Mathematica Pannonica
5/2 (1994), 145 – 168

ON A PROBLEM OF R. SCHILLING II

Wolfgang **Förg-Rob**

Institut für Mathematik, Universität Innsbruck, Technikerstraße 25/7, A-6020 Innsbruck, Austria

Received January 1994

AMS Subject Classification: 39 B 22, 26 A 30

Keywords: Schilling's equation, Schilling's problem.

Abstract: Studies of a physical problem led to the functional equation

(1) $f(qx)=\frac{1}{4q}(f(x+1)+f(x-1)+2f(x))$ for all $x\in\mathbb{R}$

with the boundary condition

(2) $f(x)=0$ for all x with $|x|>Q:=\frac{q}{1-q}$

where $q\in]0,1[$ is a fixed real number. It turns out that the behaviour of solutions of (1) which fulfill the boundary condition (2) is quite different, depending heavily on the value of q. An — in some sense "complete" — answer on the general solution of (1) under the condition (2) (including investigations on continuity, differentiability, measurability, integrability) can be given in the following cases: $0<q<\frac{1}{3}$, $q=\frac{1}{3}$, and $q=\frac{1}{2}$.

Studies of a physical problem (cf. [4]) led Prof. R. Schilling to the functional equation given below. It was known that in the case $q=\frac{1}{2}$ there is a continuous solution with bounded support. Now the question arose to find all the solutions of this equation.

Let the functional equation

$$(1)\qquad f(qx)=\frac{1}{4q}\big(f(x+1)+f(x-1)+2f(x)\big)\quad\text{for all } x\in\mathbb{R}$$

and the boundary condition

$$(2)\qquad f(x)=0\quad\text{for all } x \text{ with }\quad |x|>Q:=\frac{q}{1-q}$$

be given, where $q\in]0,1[$ is a fixed real number. In the previous paper [2] we dealt with the problem of finding solutions of equ. (1) with unbounded support. Now we turn over to some results on solutions with bounded support. As we will see, the problem (and its solutions) have a quite different behaviour depending on the values of q. Though the problem in general is far from being solved, some special cases can be treated in the sequel:

III. Solutions with bounded support

a) The case $0 < q < \frac{1}{3}$

In this case we have $0 < Q < \frac{1}{2} < 1 - Q < 1$. The following theorem will give 5 conditions (α)-(ε) equivalent to (1)-(2). Though they are more than two (as originally given) their structure is much more easier than (1)+(2):

Theorem 20. *Let $q < \frac{1}{3}$. Then the system* (1) *and* (2) *is equivalent to the system*

(α)	$f(x) = 2qf(qx)$	*for all* $x \in [-Q, Q]$
(β)	$f(x) = 4qf(q(x+1))$	*for all* $x \in [-Q, Q]$
(γ)	$f(x) = 4qf(q(x-1))$	*for all* $x \in [-Q, Q]$
(δ)	$f(x) = 0$	*for all* x *with* $qQ < \lvert x\rvert < q(1-Q)$
(ε)	$f(x) = 0$	*for* $\lvert x\rvert > Q$.

Proof. (a) Let f fulfill (1) and (2). We show that f is a solution of (α)-(ε):

(ε) is trivial, as (2)=(ε).

(α) Let $x \in [-Q, Q]$. Then $x + 1 > Q$, $x - 1 < -Q$. Thus $f(x+1) = f(x-1) = 0$ and $f(qx) = \frac{1}{4q} 2f(x)$, i.e. ($\alpha$) holds.

(β) Let $x \in [-Q, Q]$. Then $y := x + 1 \in [1 - Q, 1 + Q]$ and therefore $y > Q$, $y + 1 > Q$, $y - 1 = x \in [-Q, Q]$, $qy \in [q(1-Q), q(1+Q)]$. Remembering that $q(1+Q) = Q$ we have $qy \in [-Q, Q]$ and $f(qy) = \frac{1}{4q} f(y-1)$, thus $f(q(x+1)) = \frac{1}{4q} f(x)$.

(γ) like (β).

(δ) Let $qQ < |x| < q(1-Q)$ and put $y := \frac{x}{q}$. Then $Q < |y| < 1-Q$ and therefore $|y| > Q$, $|y+1| > Q$, $|y-1| > Q$, thus $f(y) = f(y+1) = f(y-1) = 0$. That implies $f(x) = f(qy) = 0$.

(b) On the other hand, suppose that f is a function which fulfills (α)-(ε). We show that f is a solution of (1)-(2):

(2) is trivial, as (2)=(ε).

(1) If $x \in [-Q,Q]$ then $x-1<-Q$, $x+1>Q$, and therefore $f(x+1)=f(x-1)=0$ by (ε). Thus (α) implies that $f(qx)=\frac{1}{2q}f(x)= = \frac{1}{4q}\big(f(x+1)+f(x-1)+2f(x)\big)$. If $Q<|x|<1-Q$ then $|x|>Q$, $|x-1|>Q$, $|x+1|>Q$. By (ε) $f(x)=f(x+1)=f(x-1)=0$. By (δ) $f(qx)=0$, and thus $f(qx)=\frac{1}{4q}\big(f(x+1)+f(x-1)+2f(x)\big)$. If $x\in[1-Q,1+Q]$, then $x>Q$, $x+1>Q$, $x-1\in[-Q,Q]$ and $qx\in[-Q,Q]$. (β) and (ε) imply $f(x-1)=4qf(qx)$ and therefore $f(qx)=\frac{1}{4q}\big(f(x+1)+f(x-1)+2f(x)\big)$. The case $x\in[-(1+Q),-(1-Q)]$ is treated like $[1-Q,1+Q]$ by use of (γ) and (ε). If $|x|>1+Q$, then $|x|>Q$, $|x+1|>Q$, $|x-1|>Q$, $|qx|>Q$, and therefore (ε) implies that (1) is fulfilled. $\Diamond$

As the next theorem shows, the conditions (α)–(ε) give rise to a detailed description of all the solutions:

Theorem 21. *Suppose that f is a solution of (α)–(ε). Let*

$$A_1 := \{x \mid qQ<|x|<q(1-Q)\}$$

and $\varphi_0,\varphi_1,\varphi_{-1}\colon [-Q,Q]\to[-Q,Q]$ be the functions

$$\varphi_0(x):=qx,\quad \varphi_1(x):=q(x+1),\quad \varphi_{-1}(x):=q(x-1).$$

Define the sets A_n recursively by

$$A_{n+1}:=\varphi_0(A_n)\cup\varphi_1(A_n)\cup\varphi_{-1}(A_n).$$

Then the following holds:

(a) *Each A_n and the set $A:=\bigcup_{n\in\mathbb{N}} A_n$ are open;*

(b) *$\lambda(A)=2Q$, where λ is the Lebesgue measure on $\mathbb{R}$;*

(c) *$f(x)=0$ for each $x\in A$ (i.e. $f=0$ a.e.);*

(d) *$[-Q,Q]\setminus A=\left\{x \mid x=\sum_{n=1}^{\infty} a_i q^i,\text{ where } a_i\in\{0,1,-1\}\right\}$, and this set is uncountable.*

Proof. $\varphi_0,\varphi_1,\varphi_{-1}$ are linear-affine, order-preserving homeomorphisms.

(a) As A_1 is open, by induction each A_n is open and, therefore, the set A, too.

(b) First we show that $(A_n)_{n\in\mathbb{N}}$ is a family of pairwise disjoint sets. We compute a detailed description of A_n. Let $J:=]Q,1-Q[$. Then $A_1=qJ\cup(-q)J$, and by induction one can easily see that A_n is the union of all the sets

$$\sum_{i=1}^{n-1}a_iq^i+q^nJ \quad\text{and}\quad \sum_{i=1}^{n-1}a_iq^i-q^nJ,$$

where $a_i\in\{0,1,-1\}$. We show that all these sets are disjoint: Let

$m, n \in \mathbb{N}$ and $a_i, b_j \in \{0, 1, -1\}$, and

$$x \in \Big(\sum_{i=1}^{m-1} a_i q^i \pm q^m J\Big) \cap \Big(\sum_{j=1}^{n-1} b_j q^j \pm q^n J\Big).$$

Suppose that $a_1 > b_1$ and $m, n \geq 2$: As $J \subseteq]0,1[$ we have

$$x \geq a_1 q - \sum_{i=1}^{\infty} 1 q^i = a_1 q - q\frac{q}{1-q} > a_1 q - \frac{1}{2}q;$$

on the other hand

$$x \leq b_1 q + \sum_{i=2}^{\infty} 1 q^i = b_1 q + q\frac{q}{1-q} < b_1 q + \frac{1}{2}q,$$

a contradiction. Thus, if the intersection of two such sets is nonvoid then necessarily $a_1 = b_1$, and by a simple induction argument we are reduced to the case

$$x \in \Big(\sum_{i=1}^{m-1} a_i q^i \pm q^m J\Big) \cap (qJ).$$

Suppose $m \geq 2$: If $a_1 = 0$ or $a_1 = -1$ then $x < \sum_{i=2}^{\infty} 1q^i = q\frac{q}{1-q} = qQ$, and $x \in qJ$ implies $x > qQ$, a contradiction. If $a_1 = 1$, then $x < q(1-Q)$, and $x > q - \sum_{i=2}^{\infty} 1q^i = q - q\frac{q}{1-q} = q(1-Q)$, a contradiction. Therefore, the only possible case is $m = 1$, and we have shown that all these sets are disjoint. Now the Lebesgue measure of an interval $\sum_{i=1}^{n-1} a_i q^i \pm q^n J$ is equal to $q^n \lambda(J) = q^n(1-2Q)$, and therefore we get $\lambda(A) = 2\lambda(J) \sum_{n=1}^{\infty} q^n 3^{n-1}$, because there are 3^{n-1} polynomials $\sum_{i=1}^{n-1} a_i q^i$ with $a_i \in \{0, 1, -1\}$. Thus

$$\lambda(A) = 2q\frac{1}{1-3q}(1-2Q) = 2q\frac{1}{1-3q}\,\frac{1-3q}{1-q} = 2Q.$$

(c) Using $f(x) = 0$ for $x \in A_1$ (by (δ)), by induction and (α), (β), (γ) we can show that $f(x) = 0$ for $x \in A_n$, where $n \in \mathbb{N}$. Thus $f(x) = 0$ for each $x \in A$.

