Mathematica Pannonica
5/1 (1994), 29 – 65

ON A PROBLEM OF R. SCHILLING I.

Wolfgang **Förg-Rob**

Institut für Mathematik, Universität Innsbruck, Technikerstraße 25/7, A-6020 Innsbruck, Austria

Received January 1994

AMS Subject Classification: 39 B 22, 26 A 30

Keywords: Schilling's equation, Schilling's problem.

Abstract: Studies of a physical problem (cf. [4]) led to the functional equation

(1) $$f(qx) = \frac{1}{4q}(f(x+1)+f(x-1)+2f(x)) \quad \text{for all } x \in \mathbb{R}$$

with the boundary condition

(2) $$f(x) = 0 \quad \text{for all } x \text{ with } |x| > Q := \frac{q}{1-q}$$

where $q \in]0,1[$ is a fixed real number. In this paper the general solution of (1) with unbounded support is given. It can be shown that in the case of unbounded support a function on a special interval can be chosen arbitrarily and then uniquely extended to a solution of (1). Furthermore, investigations are done on the continuity, differentiability, measurability and integrability of such solutions.

Studies of a physical problem (cf. [4]) led Prof. R. Schilling to the functional equation given below. It was known that in the case $q = \frac{1}{2}$ there is a continuous solution with bounded support. Now the question arose to find all the solutions of this equation. At the moment the problem is far from being solved completely, but in the sequel there will be given some partial answers:

Let the functional equation

(1) $$f(qx) = \frac{1}{4q}\big(f(x+1)+f(x-1)+2f(x)\big) \quad \text{for all } x \in \mathbb{R} .$$

and the boundary condition

$$(2) \qquad f(x) = 0 \quad \text{for all } x \text{ with } \quad |x| > Q := \frac{q}{1-q}$$

be given, where $q \in]0,1[$ is a fixed real number. First of all we conduct some investigations on the boundary condition (2). As in our considerations the set $\{x \mid f(x) \neq 0\}$ plays a more important role than the support $\operatorname{supp}(f)$, which denotes the closure of this set, we abbreviate $S(f) := \{x \mid f(x) \neq 0\}$.

I. The boundary condition

In this chapter we show that the boundary condition (2) is natural in some sense. This is done in the subsequent theorem. Therefore, let $q \in]0,1[$ be a fixed real number, $Q = \frac{q}{1-q}$. First we give a short lemma and start with a remark:

Remark 1. $q(Q+1) = Q$, which can easily be verified by direct computation.

Lemma 1. *Let f be a solution of* (1) *whose support is contained in the interval* $]-\infty, b]$ *for some* $b \in \mathbb{R}$. *Then the following holds:*

(i) *If $b \geq Q$, then* $\operatorname{supp}(f) \subseteq]-\infty, Q]$; *moreover, if* $q \neq \frac{1}{4}$, *then* $S(f) \subseteq]-\infty, Q[$.

(ii) *If $b < Q$, then f is identically* 0.

Proof. Let $\operatorname{supp}(f) \subseteq]-\infty, b]$. As the case $b = Q$ is evident, we only have to deal with the other two possibilities:

(i): Suppose that $b > Q$. Then $b(1-q) > q$ and therefore $b > q(b+1) > q(Q+1) = Q$. Now let $x > b+1$. Then $x+1 > x > x-1 > b$, and thus we have $f(x+1) = f(x) = f(x-1) = 0$, which implies that $f(qx) = 0$ by equation (1). Thus in this case we have $\operatorname{supp}(f) \subseteq]-\infty, q(b+1)]$. Define a sequence (b_n) by $b_0 := b$, $b_{n+1} := q(b_n + 1)$. As shown above, for $b_0 > Q$ this sequence is strictly decreasing and has the lower bound Q. Furthermore, by induction one can immediately see that $\operatorname{supp}(f) \subseteq]-\infty, b_n]$ for any $n \in \mathbb{N}$. Therefore the sequence (b_n) is convergent, the limit B fulfills $B \geq Q$ and $B = q(B+1)$, which implies that $B = Q$, and we have $\operatorname{supp}(f) \subseteq]-\infty, Q]$.

In the case $\operatorname{supp}(f) \subseteq]-\infty, Q]$ we have

$$f(Q) = f(q(Q+1)) = \frac{1}{4q}(f(Q) + f(Q+2) + 2f(Q+1)) = \frac{1}{4q} f(Q).$$

Therefore, if $q \neq \frac{1}{4}$, we have $f(Q) = 0$ and $S(f) \subseteq]-\infty, Q[$.

(ii): Suppose that $b < Q$. Then $b(1-q) < q$, and therefore $\frac{b}{q} - 1 < b$.

