1}\Delta_{ii}\Gamma_{i_ci}^{-1})$. For this we need a preliminary result:

4.4.4 Lemma. *For* $n, k, j \in N_2$, *if* $a \in \Gamma_{nk}$, $x, y \in \Gamma_{kj}$ *and* $abx - aby \in \Delta_{nj}$ *for all* $b \in \Gamma_{kk}$, *then* $a \in \Delta_{nk}$ *or* $x - y \in \Delta_{kj}$. *In particular, for* $n = k = j = i_c$, $\Delta_{i_ci_c} = \Gamma_{ii_c}^{-1}\Delta_{ii}\Gamma_{i_ci}^{-1}$ *is an equiprime ideal of* $\Gamma_{i_ci_c}$.

Proof. As every equiprime ideal is 3-semiprime, we may use the second part of Cor. 2.12 (which we often do without any further mentioning). If $a \notin \Delta_{nk} = \Gamma_{in}^{-1}\Delta_{ik}$, then $ua \notin \Delta_{ik} = \Delta_{ii}\Gamma_{ki}^{-1}$ for some $u \in \Gamma_{in}$. Thus $uav \notin \Delta_{ii}$ for some $v \in \Gamma_{ki}$. For any $q \in \Gamma_{ik}$, $c \in \Gamma_{ji}$ and $d \in \Gamma_{ii}$, $(uav)d(qxc) - (uav)d(qyc) = [u(avdqx - avdqy) + avdqy) - uavdqy]c \in (\Gamma_{in} * \Delta_{nj})\Gamma_{ji} \subseteq \Delta_{ij}\Gamma_{ji} \subseteq \Delta_{ii}$. Since Δ_{ii} is an equiprime ideal of Γ_{ii}, we have $qxc - qyc \in \Delta_{ii}$ for all $q \in \Gamma_{ik}$, $c \in \Gamma_{ji}$. Thus $qx - qy \in \Delta_{ii}\Gamma_{ji}^{-1} = \Delta_{ij}$ for all $q \in \Gamma_{ik}$. Since $1 = 1_{\Gamma_{kk}} \in \Gamma_{kk} = \overline{\Gamma_{ki}\Gamma_{ik}}$, we have $1 = \sum_{t=1}^{m} \sigma_t g_t h_t$ where $\sigma_t \in \{+, -\}$, $g_t \in \Gamma_{ki}$ and $h_t \in \Gamma_{ik}$. Now

$$x - y = 1x - 1y = \sigma_1 g_1 h_1 x + \ldots + \sigma_m g_m h_m x - \sigma_m g_m h_m y - \ldots - \sigma_1 g_1 h_1 y.$$

For each t, $\sigma_t g_t h_t x - \sigma_t g_t h_t y = (\sigma_t g_t)[(h_t x - h_t y) + h_t y] - (\sigma_t g_t)h_t y \in \Gamma_{ki} * \Delta_{ij} \subseteq \Delta_{kj}$, which is normal in Γ_{kj}, and we may conclude that $x - y \in \Delta_{kj}$. $\Diamond$

Proof (of 4.4.3). We abbreviate an element $\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$ of Γ^+ by $[x_{ij}]$. Let $A, B, C \in M_2(\Gamma)$ such that $AUB - AUC \in \Delta^*$ for all $U \in M_2(\Gamma)$. Suppose $A \notin \Delta^*$ and $B - C \notin \Delta^*$. Then $[a_{ij}] = A[z_{ij}] \notin \Delta^+$ + and $[b_{ij} - c_{ij}] = B[x_{ij}] - C[x_{ij}] = (B - C)[x_{ij}] \notin \Delta^+$ for some $[z_{ij}], [x_{ij}] \in \Gamma^+$. Suppose $a := a_{kj} \notin \Delta_{kj}$ and $b - c = b_{pq} \notin \Delta_{pq}$. Now $a \notin \Delta_{kj} = \Delta_{kp}\Gamma_{jp}^{-1}$ implies $au \notin \Delta_{kp}$ for some $u \in \Gamma_{jp}$. From the above Lemma, we know there is a $d \in \Gamma_{pp}$ such that $audb - -audc \notin \Delta_{kq}$. Let $V := (s_{1j}^{z_{ij}} + s_{2j}^{z_{2j}})s_{jp}^{ud} \in M_2(\Gamma)$. Since $AVB - AVC \in \in \Delta^*$, also $s_{kk}^1 AVB - s_{kk}^1 AVC \in \Delta^*$. Thus $(s_{kk}^1 AVB - s_{kk}^1 AVC)[x_{ij}] = = s_{kk}^1(s_{1j}^{a_{1j}} + s_{2j}^{a_{2j}})s_{jp}^{ud}[b_{ij}] - s_{kk}^1(s_{1j}^{a_{1j}} + s_{2j}^{a_{2j}})s_{jp}^{ud}[c_{ij}] = s_{kp}^{aud}[b_{ij}] - s_{kp}^{aud}[c_{ij}] = = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \in \Delta^+$ where $y_{kt} = audb_{pt} - audc_{pt}$ and $y_{k_ct} = 0$ for $t = = 1, 2$. In particular, for $t = q$, we get $y_{kq} = audb - audc \in \Delta_{kq}$ — a contradiction. ◊

References

[1] AMITSUR, S. A.: Rings of Quotients and Morita Contexts, *J. Algebra* **17** (1971), 273–298.

[2] ANDERSON, T., KAARLI, K. and WIEGANDT, R.: On left strong radicals of near-rings, *Proc. Edinburgh Math. Soc.* **31** (1988), 447-456.

[3] KYUNO, S. and VELDSMAN, S.: Morita near-rings, *Quaest. Math.* **15** (1992), 431–449.

[4] KYUNO, S. and VELDSMAN, S.: A lattice isomorphism between sets of ideals of the near-rings in a near-ring morita context, *Comm. Algebra*, to appear.

[5] MLITZ, R. and VELDSMAN, S.: Radicals and subdirect decompositions in Ω-groups, *J. Austral. Math. Soc.* **48** (1990), 171–198.

[6] SANDS, A. D.: Radicals and Morita Contexts, *J. Algebra* **24** (1973), 335–345.

[7] VELDSMAN, S.: Special radicals and matrix near-rings, *J. Austral. Math. Soc.* **52** (1992), 356–367.