(d) As A is an open set $[-Q, Q] \setminus A$ is a closed set which contains all the border points of the intervals $\sum_{i=1}^{n-1} a_i q^i \pm q^n J$, that is, all the points

$$\sum_{i=1}^{n-1} a_i q^i \pm q^n Q = \sum_{i=1}^{n-1} a_i q^i \pm \sum_{i=n+1}^{\infty} 1 q^i$$

and

$$\sum_{i=1}^{n-1} a_i q^i \pm q^n(1-Q) = \sum_{i=1}^{n-1} a_i q^i \pm \Big(q^n + \sum_{i=n+1}^{\infty} (-1) q^i\Big).$$

Furthermore, all the limit points of these border points belong to the set $[-Q,Q] \setminus A$.

On the other hand, $\lambda(A) = 2Q$, and therefore $[-Q,Q]$ contains no proper interval belonging to $[-Q,Q] \setminus A$. Thus every element of $[-Q,Q] \setminus A$ is a limit point of the points given above. Now the set $\Big\{x \,|\, x = \sum_{n=1}^{\infty} a_i q^i$, where $a_i \in \{0,1,-1\}\Big\}$ is homeomorphic to the product space $\{0,1,-1\}^{\mathbb{N}}$ via the bijection $(a_i)_{i\in\mathbb{N}} \to \sum_{i=1}^{\infty} a_i q^i$, because $q < \frac{1}{3}$. It is easy to see that the set of border points of the intervals is dense in the set

$$\Big\{x \mid x = \sum_{n=1}^{\infty} a_i q^i, \quad \text{where} \quad a_i \in \{0,1,-1\}\Big\},$$

and this set is closed. $\Diamond$

Corollary 4. *Let $q < \frac{1}{3}$. Then any solution of* (1) *and* (2) *is equal to* 0 *almost everywhere (and therefore measurable). Thus any continuous solution is identically zero.*

The next theorem gives an idea how to find all the solutions of (1) and (2) in the case $q < \frac{1}{3}$. But before we have to give a definition.

Definition 2. Let $q < \frac{1}{3}$ and use the notations of Th. 21. Let

$$B := [-Q,Q] \setminus A = \Big\{x \mid x = \sum_{n=1}^{\infty} a_i q^i, \text{ where } a_i \in \{0,1,-1\}\Big\}.$$

Define a relation $\sim$ on B by

$$\sum_{i=1}^{\infty} a_i q^i \sim \sum_{i=1}^{\infty} b_i q^i : \Leftrightarrow \exists m,n \in \mathbb{N} : a_{m+i} = b_{n+i} \text{ for all } i \in \mathbb{N}.$$

It is easy to see that $\sim$ is an equivalence relation on B. Furthermore, $\sim$ has the following property:

Lemma 4. *If $x, y \in B$, then $x \sim y$ iff there is a $z \in B$ and there are $\psi_1, \dots, \psi_k$, $\omega_1, \dots, \omega_p \in \{\varphi_1, \varphi_0, \varphi_{-1}\}$ such that $x = \psi_1(\psi_2(\dots\psi_k(z)\dots))$ and $y = \omega_1(\omega_2(\dots\omega_p(z)\dots))$.*

Proof. For $j \in \{-1, 0, 1\}$ we have $\varphi_j\Big(\sum\limits_{i=1}^{\infty} c_i q^i\Big) = jq + \sum\limits_{i=2}^{\infty} c_{i-1} q^i$. Now let $x = \sum\limits_{i=1}^{\infty} a_i q^i$, $y = \sum\limits_{i=1}^{\infty} b_i q^i$. First suppose that $x \sim y$ and let m, n be as in the definition. Let $z = \sum\limits_{i=1}^{\infty} a_{m+i} q^i = \sum\limits_{i=1}^{\infty} b_{n+i} q^i$. Then

$$x = \varphi_{a_1}\big(\varphi_{a_2}(\dots\varphi_{a_m}(z)\dots)\big) \quad\text{and}\quad y = \varphi_{b_1}\big(\varphi_{b_2}(\dots\varphi_{b_n}(z)\dots)\big).$$

On the other hand, if $z = \sum\limits_{i=1}^{\infty} c_i q^i$ and

$$x = \varphi_{\alpha_1}\big(\varphi_{\alpha_2}(\dots\varphi_{\alpha_m}(z)\dots)\big) \quad\text{and}\quad y = \varphi_{\beta_1}\big(\varphi_{\beta_2}(\dots\varphi_{\beta_n}(z)\dots)\big)$$

for $\alpha_1, \alpha_2, \dots, \alpha_m, \beta_1, \beta_2, \dots, \beta_n \in \{-1, 0, 1\}$, then $a_{m+i} = c_i = b_{n+i}$. ◊

Theorem 22. *Let f be a solution of (α)–(ε). Then the following holds:*

(a) *$f(x) = 0$ for $x \notin B$;*

(b) *for $x \in B$, the value $f(x)$ determines the values $f(y)$ on the equivalence class $[x] = \{y \mid y \sim x\}$;*

(c) *if $x = \sum\limits_{i=1}^{\infty} a_i q^i$ is periodic, i.e. there are positive integers m, p such that $a_{m+i} = a_{m+p+i}$ for all $i \in \mathbb{N}$, and if s is the number of zeroes in the period, i.e. $s = \#\{i \mid m < i \le m+p, a_i = 0\}$, then $f(x) = 0$ whenever $(4q)^p \ne 2^s$.*

Proof. (a) was shown in Th. 21.

(b) is a trivial consequence of Lemma 4 and equations (α), (β), (γ).

(c) We have

$$x \sim \sum_{i=1}^{\infty} a_{m+i} q^i = \sum_{i=1}^{\infty} a_{m+p+i} q^i =: y.$$

Then $y = \varphi_{a_{m+1}}\big(\varphi_{a_{m+2}}(\dots\varphi_{a_{m+p}}(y)\dots)\big)$ and therefore by (α), (β), (γ): $f(y) = (2q)^s(4q)^{p-s} f(y)$. Thus, $f(y) = 0$ whenever $(2q)^s(4q)^{p-s} \ne 1$, and in this case we have $f(x) = 0$ by (α)–(γ). ◊

Let us denote by B_p the set $B_p := \{x \mid x \in B \text{ and } x \text{ periodic}\}$, and by B_{np} the set $B \setminus B_p$ (those x which are not periodic), then we can state the following

Lemma 5.

(a) *If* $x \in B_p$ *and* $x \sim y$, *then* $y \in B_p$.

(b) *The set of periods of minimal length form a system of representatives for* $\sim$ *on* B_p.

The proof is easy and omitted. ◊

Now we can give the general solution of (α)–(ε) in the case $q < \frac{1}{3}$. We do this in the next 3 theorems, distinguishing between different cases.

Theorem 23. *Suppose that* $0 < q < \frac{1}{4}$. *Then the general solution can be given in the following way:*

(a) $f(x) = 0$ *for* $x \in A$ *and* $|x| > Q$.

(b) *Let* $(x_k)_{k \in K}$ *be a system of representatives for* $\sim$ *on* B_{np}, *choose* $(f(x_k))_{k \in K}$ *arbitrarily and extend* f *onto the equivalence class* $[x_k]$ *of* x_k *as described in* Th. 22(b).

(c) $f(x) = 0$ *for* $x \in B_p$.

Theorem 24. *Suppose that* $\frac{1}{4} \leq q < \frac{1}{3}$ *and that* $q = 2^r$ *for some rational* r. *Then the general solution can be given in the following way;*

(a) $f(x) = 0$ *for* $x \in A$ *and* $|x| > Q$.

(b) *Let* $(x_k)_{k \in K}$ *be a system of representatives for* $\sim$ *on* B_{np}, *choose* $(f(x_k))_{k \in K}$ *arbitrarily and extend* f *onto the equivalence class* $[x_k]$ *of* x_k *as described in* Th. 22(b).

(c) *Let* $(\beta_p)_{p \in P}$ *be the system of periods of minimal length (which is a system of representatives for* $\sim$ *on* B_p*) and denote by* $\ell(\beta_p)$ *the length and by* $z(\beta_p)$ *the number of zeroes of* β_p. *Choose values* $g(\beta_p)$ *in the following way:*

$$g(\beta_p) = \begin{cases} \text{arbitrary} & \text{if } \dfrac{z(\beta_p)}{\ell(\beta_p)} = 2 + r \\ 0 & \text{otherwise.} \end{cases}$$

For any $p \in P$ *choose an element* $x_p \in B_p$ *with period* β_p, *define* $f(x_p) := g(\beta_p)$ *and extend* f *onto* B_p *as in* (b).