We first assume $b > 0$. In this case for $x > \frac{b}{q} > b$ we have $f(x+1) = f(x) = f(qx) = 0$, and therefore by equation (1) we get $\operatorname{supp}(f) \subseteq]-\infty, \frac{b}{q} - 1]$. Define the sequence (b_n) by $b_0 := b$, $b_{n+1} := = b_n/q - 1$. This sequence is decreasing, and by induction we get $\operatorname{supp}(f) \subseteq]-\infty, b_n]$ for each n where $b_{n-1} > 0$. On the other hand, this sequence (b_n) cannot have a lower bound, because this bound would be the limit, fulfilling $\frac{B}{q} - 1 = B$, i.e. $B = Q$, a contradiction. Thus there is an n such that $b_n \leq 0$, and $\operatorname{supp}(f) \subseteq]-\infty, 0]$.

Now we assume $b \leq 0$: For $x > b$ we have $x + 1 > x > b$ and $qx > b$. Therefore $f(x) = f(qx) = f(x+1) = 0$, which implies that $f(x-1) = 0$. Thus $\operatorname{supp}(f) \subseteq]-\infty, b-1]$, and by induction we get $\operatorname{supp}(f) = \emptyset$. $\Diamond$

A similar lemma can be proved in the same way for supports bounded from below:

Lemma 2. *Let f be a solution of* (1) *whose support is contained in the interval $[a, \infty[$ for some $a \in \mathbb{R}$. Then the following holds:*

(i) *If $a \leq -Q$, then* $\operatorname{supp}(f) \subseteq [-Q, \infty[$*; moreover, if $q \neq \frac{1}{4}$, then* $S(f) \subseteq]-Q, \infty[$.

(ii) *If $a > -Q$, then f is identically* 0.

Combining these two lemmata, we get the following

Theorem 1. *Let f be a nonvanishing solution of* (1)*, then $S(f)$ is contained in exactly one of the following intervals, and it is not contained in any proper subintervals:*

(a) *for $q \neq \frac{1}{4}$:* $]-Q, Q[$ *or* $]-\infty, Q[$ *or* $]-Q, \infty[$ *or* $\mathbb{R}$*;*

(b) *for $q = \frac{1}{4}$:* $]-Q, Q[$ *or* $]-\infty, Q[$ *or* $]-Q, \infty[$ *or* $\mathbb{R}$ *or* $[-Q, Q]$ *or* $]-Q, Q]$ *or* $[-Q, Q[$ *or* $]-\infty, Q]$ *or* $[-Q, \infty[$.

Proof. In Lemma 1 it was shown that a nonempty support bounded from above has Q as its least upper bound, Lemma 2 gave the answer for bounds from below. The restriction to open intervals for $S(f)$, except for the case $q = \frac{1}{4}$, was also shown in these two lemmata. Later on it will be shown that all these cases really can occur. $\Diamond$

II. Solutions with unbounded support

a) General results

In this chapter we give some general results on the solutions of equation (1) and also present general solutions with unbounded supports. First we start with a uniqueness theorem (cf. [2]):

Theorem 2. *Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be solutions of* (1) *which coincide on the half-open interval* $[-1, 1[$. *Then they are identical.*

Proof. We give this proof by induction and show that f coincides with g on any interval $[-n, n[$, where n is a positive integer. For $n = 1$ this is true by assumption. Now suppose that f and g coincide on the interval $[-n, n[$, and let $x \in [-(n+1), n+1[\setminus[-n, n[$. Then either $n \le x < n+1$ or $-n-1 \le x < -n$. In the first case choose $y := x - 1 \in [-n, n[$. Then $y, y-1, qy \in [-n, n[$, and by (1) we have $g(x) = g(y+1) = 4qg(qy) - g(y-1) - 2g(y) = 4qf(qy) - f(y-1) - 2f(y) = f(y+1) = f(x)$. Similarly, choose $z := x + 1$ in the second case. $\Diamond$

Next we give a theorem how to get all the solutions in the case of unbounded support. By Th. 2 it is sufficient to give the restriction of the solution to the interval $[-1, 1[$.

Theorem 3 (cf. [2]). *Let $h\colon [-1, 1[\to \mathbb{R}$ be an arbitrary function. Then there exists exactly one solution of* (1) *such that the restriction of this solution to the interval* $[-1, 1[$ *coincides with h.* In other words: *Any function $h\colon [-1, 1[\to \mathbb{R}$ can be uniquely extended to a solution of* (1).