Theorem 25. *Suppose that* $\frac{1}{4} \leq q < \frac{1}{3}$ *and that* $q \neq 2^r$ *for any rational* r. *Then the general solution can be given in the following way:*

(a) $f(x) = 0$ *for* $x \in A$ *and* $|x| > Q$.

(b) *Let* $(x_k)_{k \in K}$ *be a system of representatives for* $\sim$ *on* B_{np}, *choose* $(f(x_k))_{k \in K}$ *arbitrarily and extend* f *onto the equivalence class* $[x_k]$ *of* x_k *as described in* Th. 22(b).

(c) $f(x) = 0$ *for* $x \in B_p$.

Proof. (a) was shown in Th. 21.

(b) If x is nonperiodic, $x = \sum_{i=1}^{\infty} a_i q^i$, for no $m \neq n$ the equality $a_{m+i} = a_{n+i}$ can hold for all $i \in \mathbb{N}$. Therefore it is easy to see that the computation of $f(y)$ for $y \sim x$ cannot give "different results" via the choice of a z as in Lemma 4.

(c) If $x \sim y$ and x is periodic, then y has the same period as x. By Th. 22 $f(x) \neq 0$ is possible only in the case when $(4q)^\ell = 2^z$, that is, $q = 2^r$, where $r = \frac{z}{\ell} - 2$. ◊

Corollary 5. *Let* $q = \frac{1}{4}$. *Then* $Q = \frac{1}{3} = \sum_{i=1}^{\infty} 1\left(\frac{1}{4}\right)^i$, *which is periodic with period* 1. *Also* $-Q = -\frac{1}{3} = \sum_{i=1}^{\infty}(-1)\left(\frac{1}{4}\right)^i$ *is periodic with period* 1. *Furthermore,* $-Q$ *is not equivalent to* Q. *Thus* $\ell = 1$, $z = 0$, *and as* $\frac{1}{4} = 2^{-2}$ *and* $2+(-2) = \frac{0}{1}$ *there exist solutions* f *of* (1)-(2) *in this case such that*

$$\begin{array}{llll} f(Q) \neq 0 & \text{and} & f(-Q) \neq 0 & \text{or} \\ f(Q) = 0 & \text{and} & f(-Q) \neq 0 & \text{or} \\ f(Q) \neq 0 & \text{and} & f(-Q) = 0 & \text{or} \\ f(Q) = 0 & \text{and} & f(-Q) = 0. & \end{array}$$

In other words, we have seen that any of the cases $-Q, Q \in S(f) \subseteq [-Q, Q]$, $-Q \in S(f) \subseteq [-Q, Q[$, $Q \in S(f) \subseteq]-Q, Q]$, $S(f) \subseteq]-Q, Q[$ *really can occur.*

After this investigation into the case $q < \frac{1}{3}$ we turn over to the next value for q:

b) The case $\mathbf{q = \frac{1}{3}}$

The methods used in this case to give the solutions of (1)-(2) are very similar to those used in the case $q < \frac{1}{3}$. First we give a system equivalent to (1)-(2):

Theorem 26. *Let* $q = \frac{1}{3}$. *Then* $Q = \frac{1}{2}$. *and the system* (1)-(2) *is equivalent to the system*

$$(\alpha)\qquad f(x)=\frac{2}{3}f\Big(\frac{1}{3}x\Big)\qquad \textit{for all } x\in\Big[-\frac{1}{2},\frac{1}{2}\Big]$$

$$(\beta)\qquad f(x)=\frac{4}{3}f\Big(\frac{1}{3}(x+1)\Big)\qquad \textit{for all } x\in\Big[-\frac{1}{2},\frac{1}{2}\Big]$$

$$(\gamma)\qquad f(x)=\frac{4}{3}f\Big(\frac{1}{3}(x+1)\Big)\qquad \textit{for all } x\in\Big[-\frac{1}{2},\frac{1}{2}\Big]$$

$$(\delta)\qquad f(x)=0\qquad \textit{for all } |x|\geq\frac{1}{2}.$$

Proof. This proof is nearly the same as the proof of Th. 20, but we have to take care that $x\in[-Q,Q]$ does not imply that $x+1$, $x-1\notin[-Q,Q]$, because $-Q+1=Q$.

(a) Suppose that f is a solution of (1) and (2). We show that f fulfills (α)–(δ):

(δ) follows directly from (2).

(α)–(γ) is shown in the same manner as in Th. 20 for all values $x\in\left]-\frac{1}{2},\frac{1}{2}\right[$ (for these values the same arguments as in Th. 20 hold). $x=\pm\frac{1}{2}$: As $f\big(\frac{1}{2}\big)=f\big(-\frac{1}{2}\big)=0$, (α)–(γ) are trivial consequences of (1) and (2).

(b) On the other hand, let f be a solution of (α)–(δ).

(2) is a trivial consequence of (δ), and (1) can be derived from (α)–(δ) as in the case $q<\frac{1}{3}$. $\Diamond$

As in the case $q<\frac{1}{3}$, we give an equivalence relation $\sim$ in order to describe the solutions. Therefore, let $I:=\left[-\frac{1}{2},\frac{1}{2}\right]$ and denote by $\varphi_0,\varphi_1,\varphi_{-1}\colon I\to I$ the functions $\varphi_j(x)=\frac{1}{3}(x+j)$. We denote by A the set

$$A:=\left\{\psi_1\big(\psi_2(\ldots\psi_k(z)\ldots)\big)\;\middle|\;\begin{array}{c}k\in\mathbb{Z},k\geq0,z\in\left\{0,\frac{1}{2},-\frac{1}{2}\right\},\\ \psi_1\ldots\psi_k\in\{\varphi_1,\varphi_0,\varphi_{-1}\}\end{array}\right\},$$

and let $B:=I\setminus A$. For two numbers

$$x=\sum_{i=1}^{\infty}a_i\Big(\frac{1}{3}\Big)^i\quad\text{and}\quad y=\sum_{i=1}^{\infty}b_i\Big(\frac{1}{3}\Big)^i,$$

where $a_i,b_j\in\{0,1,-1\}$ and $x,y\in B$, we define

$$x\sim y\colon\;\Leftrightarrow\;\exists m,n\colon\; a_{m+i}=b_{n+i}\;\text{ for all } i\in\mathbb{N}.$$

Lemma 6. *With the notation above the following holds:*

(a) $A=\left\{x=\sum\limits_{i=1}^{n}a_i\big(\frac{1}{3}\big)^i+\alpha\big(\frac{1}{3}\big)^n\;\middle|\;\begin{array}{c}n\in\mathbb{N},\;a_i\in\{0,1,-1\},\\ \alpha\in\left\{0,\frac{1}{2},-\frac{1}{2}\right\}\end{array}\right\};$

(b) *Each* $x \in I$ *has a representation* $x = \sum_{i=1}^{\infty} a_i\left(\frac{1}{3}\right)^i$, *where* $a_i \in \in \{0, 1, -1\}$. *This representation is unique whenever* $x \in B$. *Furthermore, the real number* $\sum_{i=1}^{\infty} a_i\left(\frac{1}{3}\right)^i$ *(where* $a_i \in \{(0, 1, -1\}$*) is an element of* B *iff this representation has no period of length* 1 *(i.e. for all* $m \in \mathbb{N}$ *there is an* $n \in \mathbb{N}$ *such that* $n > m$ *and* $a_n \neq a_m$*).*

(c) *The relation* $\sim$ *is well-defined and an equivalence relation on the set* B.

(d) *Let* $\mathbb{Q}$ *denote the set of rational numbers. Then*

$$I \cap \mathbb{Q} = A \cup \{x \in B \mid x \textit{ has a periodic representation}\}.$$

(e) *If* $x, y \in B$, x *rational and* $x \sim y$, *then* y *is rational, too.*

Proof. (a) Let

$$A' := \left\{x = \sum_{i=1}^{n} a_i\left(\frac{1}{3}\right)^i + \alpha\left(\frac{1}{3}\right)^n \;\middle|\; \begin{array}{c} n \in \mathbb{N},\ a_i \in \{0, 1, -1\}, \\ \alpha \in \{0, \frac{1}{2}, -\frac{1}{2}\} \end{array}\right\}.$$

It is easy to see that $A \subseteq A'$ and $A' \subseteq A$. (b) and (d) are well known from elementary analysis, (c) and (e) are immediate consequences. ◊

Now let $B_r := B \cap \mathbb{Q}$ be the rational points of B, and $B_{np} := := B \backslash B_r$ the set of those $x \in B$ which have a nonperiodic representation. Furthermore, let $(x_k)_{k \in K}$ be a system of representatives for $\sim$ on the set B_{np}. The preceding lemma gives the technical details for proving the following theorems on the structure of the solutions.

Theorem 27. *Let* f *be a solution of* (α)–(δ). *Then:*

(a) *If* $x, y \in B$, $x \sim y$, $x = \sum_{i=1}^{\infty} a_i\left(\frac{1}{3}\right)^i$, $y = \sum_{i=1}^{\infty} b_i\left(\frac{1}{3}\right)^i$ *and* $m, n \in \mathbb{N}$ *such that* $a_{m+i} = b_{n+i}$ *for all* $i \in \mathbb{N}$, *then* $\alpha_{a_1}\alpha_{a_2}\ldots\alpha_{a_m} f(x) = = \alpha_{b_1}\alpha_{b_2}\ldots\alpha_{b_n} f(y)$, *where* $\alpha_0 = \frac{2}{3}$ *and* $\alpha_1 = \alpha_{-1} = \frac{4}{3}$;

(b) $f(x) = 0$ *for all* $x \in \mathbb{Q}$.

Proof. (a) Let

$$z := \sum_{i=1}^{\infty} a_{m+i}\left(\frac{1}{3}\right)^i = \sum_{i=1}^{\infty} b_{n+i}\left(\frac{1}{3}\right)^i.$$

Then

$$x = \varphi_{a_1}\left(\varphi_{a_2}(\ldots(\varphi_{a_m}(z)\ldots)\right) \text{ and } y = \varphi_{b_1}\left(\varphi_{b_2}(\ldots\varphi_{b_n}(z)\ldots)\right).$$

By (α)–(γ) we have $\alpha_j f(\varphi_j(u)) = f(u)$ for all $u \in I$. Thus

$$f(z) = \alpha_{a_1} \cdot \alpha_{a_2} \cdot \ldots \cdot \alpha_{a_m} \cdot f(x) = \alpha_{b_1} \cdot \alpha_{b_2} \cdot \ldots \cdot \alpha_{b_n} \cdot f(y).$$

(b) If $x \in \mathbb{Q}$, then x has a (not necessarily unique) periodic representation $x = \sum_{i=1}^{\infty} a_i \left(\frac{1}{3}\right)^i$. Let $m, p \in \mathbb{N}$, $p \geq 1$, such that $a_{m+i} = a_{m+p+i}$ for all $i \in \mathbb{N}$. Because of (α)–(γ) we have

$$\alpha_{a_i} \cdot \alpha_{a_2} \cdot \ldots \cdot \alpha_{a_m} \cdot f(x) = \alpha_{a_1} \cdot \alpha_{a_2} \cdot \ldots \cdot \alpha_{a_{m+p}} \cdot f(x),$$

i.e.,

$$\left(\alpha_{a_{m+1}} \cdot \alpha_{a_{m+2}} \cdot \ldots \cdot \alpha_{a_{m+p}} - 1\right) \cdot f(x) = 0.$$

Now $\alpha_{a_{m+1}} \cdot \alpha_{a_{m+2}} \cdot \ldots \cdot \alpha_{a_{m+p}} = \left(\frac{2}{3}\right)^k \left(\frac{4}{3}\right)^{p-k}$ for some $k \in \mathbb{N}$. As this product cannot be equal to 1 we must conclude that $f(x) = 0$. ◊

Corollary 6. *In the case $q = \frac{1}{3}$, the only continuous solution of* (1) *and* (2) *is identically* 0.

Proof. By Th. 27, $f(x) = 0$ for all $x \in \mathbb{R} \setminus I$ and for $x \in I \cap \mathbb{Q}$. ◊

On the other hand, we can give the general solution of equations (α)–(δ):

Theorem 28. *Let $g\colon \{x_k \mid k \in K\} \to \mathbb{R}$ be given arbitrarily and define $f\colon I \to \mathbb{R}$ by*

$$f(x) := \begin{cases} g(x_k) & \text{if } x = x_k \\ \text{defined by the formula given in Th. 26(a)} & \text{if } x \sim x_k \\ 0 & \text{if } x \in I \cap \mathbb{Q} \\ 0 & \text{if } |x| \geq \frac{1}{2}. \end{cases}$$

Then f is a solution of (α)–(δ).

Proof. As any x which is equivalent to some x_k is an element of B_{np}, the representation $x = \sum_{i=1}^{\infty} a_i \left(\frac{1}{3}\right)^i$ is unique. Therefore, f is well-defined. It is easy to see that f fulfills (α)–(δ). By Th. 27, the function f given above is the only possible extension of the given function g. ◊

Next we will show that any measurable solution vanishes almost everywhere.

Theorem 29. *Let $q = \frac{1}{3}$. Then any (Lebesgue-)measurable solution of* (1) *and* (2) *vanishes almost everywhere.*

Proof. Let f be a measurable solution of (α)–(δ), and denote by $A_r := \{x \mid x \in I \text{ and } |f(x)| > r\}$ for any real $r > 0$. As $A_r \subseteq B \subseteq I$ and $\bigcap_{r>0} A_r = \emptyset$, the function $\mu\colon r \to \mu(r) := \lambda(A_r)$ (Lebesgue measure of A_r) is nonincreasing, bounded by 1, and $\lim\limits_{r\to\infty}\mu(r)=0$. Now let $t(s):=\frac{3}{2}s$ and $u(s):=\frac{3}{4}s$. Then one can immediately see that

$$\begin{aligned}\varphi_0(A_s)&=\left]-\frac{1}{6},\frac{1}{6}\right[\cap A_{t(s)} &&\text{by } (\alpha)\\ \varphi_1(A_s)&=\left]\frac{1}{6},\frac{1}{2}\right[\cap A_{u(s)} &&\text{by } (\beta)\\ \varphi_{-1}(A_s)&=\left]-\frac{1}{2},-\frac{1}{6}\right[\cap A_{u(s)} &&\text{by } (\gamma)\end{aligned}$$

for each $s>0$. Using the numbers α_j of Th. 27 we have $A_r = \varphi_0(A_{\alpha_0 r})\cup\varphi_1(A_{\alpha_1 r})\cup\varphi_{-1}(A_{\alpha_{-1} r})$, and the sets $\varphi_j(A_{\alpha_j r})$ are pairwise disjoint. Therefore, for each $r>0$ the equation

$$\mu(r)=\frac{1}{3}\mu\Big(\frac{2}{3}r\Big)+\frac{2}{3}\mu\Big(\frac{4}{3}r\Big)$$

holds. By induction, we get the equation

$$\mu(r)=\sum_{k=0}^{n}\binom{n}{k}\Big(\frac{2}{3}\Big)^k\Big(\frac{1}{3}\Big)^{n-k}\mu\Big(2^k\Big(\frac{2}{3}\Big)^n r\Big)$$

for each $r>0$, $n\in\mathbb{N}$. Now $2^2 2^3=32>27=3^3$, thus $2^{2/3}\big(\frac{2}{3}\big)>1$. As the function $x\to\frac{2}{3}2^x$ is continuous, there is a $v=\frac{p}{q}\in\mathbb{Q}$ such that $p,q\in\mathbb{N}$, $v<\frac{2}{3}$ and $1<2^p\big(\frac{2}{3}\big)^q<\Big(2^{2/3}\big(\frac{2}{3}\big)\Big)^q$. Thus for any $n\in\mathbb{N}$ we get

$$\begin{aligned}\mu(r)=&\sum_{k=0}^{pn-1}\binom{qn}{k}\Big(\frac{2}{3}\Big)^k\Big(\frac{1}{3}\Big)^{qn-k}\mu\Big(2^k\Big(\frac{2}{3}\Big)^{qn}r\Big)+\\ &+\sum_{k=pn}^{qn}\binom{qn}{k}\Big(\frac{2}{3}\Big)^k\Big(\frac{1}{3}\Big)^{qn-k}\mu\Big(2^k\Big(\frac{2}{3}\Big)^{qn}r\Big)\le\\ &\le\sum_{k=0}^{pn-1}\binom{qn}{k}\Big(\frac{2}{3}\Big)^k\Big(\frac{1}{3}\Big)^{qn-k}1+\\ &+\sum_{k=pn}^{qn}\binom{qn}{k}\Big(\frac{2}{3}\Big)^k\Big(\frac{1}{3}\Big)^{qn-k}\mu\Big(2^{pn}\Big(\frac{2}{3}\Big)^{qn}r\Big)\le\\ &\le\sum_{k=0}^{pn-1}\binom{qn}{k}\Big(\frac{2}{3}\Big)^k\Big(\frac{1}{3}\Big)^{qn-k}1+\\ &+\mu\Big(2^{pn}\Big(\frac{2}{3}\Big)^{qn}r\Big)\sum_{k=0}^{qn}\binom{qn}{k}\Big(\frac{2}{3}\Big)^k\Big(\frac{1}{3}\Big)^{qn-k}=\end{aligned}$$

$$= \sum_{k=0}^{pn-1} \binom{qn}{k} \left(\frac{2}{3}\right)^k \left(\frac{1}{3}\right)^{qn-k} 1 + \mu\Big(\Big(2^p \left(\frac{2}{3}\right)^q\Big)^n r\Big).$$

Now choose a real number w such that $v < w < \frac{2}{3}$ and define the continuous function g on the interval $[0,1]$ by

$$g(x) := \begin{cases} 1 & \text{for } 0 \le x \le v \\ \frac{w-x}{w-v} & \text{for } v < x < w \\ 0 & \text{for } w \le x \le 1. \end{cases}$$

Then

$$\sum_{k=0}^{pn-1} \binom{qn}{k} \left(\frac{2}{3}\right)^k \left(\frac{1}{3}\right)^{qn-k} 1 =$$

$$= \sum_{k=0}^{pn-1} \binom{qn}{k} \left(\frac{2}{3}\right)^k \left(\frac{1}{3}\right)^{qn-k} g\left(\frac{k}{qn}\right) \le$$

$$\le \sum_{k=0}^{qn} \binom{qn}{k} \left(\frac{2}{3}\right)^k \left(\frac{1}{3}\right)^{qn-k} g\left(\frac{k}{qn}\right).$$

The last expression is the approximation of g by Bernstein polynomials at $x = \frac{2}{3}$, and, therefore,

$$\lim_{n\to\infty} \sum_{k=0}^{qn} \binom{qn}{k} \left(\frac{2}{3}\right)^k \left(\frac{1}{3}\right)^{qn-k} g\left(\frac{k}{qn}\right) = g\left(\frac{2}{3}\right) = 0.$$

Furthermore, $2^p \left(\frac{2}{3}\right)^q > 1$, and therefore $\lim \mu\big(\big(2^p \left(\frac{2}{3}\right)^q\big)^n r\big) = 0$. Thus we see that $\mu(r) \le 0$ for each $r > 0$, and this fact implies that the solution f has to vanish almost everywhere. ◊

Corollary 7. *As any continuous function is measurable, the preceding theorem gives another proof for the fact that in the case $q = \frac{1}{3}$ the zero function is the only continuous solution of the system* (1)-(2).