Proof. Let $h\colon [-1, 1[\to \mathbb{R}$ be given. We first extend h by induction to the intervals $[-1, n[$ for each natural number n and then to the intervals $[-n, \infty[$:

Let $f_1 := h\colon [-1, 1[\to \mathbb{R}$. Suppose that f_n is given on $[-1, n[$ for some nonnegative integer n. We define f_{n+1} on the interval $[-1, n+1[$ by

$$f_{n+1}(x) := \begin{cases} f_n(x) & \text{for } x \in [-1, n[\\ 4qf_n(q(x-1)) - f_n(x-2) - 2f_n(x-1) & \text{otherwise} \end{cases}$$

(it is easy to see that for $n \le x < n+1$ we have $q(x-1), x-2, x-1 \in [-1, n[$). As — by definition — any two functions f_m, f_n coincide on the intersection of their domains, this family of functions uniquely defines a function F_1 on the interval $[-1, \infty[$. We continue like before: Suppose that F_n is given on the interval $[-n, \infty[$. We define F_{n+1} on the interval $[-(n+1), \infty[$ by

$$F_{n+1}(x) := \begin{cases} F_n(x) & \text{for } x \in [-n, \infty[\\ 4qF_n(q(x+1)) - F_n(x+2) - 2F_n(x+1) & \text{otherwise.} \end{cases}$$

Like before, this family of functions uniquely defines a function f on the whole real line. We only have to show that f is a solution of (1):

Let $x \in \mathbb{R}$. (α) If $x < 0$, then there is an $n \in \mathbb{N}$ such that $-(n+1) \le x - 1 < -n$. By definition, f coincides with F_{n+1} on the interval $[-(n+1),\infty[$. Thus — by definition of F_{n+1} — we have $f(x-1)=F_{n+1}(x-1)=4qF_n(qx)-F_n(x+1)-2F_n(x)=4qf(qx)-f(x+1)-2f(x)$, which is nothing else but equation (1).

(β) If $x\geq 0$, then there is an $n\in\mathbb{N}$ such that $n\leq x+1<n+1$. Like before, we have $f(x+1)=f_{n+1}(x+1)=4qf_n(qx)-f_n(x-1)-2f_n(x)=4qf(qx)-f(x-1)-2f(x)$, and (1) is fulfilled, too. ◊

The next two theorems deal with solutions whose support is bounded from above. First we give a uniqueness theorem.

Theorem 4. *Let f,g be solutions of (1) whose supports are contained in the interval $]-\infty,Q]$. Then $f=g$ iff the restrictions of f and g to the interval $]Q-1,qQ]$ coincide and $f(Q)=g(Q)$. (The second condition is necessary only in the case $q=\frac{1}{4}$.)*

Proof. We define a sequence (x_n) by $x_0:=Q-1$, $x_{n+1}:=q(x_n+1)$. As $x_0<Q$, we have $x_n<x_{n+1}<Q$ for any $n\in\mathbb{N}$, and $\lim_{n\to\infty}x_n=Q$. Now suppose that f and g coincide on the interval $]x_n,x_{n+1}]$ for some nonnegative integer n, and let $x\in]x_{n+1},x_{n+2}]$. By definition of the sequence (x_n) we have $x=q(y+1)$ for some $y\in]x_n,x_{n+1}]$, which also implies that $y+2>y+1>Q$. Using equation (1) for the value $y+1$, we get $g(x)=g(q(y+1))=\frac{1}{4q}(g(y)+g(y+2)+2g(y+1))=\frac{1}{4q}g(y)=\frac{1}{4q}f(y)=f(x)$. Thus by induction we get the result that f and g coincide on the interval $]Q-1,Q[$ and — by assumption — their values at the point Q are identical, thus they coincide on the interval $]Q-1,Q]$ and, therefore, on the interval $]Q-1,\infty[$.

Now let $y_0:=Q-1$ and $y_{n+1}:=\frac{1}{q}y_n-1$. The sequence (y_n) is strictly decreasing and unbounded, thus there is a nonnegative integer k such that $y_k<0$, $y_{k-1}\geq 0$. We will show that f and g coincide on the interval $]y_k,\infty[$. Let $0\leq m<k$, and suppose that f and g coincide on $]y_m,\infty[$. For $x\in]y_{m+1},y_m]$ we have $x+2>x+1>q(x+1)>y_m$ and can derive from equation (1) that $f(x)=g(x)$, i.e., f and g coincide on $]y_{m+1},\infty[$. A usual induction argument shows that f and g coincide on $]y_k,\infty[$ and, therefore, on $[0,\infty[$.

We finish the proof by one more induction process: Suppose that f and g coincide on $[-n,\infty[$ for some nonnegative integer n. Then for $x\in[-(n+1),-n[$ we have $x+2>x+1\geq -n$, $q(x+1)\geq -n$, thus equation (1) gives $f(x)=g(x)$, and f and g coincide on $[-(n+1),\infty[$. Thus $f=g$. ◊

We can use the same ideas to give all the solutions of equation (1) under the assumption that the support is bounded from above:

Theorem 5. *Let $h\colon]Q-1, qQ] \to \mathbb{R}$ be an arbitrary function, and α a real number which is arbitrary in the case $q = \frac{1}{4}$ and 0 otherwise. Then there exists a unique solution f of (1) such that the restriction of f to the interval $]Q-1, qQ]$ is identical to h, $f(Q) = \alpha$ and* $\operatorname{supp}(f) \subseteq\ \subseteq]-\infty, Q]$.