Finally, the last case which can be said to have been completely solved is

c) The case $\mathbf{q} = \frac{1}{2}$

Theorem 30. *Let $q = \frac{1}{2}$. Then the system* (1) *and* (2) *is equivalent to the system*

(α)	$f(x)=0$	for $\lvert x\rvert \geq 1$
(β)	$f(x)=2f\left(\frac{x+1}{2}\right)-2f(x+1)$	for $x\in[-1,0]$
(γ)	$f\left(\frac{x}{4}\right)=\frac{3}{2}f\left(\frac{x}{2}\right)-\frac{1}{2}f(x)$	for $x\in[0,1]$
(δ)	$f\left(\frac{x+1}{2}\right)=\frac{1}{2}f(x)$	for $x\in[0,1]$
(ε)	$f\left(\frac{x+1}{4}\right)=f\left(\frac{x}{2}\right)-\frac{1}{4}f(x)$	for $x\in[0,1]$.

Proof. (a) Let f be a solution of (1) and (2). We show that f fulfills (α)–(ε):

(α) Let $x=2$. By (2), $f(2)=f(3)=0$, and therefore by (1) $f(1)=f\left(\frac{1}{2}2\right)=\frac{1}{2}\big(f(3)+f(1)+2f(2)\big)=\frac{1}{2}f(1)$. Thus $f(1)=0$, and $f(-1)=0$ is shown in the same way.

(β) Let $x\in[-1,0]$ and let $y:=x+1$. Then $f(y+1)=0$ by (α) and $f\left(\frac{y}{2}\right)=\frac{1}{2}\big(f(y+1)+f(y-1)+2f(y)\big)$, thus $f\left(\frac{x+1}{2}\right)=\frac{1}{2}\big(f(x)+ + 2f(x+1)\big)$ and $f(x)=2f\left(\frac{x+1}{2}\right)-2f(x+1)$.

(δ) Let $x\in[0,1]$. Then $x+2>x+1\geq 1$ and $\frac{x+1}{2}\in[0,1]$. By (1) and (α) — used for the real number $x+1$ — we have $f\left(\frac{x+1}{2}\right)= =\frac{1}{2}\big(f(x)+f(x+2)+2f(x+1)\big)=\frac{1}{2}f(x)$.

(ε) Let $x\in[0,1]$ and $y:=x-1\in[-1,0]$. By (1) and (α) we have $f(y-1)=0$ and therefore $f\left(\frac{y}{2}\right)=\frac{1}{2}\big(f(y+1)+2f(y)\big)$. By (β) we get $2f\left(\frac{\frac{y}{2}+1}{2}\right)-2f\left(\frac{y}{2}+1\right)=\frac{1}{2}\big(f(y+1)+4f\left(\frac{y+1}{2}\right)-4f(y+1)\big)$, or $2f\left(\frac{x+1}{4}\right)-2f\left(\frac{x+1}{2}\right)=\frac{1}{2}f(x)+2f\left(\frac{x}{2}\right)-2f(x)$, which implies (ε) by use of (δ).

(γ) Let $x\in[0,1]$ and $y:=x-1$, then $y-2<y-1\leq -1$, and $\frac{y-1}{2}\in[-1,0]$. By (1) and (2) (used for the number $y-1$) we have $f\left(\frac{y-1}{2}\right)=\frac{1}{2}\big(f(y-2)+f(y)+2f(y-1)\big)=\frac{1}{2}f(y)$. By (β),

$$2f\left(\frac{\frac{y-1}{2}+1}{2}\right)-2f\left(\frac{y-1}{2}+1\right)=$$
$$=f\left(\frac{y-1}{2}\right)=\frac{1}{2}f(y)=f\left(\frac{y+1}{2}\right)-f(y+1).$$

Thus $2f\left(\frac{x}{4}\right)-2f\left(\frac{x}{2}\right)=f\left(\frac{x}{2}\right)-f(x)$, which implies (γ).

(b) On the other hand, let f be a solution of (α)–(ε). We show that f fulfills (1) and (2):

(2) is an immediate consequence of (α).

(1): We show that (1) is fulfilled for any $x \in \mathbb{R}$:

(1.1) Let $|x| \geq 2$. Then $f(x) = f(\frac{x}{2}) = f(x+1) = f(x-1) = 0$, and (1) is fulfilled.

(1.2) Let $x \in [1,2]$ and $y := x-1$. Then

$$(1) \Leftrightarrow f\left(\frac{x}{2}\right) = \frac{1}{2}f(x-1) \text{ by } (\alpha) \Leftrightarrow f\left(\frac{y+1}{2}\right) = \frac{1}{2}f(y),$$

which is fulfilled by (δ).

(1.3) Let $x \in [0,1]$, $y := x-1$. Then

$$(1) \Leftrightarrow f\left(\frac{x}{2}\right) = \frac{1}{2}(f(x-1) + 2f(x)) \text{ by } (\alpha) \Leftrightarrow$$
$$\Leftrightarrow f(y) = 2f\left(\frac{y+1}{2}\right) - 2f(y+1) \Leftrightarrow (\beta).$$

(1.4) Let $x \in [-1,0]$ and $y := x+1$. Then

$$(1) \Leftrightarrow f\left(\frac{x}{2}\right) = \frac{1}{2}(f(x+1) + 2f(x)) \text{ by } (\alpha) \Leftrightarrow$$
$$\Leftrightarrow 2f\left(\frac{x+2}{4}\right) - 2f\left(\frac{x+2}{2}\right) =$$
$$= \frac{1}{2}f(x+1) + 2f\left(\frac{x+1}{2}\right) - 2f(x+1) \text{ by } (\beta) \Leftrightarrow$$
$$\Leftrightarrow 4f\left(\frac{y+1}{4}\right) - 4f\left(\frac{y+1}{2}\right) = 4f\left(\frac{y}{2}\right) - 3f(y) \Leftrightarrow$$
$$\Leftrightarrow 4f\left(\frac{y+1}{4}\right) = 4f\left(\frac{y}{2}\right) - f(y) \text{ by } (\delta) \Leftrightarrow (\varepsilon).$$

(1.5) Let $x \in [-2,-1]$, $y := x+1$, $w := x+2$. Then

$$(1) \Leftrightarrow f\left(\frac{x}{2}\right) = \frac{1}{2}f(x+1) \text{ by } (\alpha) \Leftrightarrow$$
$$\Leftrightarrow 2f\left(\frac{x+2}{4}\right) - 2f\left(\frac{x+2}{2}\right) = f\left(\frac{x+1+1}{2}\right) - f(x+1+1) \text{ by } (\beta) \Leftrightarrow$$
$$\Leftrightarrow f\left(\frac{w}{4}\right) = \frac{3}{2}f\left(\frac{w}{2}\right) - \frac{1}{2}f(w) \Leftrightarrow (\gamma). \ \Diamond$$

Corollary 8. *Let $q = \frac{1}{2}$, and (α)–(ε) as in the preceding theorem. Then any function $g\colon [0,1] \to \mathbb{R}$ which fulfills (γ)–(ε) has a unique extension to a solution $f\colon \mathbb{R} \to \mathbb{R}$ of (1) and (2).*

Proof. As (1)-(2) is equivalent to (α)–(ε), the statement is trivial because such an f is given on $[-1,0]$ by (β), and for $|x| \geq 1$ by (α). The only problem might arise at the points $0, 1, -1$ because of some fact of "confusion". But:

(δ) implies that $g(1) = g\left(\frac{1+1}{2}\right) = \frac{1}{2}g(1)$, therefore $g(1) = 0$;

(β) gives $f(0) = 2f\left(\frac{1}{2}\right)$, the same as ($\delta$), and $f(-1) = 2f(0) - - 2f(0) = 0$. ◊

Remark 4. As the proofs of Th. 30 and Cor. 8 show, the other case — restriction to the interval $[-1, 0]$ instead of $[0, 1]$ — can be dealt in the same way. Thus for describing solutions of (1)-(2) we always may restrict ourselves to solutions of (γ)–(ε) on the interval $[0, 1]$.

Theorem 31. *Let* $q = \frac{1}{2}$, *and* (α)–(ε) *as in* Th. 30. *A function* $f\colon [0,1] \to \mathbb{R}$ *is a solution of* (γ)–(ε) *iff*

$$(*) \qquad f\left(\frac{x+m}{2^k}\right) = \left(2 - \frac{m+1}{2^{k-1}}\right) f\left(\frac{x}{2}\right) + \left(\frac{m+2}{2^k} - 1\right) f(x)$$

holds for any $x \in [0,1]$ *and any nonnegative integers* k, m *such that* $0 \le m < 2^k$.

Proof. (a) Suppose that $(*)$ is fulfilled. We show that (γ)–(ε) hold. (γ): Put $m = 0$, $k = 2$. (δ): Put $m = 1$, $k = 1$. (ε): Put $m = 1$, $k = 2$.

(b) On the other hand, suppose that (γ)–(ε) are fulfilled. We show that $(*)$ holds.

$k = 0$: Then $m = 0$, and $(*)$ is nothing else but $f(x) = f(x)$.

$m = 0$: $f\left(\frac{x}{2}\right) = f\left(\frac{x}{2}\right)$, a trivial statement.

$m = 1$: nothing else but (δ).

$k > 1$: We do the proof by induction on k:

$m = 2n$: Then $\frac{x+m}{2^k} = \frac{x+2n}{2^k} = \frac{\frac{x}{2}+n}{2^{k-1}}$, and $0 \le m < 2^k$ implies that $0 \le n < 2^{k-1}$, thus

$$f\left(\frac{x+m}{2^k}\right) = f\left(\frac{\frac{x}{2}+n}{2^{k-1}}\right) = \text{(by induction)}$$

$$= \left(2 - \frac{n+1}{2^{k-2}}\right) f\left(\frac{x}{4}\right) + \left(\frac{n+2}{2^{k-1}} - 1\right) f\left(\frac{x}{2}\right) =$$

$$= \left(2 - \frac{n+1}{2^{k-2}}\right)\left(\frac{3}{2} f\left(\frac{x}{2}\right) - \frac{1}{2} f(x)\right) + \left(\frac{n+2}{2^{k-1}} - 1\right) f\left(\frac{x}{2}\right) =$$

$$= \left(2 - \frac{2n+1}{2^{k-1}}\right) f\left(\frac{x}{2}\right) + \left(\frac{2n+2}{2^k} - 1\right) f(x) \Leftrightarrow (*).$$

$m = 2n+1$: Then $\frac{x+m}{2^k} = \frac{x+2n+1}{2^k} = \frac{\frac{x+1}{2}+n}{2^{k-1}}$, and $0 \le m < 2^k$ implies that $0 \le n < 2^{k-1}$, thus

$$f\left(\frac{x+m}{2^k}\right) = f\left(\frac{\frac{x+1}{2}+n}{2^{k-1}}\right) = \text{(by induction)}$$

$$= \Big(2 - \frac{n+1}{2^{k-2}}\Big) f\Big(\frac{x+1}{4}\Big) + \Big(\frac{n+2}{2^{k-1}} - 1\Big) f\Big(\frac{x+1}{2}\Big) =$$

$$= \Big(2 - \frac{n+1}{2^{k-2}}\Big)\Big(f\Big(\frac{x}{2}\Big) - \frac{1}{4}f(x)\Big) + \Big(\frac{n+2}{2^{k-1}} - 1\Big)\frac{1}{2}f(x) =$$

$$= \Big(2 - \frac{2n+2}{2^{k-1}}\Big) f\Big(\frac{x}{2}\Big) + \Big(\frac{2n+3}{2^k} - 1\Big) f(x) \Leftrightarrow (*). \ \Diamond$$

Remark 5. The equation $(*)$ can be written in the following way: let $z = \frac{x+m}{2^k}$, then we have

$$f(z) = \Big(1 - z - \frac{1-x}{2^k}\Big)\Big(2f\Big(\frac{x}{2}\Big) - f(x)\Big) + \frac{1}{2^k}f(x).$$

Proof. We have $2^k z - x = m$. Using this equation in $(*)$, we get

$$f(z) = f\Big(\frac{x+m}{2^k}\Big) = \frac{2^k - 2^k z + x - 1}{2^{k-1}} f\Big(\frac{x}{2}\Big) + \frac{2^k z - x + 2 - 2^k}{2^k} f(x) =$$

$$= \Big(1 - z - \frac{1-x}{2^k}\Big)\Big(2f\Big(\frac{x}{2}\Big) - f(x)\Big) + \frac{1}{2^k}f(x). \ \Diamond$$

In order to describe the solutions of (α)–(ε) we introduce some equivalence relations on the interval $[0,1[$. Let $\mathbf{M}$ be the set

$$\mathbf{M} := \Big\{\frac{p}{2^k} \mid p, k \in \mathbb{Z}\Big\}.$$

It is easy to see that $\mathbf{M}$ is dense in $\mathbb{R}$ and a group with respect to addition. Furthermore, $\mathbf{M}$ is invariant under multiplication with powers of 2 resp. $\frac{1}{2}$.

Definition. Let $x, y \in [0,1[$. We define

$$x \sim y :\Leftrightarrow x - y \in \mathbf{M}$$

and

$$x \approx y :\Leftrightarrow \exists k \in \mathbb{Z}:\ 2^k x - y \in \mathbf{M}.$$

Lemma 7. *The relations $\sim$ and $\approx$ on $[0,1[$ are equivalence relations, and for $x, y \in [0,1[$ we have*
$x \approx y \Leftrightarrow$ *there is a $z \in [0,1[$ and there are nonnegative integers k, m, n, p such that $0 \le m < 2^k$, $0 \le n < 2^p$ and $x = \frac{z+m}{2^k}$ and $y = \frac{z+n}{2^p}$.*

Proof. $\sim$ is an equivalence relation because $\mathbf{M}$ is a group.

$\approx$: Reflexivity is evident, for symmetry we have $2^{-k}y - x = -2^{-k}(2^k x - y) \in -2^{-k}\mathbf{M} = \mathbf{M}$. Transitivity: $2^k x - y = m \in \mathbf{M}$, $2^p y - z = n \in \mathbf{M} \Rightarrow 2^{k+p}x - z = 2^p m + n \in \mathbf{M}$.

As to the last statement of the lemma:

a) Suppose that there are z, k, m, n, p for x and y. Then $x = \frac{z+m}{2^k}$, $y = \frac{z+n}{2^p}$, thus $2^k x - 2^p y = (z+m)-(z+n) = m-n \in \mathbb{Z}$. Furthermore, $2^{k-p}x - y = \frac{m-n}{2^p} \in \mathbf{M} \Rightarrow x \approx y$.

b) Suppose that $x \approx y$ and $2^k x - y = \frac{m}{2^p} \in \mathbf{M}$. Because of symmetry we may assume that $k \geq 0$ and, of course, $p \geq 0$. Then $2^{k+p}x = 2^p y + m$. We put $z := 2^{k+p}x (\mathrm{mod}\, 1)$. Then $z \in [0,1[$, and $r := 2^{k+p}x - z \in \mathbb{Z}$, $0 \leq r < 2^{k+p}$, $x = \frac{z+r}{2^{k+p}}$. Furthermore, $2^p y = = 2^{k+p}x - m = z + r - m$. Thus $0 \leq r - m < 2^p$ and $y = \frac{z+r-m}{2^p}$. ◊

In the following $[x]$ will denote the equivalence class of $x \in [0,1[$ with respect to the relation $\approx$. We distinguish two types of these classes:

"type 1": there is an integer k, $k > 0$ and a $y \in [x]$ such that $2^k y - y \in \mathbf{M}$,

"type 2": for any $k \in \mathbb{N}$ and any $y \in [x]$ we have $2^k y - y \notin \mathbf{M}$.

The following remark shows that these types exclude one another.

Remark 6. Suppose that $[x]$ is of type 1 and $2^k y - y \in \mathbf{M}$ for some $k \in \mathbb{N}$, $y \in [x]$. Then:

α) For any $z \in [x]$ $2^k z - z \in \mathbf{M}$ holds;

β) $x \approx z = \frac{m}{2^k - 1}$ for some integer m, $0 \leq m < 2^k - 1$. m can be chosen in such a way that $x = \frac{z+n}{2^p}$ for some nonnegative integers n and p.

Proof. α) $2^k y - y = m \in \mathbf{M}$, $y \approx z \Rightarrow 2^p y - z = n \in \mathbf{M}$ for some $p \in \mathbb{Z}$. Thus $z = 2^p y - n$ and $2^k z - z = 2^{k+p}y - 2^k n - 2^p y + n = 2^p(2^k y - y) - - 2^k n + n = 2^p m - 2^k n + n \in \mathbf{M}$.

β) By α), $2^k x - x = \frac{m}{2^p}$ for some integers m, p. Let $z := 2^p x (\mathrm{mod}\, 1)$, $n = 2^p x - z$. Then $n \in \mathbb{Z}$ and therefore $z \approx x$. Furthermore, $2^k z - z = = 2^{k+p}x - 2^k n - 2^p x + n = 2^p(2^k x - x) - 2^k n + n = m - 2^k n + n \in \mathbb{Z}$. Furthermore, $x = \frac{z+n}{2^p}$. ◊

Remark 7. Let us denote by $\langle x \rangle$ the equivalence class of x with respect to $\sim$. Then $\langle x \rangle \subseteq [x]$ for any $x \in [0,1[$, and all the sets $\langle x \rangle$, $[x]$ are countable and dense subsets of $[0,1[$. Furthermore, if $[x]$ is of type 2, for two numbers $y, z \in [x]$ for which $2^k y - x \in \mathbf{M}$ and $2^p z - x \in \mathbf{M}$ we have $y \sim z$ if and only if $k = p$.