Proof. The uniqueness has been shown in the preceding theorem. For the existence, we will make an extension of h: As in Th. 4, let (x_n) be the sequence given by $x_0 := Q-1$, $x_{n+1} := q(x_n+1)$. Let $h_0 := h$, and h_n defined on the interval $]x_0, x_{n+1}]$ by induction:

$$h_{n+1}(x) := \begin{cases} h_n(x) & \text{for } x \in]x_0, x_{n+1}] \\ \frac{1}{4q} h_n(y) & \text{for } x = q(y+1) \in]x_{n+1}, x_{n+2}]. \end{cases}$$

As any two of the functions h_n, h_m coincide on the intersection of their domains, they uniquely define a function $h_\infty\colon]x_0, Q[\to \mathbb{R}$. Next we extend to the interval $]x_0, \infty[$ by the formula

$$g_0(x) := \begin{cases} h_\infty(x) & \text{for } x \in]x_0, Q[\\ \alpha & \text{for } x = Q \\ 0 & \text{for } x > Q. \end{cases}$$

Now we use the sequence (y_n) defined as in Th. 4 by $y_0 := Q-1$, $y_{n+1} := \frac{1}{q} y_n - 1$, which is strictly decreasing and unbounded, thus there is a nonnegative integer k such that $y_k < 0$, $y_{k-1} \geq 0$. Let m be an integer, $0 \leq m < k$, and suppose that g_m is defined on $]y_m, \infty[$. For $x \in]y_{m+1}, y_m]$ we have $x+2 > x+1 > q(x+1) > y_m$, thus we may define g_{m+1} on $]y_{m+1}, \infty[$ by

$$g_{m+1}(x) := \begin{cases} g_m(x) & \text{for } x \in]y_m, \infty[\\ 4q g_m(q(x+1)) - g_m(x+2) - 2g_m(x+1) & \text{for } x \in]y_{m+1}, y_m]. \end{cases}$$

By this process we get an extension of h to the interval $]y_k, \infty[$, which we call $f_0\colon]y_k, \infty[\to \mathbb{R}$.

Now suppose that $f_n\colon]y_k - n, \infty[\to \mathbb{R}$ is defined for a nonnegative integer n. Then we define $f_{n+1}\colon]y_k - n - 1, \infty[\to \mathbb{R}$ by

$$f_{n+1}(x) := \begin{cases} f_n(x) & \text{for } x \in]y_k - n, \infty[\\ 4q f_n(q(x+1)) - f_n(x+2) - 2f_n(x+1) & \text{otherwise.} \end{cases}$$

(As $y_k < 0$, it is easy to check that the numbers $x+2$, $x+1$, $q(x+1)$ are greater than $y_k - n$ for $x \in]y_k - n - 1, y_k - n]$.)

By the same arguments as before the family of functions (f_n) uniquely defines a function $f\colon \mathbb{R} \to \mathbb{R}$. We only have to check that this function f fulfills equation (1). From the construction it is evident that $f(Q) = \alpha$, $\operatorname{supp}(f) \subseteq]-\infty, Q]$ and f coincides with h on $]Q-1, qQ]$. Let $x \in \mathbb{R}$:

(α) $x \leq y_k + 1$: There is a nonnegative integer n such that $y_k - n - 1 < x - 1 \leq y_k - n$. As f coincides with f_{n+1} on $]y_k - n - 1, \infty[$, and from the definition of f_{n+1} (the formula given above defines the value at $x-1$) we immediately get that (1) is fulfilled.

(β) $y_k + 1 < x \leq y_0 + 1 = Q$: Once more the definition of the functions g_m shows that (1) is fulfilled.

(γ) $Q < x \leq Q+1$: Here we can use the definition of the functions h_n to show that equation (1) is fulfilled.

(δ) $x > Q + 1$: As $\operatorname{supp}(f) \subseteq]-\infty, Q]$, equation (1) is trivially fulfilled. $\Diamond$

From equation (1) it is evident that in any case when f is a solution of (1), then also the function $x \to f(-x)$ is a solution of (1). Therefore, without giving any new proofs we can reformulate Ths. 4 and 5 for the case that $\operatorname{supp}(f) \subseteq [-Q, \infty[$:

Theorem 6. *Let f, g be solutions of (1) whose supports are contained in the interval $[-Q, \infty[$. Then $f = g$ iff the restrictions of f and g to the interval $[-qQ, 1-Q[$ coincide and $f(-Q) = g(-Q)$. (The second condition is only necessary in the case $q = \frac{1}{4}$.)*