After these remarks we are able to describe the structure of the general solution of $(*)$:

Theorem 32. *Let f be a solution of $(*)$ on the interval $[0,1]$, and let $x \in [0,1[$. Then the following holds:*

(a) *The function $y \rightarrow 2f\left(\frac{y}{2}\right) - f(y) =: c_y$ is constant on $[x]$.*

(b) *If* $m \in \mathbf{M}$ *and* $x+m \in [0,1[$ *then in any case*

$$(**) \qquad f(x) - f(x+m) = mc_x.$$

(c) f *is given on* $\langle x \rangle$ *by the formula*

$$f(y) = c(1-y) + d, \text{ for constants } c \text{ and } d.$$

(d) *If* $[x]$ *is of* type 1, *then* f *is uniquely determined on* $[x]$ *by the value* $f(x)$, *and each value of* $f(x)$ *gives rise to a solution* f *on the set* $[x]$. *The value* $f(y)$ *for* $y \approx x$ *can be computed by the formula*

$$f(y) = f(x)\frac{1-y}{1-x}.$$

(e) *If* $[x]$ *is of* type 2, *then* f *is uniquely determined on* $[x]$ *by the values* $f(x)$ *and* $f\left(\frac{x}{2}\right)$, *and any choice of values of* $f(x)$ *and* $f\left(\frac{x}{2}\right)$ *gives rise to a solution* f *on the set* $[x]$.

Proof. (a) We show that for each $y \approx x$ the value $f(y)$ can be computed from the values $f(x)$ and $f\left(\frac{x}{2}\right)$: As $y \approx x$, there is a $z \in [0,1[$ and there are nonnegative integers k, m, n, p such that $x = \frac{z+m}{2^k}$ and $y = \frac{z+n}{2^p}$. Now $(*)$ implies that

$$f(x) = f\left(\frac{z+m}{2^k}\right) = \left(2 - \frac{m+1}{2^{k-1}}\right) f\left(\frac{z}{2}\right) + \left(\frac{m+2}{2^k} - 1\right) f(z)$$

and

$$f\left(\frac{x}{2}\right) = f\left(\frac{z+m}{2^{k+1}}\right) = \left(2 - \frac{m+1}{2^k}\right) f\left(\frac{z}{2}\right) + \left(\frac{m+2}{2^{k+1}} - 1\right) f(z).$$

This system of two linear equations in the unknowns $f(z), f\left(\frac{z}{2}\right)$ has a unique solution for any given values of $f(x)$, $f\left(\frac{x}{2}\right)$ because the determinant of the coefficients is nonzero. Computation of the determinant of the coefficients:

$$\det = \left(2 - \frac{m+1}{2^{k-1}}\right)\left(\frac{m+2}{2^{k+1}} - 1\right) - \left(2 - \frac{m+1}{2^k}\right)\left(\frac{m+2}{2^k} - 1\right) =$$

$$= \frac{1}{2^{2k}}((2^k - m - 1)(m + 2 - 2^{k+1}) - (2^{k+1} - m - 1)(m + 2 - 2^k)) =$$

$$= \frac{1}{2^{2k}}(-2^{2k+1} + 2^k(m+2+2m+2) - (m+1)(m+2) +$$

$$+ 2^{2k+1} - 2^k(m+1+2m+4) + (m+1)(m+2)) = -\frac{1}{2^k}.$$

By $(*)$, the value $f(y)$ can be computed from the values $f(z)$ and $f\left(\frac{z}{2}\right)$. Furthermore, we see that $2f\left(\frac{x}{2}\right) - f(x) = 2f\left(\frac{z}{2}\right) - f(z)$. Thus the function $y \to 2f\left(\frac{y}{2}\right) - f(y)$ is constant on each equivalence class $[x]$.

(b) Let $m \in \mathbf{M}$ and suppose that $x+m \in [0,1[$. As $x \approx x+m$, we have $c_x = c_{x+m}$. Thus we may assume for proving (b) that $m = \frac{p}{2^k} > 0$. Let $z := 2^k x (\mathrm{mod}\ 1)$, $n := 2^k x - z \in \mathbb{Z}$. Then $x = \frac{z+n}{2^k}$, $x+m = \frac{z+n+p}{2^k}$. By $(*)$, we have

$$f(x) = f\Big(\frac{z+n}{2^k}\Big) = \Big(2 - \frac{n+1}{2^{k-1}}\Big) f\Big(\frac{z}{2}\Big) + \Big(\frac{n+2}{2^k} - 1\Big) f(z)$$

and

$$f(x+m) = f\Big(\frac{z+n+p}{2^k}\Big) = \Big(2 - \frac{n+p+1}{2^{k-1}}\Big) f\Big(\frac{z}{2}\Big) + \Big(\frac{n+p+2}{2^k} - 1\Big) f(z).$$

Thus

$$f(x) - f(x+m) = \frac{p}{2^k}\Big(2f\Big(\frac{z}{2}\Big) - f(z)\Big) = mc_z = mc_x.$$

(c) Let $c := c_x = 2f\big(\frac{x}{2}\big) - f(x)$, $d = f(x) - c(1-x)$. Then $f(x) = c(1-x) + d$, and for $y \in \langle x \rangle$ we have (by (b)) $f(y) = f(x) + (x-y)c = d + c(1-x) + c(x-y) = d + c(1-y)$.

(d) Suppose that $[x]$ is of type 1. Then $x = \frac{z+n}{2^p}$, where $z = \frac{m}{2^k - 1}$, for some integers k, m, n, p, i.e. $[x] = [z]$. Now $z = \frac{z+m}{2^k}$ and $(*)$ imply that

$$f(z) = f\Big(\frac{z+m}{2^k}\Big) = \Big(1 - z - \frac{1-z}{2^k}\Big)\Big(2f\Big(\frac{z}{2}\Big) - f(z)\Big) + \frac{1}{2^k} f(z).$$

Thus

$$\Big(1 - \frac{1}{2^k}\Big) f(z) = (1-z)\Big(1 - \frac{1}{2^k}\Big)\Big(2f\Big(\frac{z}{2}\Big) - f(z)\Big),$$

and $2f\big(\frac{z}{2}\big) - f(z) = \frac{f(z)}{1-z}$. Now we use this result and $(*)$ for x:

$$f(x) = f\Big(\frac{z+n}{2^p}\Big) = \Big(1 - x - \frac{1-z}{2^p}\Big)\Big(2f\Big(\frac{z}{2}\Big) - f(z)\Big) + \frac{1}{2^p} f(z) =$$

$$= \Big(1 - x - \frac{1-z}{2^p}\Big)\frac{f(z)}{1-z} + \frac{1}{2^p} f(z)$$

$$f\Big(\frac{x}{2}\Big) = f\Big(\frac{z+n}{2^{p+1}}\Big) = \Big(1 - \frac{x}{2} - \frac{1-z}{2^{p+1}}\Big)\Big(2f\Big(\frac{z}{2}\Big) - f(z)\Big) + \frac{1}{2^{p+1}} f(z) =$$

$$= \Big(1 - \frac{x}{2} - \frac{1-z}{2^{p+1}}\Big)\frac{f(z)}{1-z} + \frac{1}{2^{p+1}} f(z).$$

Thus $(1-x) f\big(\frac{x}{2}\big) - \big(1 - \frac{x}{2}\big) f(x) = 0$. Therefore, the value $f\big(\frac{x}{2}\big)$ is uniquely determined by $f(x)$. In order to show that there is a solution on $[x]$ for any choice of the value $f(x)$, one has to check that $y \to \frac{1-y}{1-x} f(x)$ is a solution of $(*)$ on $[x]$. This is easy and omitted.

(e) Suppose that $[x]$ is of type 2. We only have to show that any choice of values of $f(x)$ and $f\left(\frac{x}{2}\right)$ gives rise to a solution on the set $[x]$. As it has been shown in (1), for any $y \in [x]$ the value $f(y)$ can be computed from $f(x)$ and $f\left(\frac{x}{2}\right)$. Thus it only has to be shown that $f(x)$ and $f\left(\frac{x}{2}\right)$ can be chosen arbitrarily. Here we indicate an explicit construction of the solution: Let $f(x)$ and $f\left(\frac{x}{2}\right)$ be given and define $c := 2f\left(\frac{x}{2}\right) - f(x)$, $d := f(x) - c(1-x)$ and $g\colon [x] \to \mathbb{R}$ by $g(y) := c(1-y) + d2^k$, where $2^k x - y \in \mathbf{M}$. Then:

$$g(x) = c(1-x) + d1 = f(x)$$

$$g\left(\frac{x}{2}\right) = c\left(1 - \frac{x}{2}\right) + d\frac{1}{2} =$$

$$= c\frac{2-x}{2} + \frac{1}{2}f(x) - c\frac{1-x}{2} = \frac{1}{2}c + \frac{1}{2}f(x) = f\left(\frac{x}{2}\right).$$

g is a solution of (γ)–(ε). In fact, let $y \in [x]$, $2^k x - y \in \mathbf{M}$:

$$(\gamma) \qquad \begin{aligned} &\frac{3}{2}g\left(\frac{y}{2}\right) - \frac{1}{2}g(y) = \\ &= \frac{3}{2}c\left(1 - \frac{y}{2}\right) + \frac{3}{2}d2^{k-1} - \frac{1}{2}c(1-y) - \frac{1}{2}d2^k = \\ &= c\left(1 - \frac{y}{4}\right) + d2^{k-2} = g\left(\frac{y}{4}\right) \end{aligned}$$

$$(\delta) \qquad \begin{aligned} g\left(\frac{y+1}{2}\right) &= c\left(1 - \frac{y+1}{2}\right) + d2^{k-1} = \\ &= \frac{1}{2}\left(c(1-y) + d2^k\right) = \frac{1}{2}g(y) \end{aligned}$$

$$(\varepsilon) \qquad \begin{aligned} g\left(\frac{y}{2}\right) - \frac{1}{4}g(y) &= c\left(1 - \frac{y}{2}\right) + d2^{k-1} - \frac{1}{4}c(1-y) - \frac{1}{4}d2^k = \\ &= c\left(1 - \frac{y+1}{4}\right) + d2^{k-2} = g\left(\frac{y+1}{4}\right). \end{aligned}$$

Thus g is a solution and, therefore, $g = f$.