Theorem 7. *Let $h\colon [-qQ, 1-Q[\to \mathbb{R}$ be an arbitrary function, and α a real number which is arbitrary in the case $q = \frac{1}{4}$ and 0 otherwise. Then there exists a unique solution f of (1) such that the restriction of f to the interval $[-qQ, 1-Q[$ is identical to h, $f(-Q) = \alpha$ and $\operatorname{supp}(f) \subseteq [-Q, \infty[$.*

Remark 2. Ths. 5 and 7 show that *in the case $q = \frac{1}{4}$ really both cases $S(f) \subseteq]-\infty, Q[$ and $Q \in S(f) \subseteq]-\infty, Q]$ (resp. $S(f) \subseteq]-Q, \infty[$ and $-Q \in S(f) \subseteq [-Q, \infty[$) can occur.*

Next we conduct investigations on the solutions of (1) under special conditions like continuity, differentiability, measurability, integrability. With respect to the remark before Th. 6, we may restrict ourselves to the cases $S(f) \subseteq \mathbb{R}$ and $S(f) \subseteq]-\infty, Q]$. The main question will be: Which conditions have to be imposed on the defining function h (cf. Th. 3 resp. 5) in order that the solution f has the desired property?

b) Continuous solutions

It is evident that in this case h has to be continuous. The answer concerning the necessity of further properties on h is given below:

Theorem 8 (The case $S(f) \subseteq \mathbb{R}$). *Let* $h\colon [-1,1[\to \mathbb{R}$ *be continuous. Then the unique solution* f *of* (1) *defined by* h (unique extension by Th. 3) *is continuous iff* $\lim_{x\to 1} h(x) = (4q-2)h(0) - h(-1)$.

Proof. We use the notations f_n and F_n of Th. 3.

"only if": $\lim_{x\to 1} h(x) = f(1) = (4q-2)h(0) - h(-1)$ by equation (1).

"if": The construction of f given in Th. 3 is very useful: $f_1 := h\colon [-1,1[\to \mathbb{R}$. If f_n is given on $[-1,n[$ for some nonnegative integer n, then f_{n+1} is defined on $[-1,n+1[$ by

$$f_{n+1}(x) := \begin{cases} f_n(x) & \text{for } x \in [-1,n[\\ 4qf_n(q(x-1)) - f_n(x-2) - 2f_n(x-1) & \text{otherwise.} \end{cases}$$

As f_n is supposed to be continuous (induction hypothesis), we only have to show that f_{n+1} is continuous at the point n (in the neighbourhoods of any other point f_{n+1} is given as a composition of continuous functions). To be more precise: We only have to show that $\lim_{x\nearrow n} f_{n+1}(x) = f_{n+1}(n)$, i.e.,

$$(*) \qquad \lim_{x\nearrow n} f_n(x) = 4qf_n(q(n-1)) - f_n(n-2) - 2f_n(n-1).$$

$n=1$: $(*)$ is fulfilled because of our assumption on h.

$n>1$: By definition of f_n we have

$$\begin{aligned} \lim_{x\nearrow n} f_n(x) &= \lim_{x\nearrow n}\big(4qf_{n-1}(q(x-1)) - f_{n-1}(x-2) - 2f_{n-1}(x-1)\big) = \\ &= \lim_{x\nearrow n}\big(4qf_n(q(x-1)) - f_n(x-2) - 2f_n(x-1)\big) = \\ &\quad \text{(because } f_n \text{ coincides with } f_{n-1} \text{ on } [-1,n-1[) \\ &= \lim_{x\nearrow n-1}\big(4qf_n(q(x)) - f_n(x-1) - 2f_n(x)\big) = \\ &= 4qf_n(q(n-1)) - f_n(n-2) - 2f_n(n-1) = f_{n+1}(n) \\ &\quad \text{(because } n-1 \text{ is an interior point of } [-1,n[\text{, and} \\ &\quad f_n \text{ is continuous on } [-1,n[). \end{aligned}$$

Thus the function F_1 of Th. 3 is continuous on $[-1,\infty[$. We proceed once more by induction, showing that each F_n is continuous. These functions are inductively defined by

$$F_{n+1}(x) := \begin{cases} F_n(x) & \text{for } x \in [-n,\infty[\\ 4qF_n(q(x+1)) - F_n(x+2) - 2F_n(x+1) & \text{otherwise.} \end{cases}$$

Here we have to show that the function F_{n+1} is continuous at the point $-n$, to be more precise, we have to show:

$$\lim_{x\nearrow -n} F_{n+1}(x) = F_n(-n).$$

But

$$\lim_{x\nearrow -n} F_{n+1}(x) = \lim_{x\nearrow -n} \big(4qF_n(q(x+1)) - F_n(x+2) - 2F_n(x+1)\big) =$$
$$= \lim_{x\nearrow -n+1} \big(4qF_n(q(x)) - F_n(x+1) - 2F_n(x)\big) =$$
$$= \big(4qF_n(q(-n+1)) - F_n(-n+2) - 2F_n(-n+1)\big) =$$
$$= 4qF_{n-1}(q(-n+1)) - F_{n-1}(-n+2) - 2F_{n-1}(-n+1) = F_n(-n) \text{ for } n \geq 2.$$