The only fact to show is that g is well defined (i.e. the integer k is unique). Suppose that $2^k x - y = m \in \mathbf{M}$, $2^p x - y = n \in \mathbf{M}$, where $k > p$. Then $2^k x - 2^p x = m - n \in \mathbf{M}$ and, therefore, $2^{k-p}x - x \in \mathbf{M}$ – a contradiction to the assumption that $[x]$ is of type 2. $\lozenge$

Theorem 33. *The general solution of* $(*)$ *on the interval* $[0,1[$ *is given in the following way: Let* $\{x_i\}_{i \in T}$ *be a system of representatives for the relation* $\approx$. *For each* $i \in T$ *choose* $c_i \in \mathbb{R}$ *arbitrarily and*

$$d_i \begin{cases} \textit{arbitrarily} & \textit{if } [x_i] \textit{ is of type } 2 \\ = 0 & \textit{if } [x_i] \textit{ is of type } 1. \end{cases}$$

Define $f\colon [0,1[\to \mathbb{R}$ *by* $f(y) = c_i(1-y) + d_i 2^k$ *whenever* $2^k x_i - y \in \mathbf{M}$.

The **proof** has been given in the preceding theorem. ◊

From this result one can easily deduce the general structure of continuous solutions.

Corollary 9. *Let* f *be a solution of* $(*)$ *on* $[0,1[$ *which is continuous on a nondegenerate interval* J. *Then* $f(x) = c(1-x)$ *for some real constant* c.

Proof. As $20 - 0 = 0 \in \mathbf{M}$, the set $[0]$ is of type 1. Thus f is given on $[0]$ by $f(x) = c(1-x)$ for some constant c. As $[0]$ is dense in $[0,1[$, $f(x) = c(1-x)$ on J. Now let $y \in [0,1[$ be arbitrary. On $\langle y \rangle$ f is given by $f(z) = c'(1-z) + d'$ for some constants c', d'. As $\langle y \rangle$ is dense in $[0,1[$, there are at least two elements $u, u' \in \langle y \rangle \cap J$, $u \neq u'$. Thus we have

$$c'(1-u) + d' = f(u) = c(1-u)$$

$$c'(1-u') + d' = f(u') = c(1-u')$$

and therefore $c' = c$, $d' = 0$. As y was chosen arbitrarily we have $f(x) = c(1-x)$ on $[0,1[$. ◊

We also can use the result of Th. 33 to give the structure of solutions of $(*)$ which are continuous at one point:

Theorem 34. *Let* $f\colon [0,1[\to \mathbb{R}$ *be a solution of* $(*)$ *which is continuous at a point* $x_0 \in [0,1[$. *Then* $f(x) = c(1-x)$ *for some constant* c.

Proof. Let $y \in [0,1[$ and $z = \frac{y}{2}$. Then $[z] = [y]$, and we have

$$f(t) = c_y(1-t) + d_y \quad \text{for} \quad t \in \langle y \rangle$$

and

$$f(t) = c_y(1-t) + d_y \frac{1}{2} \quad \text{for} \quad t \in \langle z \rangle.$$

As $\langle y \rangle$ and $\langle z \rangle$ are dense in $[0,1[$ and as f is continuous at x_0 we have

$$c_y(1-x_0) + d_y \frac{1}{2} = \lim_{\substack{t \to x \\ t \in \langle z \rangle}} f(t) = f(x_0) =$$

$$= \lim_{\substack{t \to x \\ t \in \langle y \rangle}} f(t) = c_y(1-x_0) + d_y.$$

Therefore, $d_y = 0$ and $c_y = \frac{f(x_0)}{1-x_0}$. As y was arbitrary, $f(x) = c(1-x)$, where $c = \frac{f(x_0)}{1-x_0}$. ◊

Remark 8. For getting the result of Th. 34 it is essential that the point of continuity is an element of the open interval $]-1,1[$. The following example shows that continuity at the point $x = 1$ is not sufficient to guarantee the continuity of the solution f: Let $\{x_i\}_{i \in T}$ be a system of representatives for the relation $\approx$, choose $c_i := x_i$, $d_i := 0$ for each $i \in T$ and define $f\colon [0,1[\to \mathbb{R}$ by $f(x) = c_i(1-x)$ whenever $x \approx x_i$. By Th. 33, f is a solution of $(*)$ and, by Th. 34, f is not continuous at any point $x \in [0,1[$. (As Th. 35 will show, f is not measurable, too.) As $c_i \in [0,1[$ for all $i \in T$ we have $0 \leq f(x) < 1-x$ on the interval $[0,1[$, which implies that f is continuous at $x = 1$.

Next we deal with measurable solutions. A heavy instrument for treating this question is Šmital's lemma, which can be written in the following way:

Lemma 8. *Let $A, B \subseteq \mathbb{R}$ be such that A has positive Lebesgue measure and B is dense in $\mathbb{R}$. Then the set $A + B$ has full Lebesgue measure, i.e. the complement of $A + B$ has measure 0.*

A proof can be found in [3].

Before we give the theorem on measurable solutions we need a lemma which bases on Šmital's lemma:

Lemma 9. *Let $J \subseteq [0,1[$ be a nondegenerate interval, $g\colon J \to \mathbb{R}$ a measurable function which is constant on the equivalence classes $\langle x\rangle$ for any $x \in [0,1[\cap J$. Then g is constant a.e.*

Proof. Suppose that g is not constant a.e. As g is real there must be a number $c \in \mathbb{R}$ such that both of the sets $A := \{x \mid f(x) \leq c\}$ and $B := \{x \mid f(x) > c\}$ have positive Lebesgue measure. As g is constant on each equivalence class $\langle x\rangle$ we have $A = J \cap (A + \mathbf{M})$ and $B = J \cap$ $\cap (B + \mathbf{M})$. By Šmital's lemma $\lambda(A) = \lambda(B) = \lambda(J)$, a contradiction to the fact that $A \cap B = \emptyset$. Thus g must be constant a.e. $\Diamond$

Theorem 35. *Let $f\colon [0,1[\to \mathbb{R}$ be a solution of $(*)$ which is measurable on a measurable set $S \subseteq [0,1[$ of positive Lebesgue measure. Then there is a constant $c \in \mathbb{R}$ such that $f(x) = c(1-x)$ almost everywhere.*

Proof. We give the proof in several steps:

(a) By Šmital's lemma the set $S + (\mathbf{M}\setminus\{0\})$ has full Lebesgue measure, thus $\lambda(S \cap (S + (\mathbf{M}\setminus\{0\}))) = \lambda(S) > 0$. Now $S + (\mathbf{M}\setminus\{0\}) =$ $= \bigcup_{m \in \mathbf{M}\setminus\{0\}} (S + m)$, and $\mathbf{M}$ is countable. Therefore, there is an $m \in \mathbf{M}$, $m \neq 0$, such that $\lambda(S \cap (S+m)) > 0$. Now choose such an m and let $A := (S \cap (S+m)) - m$. Then $A \subseteq S$ and $A + m \subseteq S$, thus the functions $x \to f(x)$, $x \to f(x+m)$ are both measurable on the set A.

(b) As it was shown, on A the function

$$x \to c_x = 2f\left(\frac{x}{2}\right) - f(x) \text{ is given by } c_x = \frac{f(x) - f(x+m)}{m}.$$

Now let $n \in \mathbf{M}$ be arbitrary. Then we have

$$f(x+n) = f(x) - nc_x = f(x) + \frac{n}{m}\big(f(x+m) - f(x)\big).$$

Thus f is measurable on the set $\big((A+n) \cap [0,1[\big)$, for arbitrary $n \in \mathbf{M}$, and therefore measurable on $B := [0,1[\cap \bigcup_{n \in \mathbf{M}} (A+n)$. By Šmital's lemma, $\lambda(B) = 1$, and therefore f is measurable on the whole interval $[0,1[$.

(c) By Th. 33, f is given by $f(x) = c(x)(1-x) + d(x)$, where c is constant on the set $[x]$ and d is constant on the set $\langle x \rangle$ for any $x \in [0,1[$. As f is measurable the function $c(x) = 2f\left(\frac{x}{2}\right) - f(x)$ is measurable, too, which implies that d is also measurable. By Lemma 9, c and d are constant functions a.e. Keeping in mind the structure of the function d as it is given in Th. 33 ($d(x) = d_i 2^k$), the only possible case for d to be constant a.e. is that d vanishes a.e. Thus $f(x) = c(1-x) + 0$ a.e. $\Diamond$

References

[1] BARON, K.: On a problem of R. Schilling, *Berichte der Mathematisch-statistischen Sektion in der Forschungsgesellschaft Joanneum – Graz, Bericht Nr.* **286** (1988).

[2] FÖRG-ROB, W.: On a problem of R. Schilling I., *Mathematica Pannonica* **5**/1 (1994), 29–66.

[3] KUCZMA, M.: An Introduction to the Theory of Functional Equations and Inequalities, Panstwowe Wydawnictwo Naukowe, Uniwersytet Sląski, Warszawa–Kraków–Katowice, 1985.

[4] SCHILLING, R.: Spatially-chaotic Structures, in: H. Thomas (ed.), Nonlinear Dynamics in Solids, Springer Verlag, Berlin – Heidelberg – New York, 1992.