For $n = 1$ we compute like before

$$\lim_{x\nearrow -n} F_{n+1}(x) = 4qF_n(q(-n+1)) - F_n(-n+2) - 2F_n(-n+1) =$$
$$= 4qF_1(0) - F_1(1) - 2F_1(0) = F_1(-1) \quad \text{by the condition on } h.$$

Thus each F_n is continuous, and therefore the solution f is continuous. ◊

Before we deal with the case $S(f) \subseteq]-\infty, Q]$, we introduce some notation. This will be useful to make the theorems on this case $S(f) \subseteq$ $\subseteq]-\infty, Q]$ more easily readable — and the same notation is also useful to treat the question of differentiable solutions.

Definition 1. Let $q \in]0,1[$.

$$E(q) := \left\{ \frac{Q - a_0 - a_1 q^1 - \ldots - a_m q^m}{q^m} \,\middle|\, m, a_i \in \mathbb{Z},\ m \geq 0,\ a_i \geq 1 \right\} \cup \{Q\}$$

(*exceptional points*). For real x and integers $m \geq 1$ we define the set $M(q, x, m)$ by

$$M(q,x,m) := \big\{(l_1, \ldots, l_m) \mid q^m x + l_m q^m + \ldots + l_1 q = Q,$$
$$l_i \in \mathbb{Z},\ l_1, \ldots, l_m > 0\big\}$$

(*m-tuples*). Furthermore, let $S(q, x, m)$ denote the sum

$$S(q,x,m) := \sum_{(l_1,\ldots,l_m) \in M(q,x,m)} (-1)^{l_1 + \ldots + l_m} \cdot l_1 \ldots l_m.$$

(As usual $S(q, x, m) = 0$ whenever $M(q, x, m) = \emptyset$.) Finally, let $C(q)$ denote the set

$$C(q) := \big\{x \in \mathbb{R} \mid \lim_{m\to\infty} S(q,x,m) = 0\big\}$$

(*points of continuity*, as we will see later on).

Proposition 1.

(a) *For $x \in \mathbb{R}$ the following relation holds: $x \in E(q)$ iff $M(q,x,m) \neq \neq \emptyset$ for some $m \in \mathbb{N}$.*

(b) $\{1\} \times M(q,x,m) \subseteq M(q,x,m+1)$ *for any m and for any x.*

(c) *For any $x \in \mathbb{R}$ there exists a natural number m_0 such that for any $m \geq m_0$ the relations $M(q,x,m+1) = \{1\} \times M(q,x,m)$ and $S(q,x,m+1) = -S(q,x,m)$ hold. Furthermore, the sets $M(q,x,m)$ are finite.*

(d) *The set $E(q)$ only contains isolated points, and for any compact interval J the set $J \cap E(q)$ is finite.*

(e) $\mathbb{R} \backslash E(q) \subseteq C(q)$.

(f) *Recursion formula for $S(q,x,m)$:*

$$S(q,x,m+1) = \sum_{l=1}^{u} (-1)^l l \cdot S\Big(q, x + \frac{l-1}{q^m}, m\Big),$$

where u denotes the greatest integer with $u \leq Q + 1 - q^m x$.

Proof. (a) First suppose that $x \in E(q)$.

Case 1: $x = Q$. Then $qx + q = q(Q+1) = Q$, which implies that $M(q,x,1) \neq \emptyset$.

Case 2: $x = \frac{Q - a_0 - a_1 q^1 - \ldots - a_m q^m}{q^m}$, where $m, a_i \in \mathbb{Z}$, $m \geq 0$, $a_i \geq 1$. Then

$$q^{m+1}x + (a_0+1)q^1 + a_1 q^2 + \ldots + a_m q^{m+1} =$$

$$= q\big(Q+1-1-a_0-a_1q^1-\ldots-a_mq^m\big) + (a_0+1)q^1 + a_1q^2 + \ldots + a_mq^{m+1} =$$

$$= q(Q+1) = Q.$$

Thus $M(q,x,m+1) \neq \emptyset$. Now suppose that $M(q,x,m) \neq \emptyset$ for some $m \geq 1$: Then there are integers $l_1, \ldots, l_m \geq 1$ such that $q^m x + l_m q^m + \ldots + l_1 q = Q$, which implies that

$$x = \frac{Q - l_1 q^1 - \ldots - l_m q^m}{q^m} = \frac{q(Q+1) - l_1 q^1 - \ldots - l_m q^m}{q^m}.$$

Cancellation of the factor q gives

$$x = \frac{Q - (l_1 - 1) - \ldots - l_m q^{m-1}}{q^{m-1}}.$$

If $l_1 > 1$, this expression shows that $x \in E(q)$, otherwise we proceed cancelling like before until we get the desired expression.

(b) Let $(l_1, \dots, l_m) \in M(q,x,m)$. Then $q^m x + l_m q^m + \dots + l_1 q = Q$, which implies that

$$q^{m+1}x + l_m q^{m+1} + \dots + l_1 q^2 + 1q = qQ + q = q(Q+1) = Q,$$

in other words, $(1, l_1, \dots, l_m) \in M(q,x,m+1)$.

(c) *Case* 1: $x > Q$. In this case $M(q,x,m) = \emptyset$ for any m, because the relation $q(Q+1) = Q$ immediately implies that $q^m Q + q^m + \dots + q^1 = Q$. Therefore, as $x > Q$ and $l_1, \dots, l_m \geq 1$, we immediately get $q^m x + l_m q^m + \dots + l_1 q > q^m Q + q^m + \dots + q^1 = Q$. Thus it is impossible to find elements belonging to $M(q,x,m)$.

Case 2: $x = Q$. From the computation of Case 1 it follows immediately that $M(q,Q,m) = \{(1, \dots, 1)\}$.

Case 3: $x < Q$, $x \notin E(q)$. Then $M(q,x,m) = \emptyset$ for any m.

Case 4: $x < Q$, $x \in E(q)$. Then $0 < q(Q-x) = q\big((Q+1)-(x+1)\big) = Q - q(x+1)$ and $Q - q(x+1) = q(Q-x) < Q - x$, which implies that $x < q(x+1) < Q$, and a usual induction argument shows that $x < qx + q < q^2x + q^2 + q < \dots < q^m x + q^m + \dots + q < \dots < Q$, and this strictly increasing sequence tends to Q. Now choose an integer n such that

$$Q - q < q^n x + q^n + \dots + q < Q.$$

Let $m \geq n$ be an arbitrary integer and $(l_1, \dots, l_m) \in M(q,x,m)$. Suppose that $l_1 \geq 2$, then

$$Q = q^m x + l_m q^m + \dots + l_1 q \geq q^m x + q^m + \dots + q + q > Q - q + q = Q,$$

a contradiction. Thus the only possiblity is that $l_1 = 1$. From this fact we deduce that $Q = q(Q+1) = q^m x + l_m q^m + \dots + 1q$. Cancellation of the factor q gives $Q = q^{m-1}x + l_m q^{m-1} + \dots + l_2 q$, which implies that $(l_2, \dots, l_m) \in M(q,x,m-1)$, in other words: We have shown that $M(q,x,m) \subseteq \{1\} \times M(q,x,m-1)$.

Thus in any case $M(q,x,m+1) = \{1\} \times M(q,x,m)$ for $m \geq m_0$ holds. Furthermore, for any fixed natural number m the set $M(q,x,m)$ is finite, because all the numbers $q, q^2, \dots, q^m$ are positive. The formula for the sum $S(q,x,m)$ is a trivial consequence of the equation for $M(q,x,m)$ given above.

(d) From the proof of (c) we see that $E(q)$ is a subset of the interval $[-\infty, Q]$. Thus we only have to show that the intersection of $E(q)$ with any interval $[a, Q[$ (for $a < Q$) is finite. Let $a < Q$, and as in the proof of (c) choose an n such that $Q - q < q^n a + q^n + \dots + q < q$. Now suppose that $x \in [a, Q[$. Then $Q - q < q^n x + q^n + \ldots + q < Q$, and combining (a) and the computation in the proof of (c) we immediately see that $x \in E(q)$ iff $M(q, x, n) \neq \emptyset$. A simple computation also immediately gives that the intersection of $M(q, x, n)$ and $M(q, y, n)$ is empty, if $x \neq y$. If $M(q, x, n)$ is nonempty, then there is an n-tuple $(l_1, \ldots, l_n)$ such that $Q = q^n x + l_n q^n + \ldots + l_1 q$. Thus

$$Q - q < q^n a + q^n + \ldots + q \leq q^n a + l_n q^n + \ldots + l_1 q \leq q^n x + l_n q^n + \ldots + l_1 q = Q.$$

As the numbers $q, q^2, \ldots, q^n$ are positive, there are only finitely many n-tuples $(l_1, \ldots, l_n)$ which fulfill the inequality $q^n a + l_n q^n + \ldots + l_1 q \leq Q$. Thus there are only finitely many points $x \in [a, Q[$ such that $M(q, x, n) \neq \emptyset$.

(e) By (a), for any $x \in \mathbb{R} \backslash E(q)$ we have $M(q, x, m) = \emptyset$ for any m. Thus for any m and any such x we get $S(q, x, m) = 0$.

(f)
$$M(q, x, m+1) := \\ := \left\{(l_1, \ldots, l_m, l_{m+1}) \,\middle|\, {}^{q^{m+1}x + l_{m+1}q^{m+1} + l_m q^m + \ldots + l_1 q = Q,}_{l_i \in \mathbb{Z}, l_1, \ldots, l_{m+1} > 0} \right\}.$$

Fixing l_1, the condition can be written as

$$q^{m+1}x + l_1 q + l_{m+1}q^{m+1} + \ldots + l_2 q^2 = Q = q(Q+1) \quad \text{or}$$
$$q^{m+1}x + (l_1 - 1)q + l_{m+1}q^{m+1} + \ldots + l_2 q^2 = qQ \quad \text{or}$$
$$q^m\left(x + \frac{l_1 - 1}{q^m}\right) + l_{m+1}q^m + \ldots + l_2 q^1 = Q.$$

From the last condition we immediately get

$$M(q, x, m+1) = \bigcup_{l=1}^{\infty} \{l\} \times M\left(q, x + \tfrac{l-1}{q^m}, m\right).$$

The formula for $S(q, x, m+1)$ is a trivial consequence, because the union above is disjoint and $M(q, y, m)$ is empty for $y > Q$. $\Diamond$

We will need the set $C(q)$ in order to describe continuity resp. differentiability properties. As the description of the set $E(q)$ is much easier to handle than the definition of $C(q)$, we try to find a relation "easy to handle" between the sets $\mathbb{R} \backslash E(q)$ and $C(q)$. In Prop. 1(c) it has been shown that $\mathbb{R} \backslash E(q) \subseteq C(q)$. Thus the question arises whether this inclusion is proper or not. At the moment only a partial answer can be given:

Proposition 2.

(a) *If $q \in]0, 1[$ is transcendental over the field $\mathbb{Q}$, then $C(q) = \mathbb{R} \backslash E(q)$.*

(b) *If $q \in]0, 1[$ is algebraic over the field $\mathbb{Q}$, it has a minimal polynomial in the algebra $\mathbb{Q}[Z]$. We normalize this polynomial not as* *usual (leading coefficient = 1), but with integer coefficients whose g.c.d. is equal to* 1 *(this polynomial is unique up to the factor* ± 1*). Let this polynomial be denoted by* $p(z)$.

(b1) *If* $p(1)$ *is even (i.e., the sum of the coefficients is even), then* $C(q) = \mathbb{R}\backslash E(q)$.

(b2) *If* $q \leq \frac{1}{3}$ *and* $p(z) = a - bz^k$, *where* a, b, k *are positive integers and* $a + b$ *is odd, then* $\mathbb{R}\backslash E(q)$ *is a proper subset of* $C(q)$.

Proof. Let $x \in E(q)$. Suppose that $(l_1, l_2, \ldots, l_m) \in M(q, x, m)$ and $(l'_1, l'_2, \ldots, l'_m) \in M(q, x, m)$. Then

$$0 = Q - Q = (q^m x + l_m q^m + \ldots + l_1 q) - (q^m x + l'_m q^m + \ldots + l'_1 q) =$$
$$= (l_m - l'_m) q^m + \ldots + (l_1 - l'_1) q.$$

(a) Let q be transcendental over the field $\mathbb{Q}$. Then no nonzero polynomial with integer coefficients can have q as a zero. Therefore, $M(q, x, m)$ contains exactly one element, and $S(q, x, m) \neq 0$.

(b1) For 2 elements $(l_1, l_2, \ldots, l_m) \in M(q, x, m)$ and $(l'_1, l'_2, \ldots, l'_m) \in M(q, x, m)$ the minimal polynomial $p(z)$ of q is a divisor of the polynomial $(l_m - l'_m) z^m + \ldots + (l_1 - l'_1) z$ (by Gauss' lemma from elementary algebra). Thus the even number $p(1)$ is a divisor of $(l_m - l'_m) + \ldots + (l_1 - l'_1)$, in other words: $l_1 + \ldots + l_m$ and $l'_1 + \ldots + l'_m$ are either both even or both odd. Therefore, all the terms in the sum $S(q, x, m)$ have the same sign, which implies that $S(q, x, m) \neq 0$.

(b2) In order to show that in this case $\mathbb{R}\backslash E(q)$ is a proper subset of $C(q)$, we give an element $x \in E(q)$ with $x \in C(q)$: Let r be the smallest positive integer such that

$$\frac{q - q^{k+r}}{1 - q} + (3a - 1) q^r < 1,$$

and let $m := k + r$. (Such an integer r exists, because $Q = \frac{q}{1-q} < 1$.) Now define $(l_1, \ldots, l_m)$ by

$$l_i := \begin{cases} 2a, & \text{if } i = r \\ b & \text{if } i = m \\ 1 & \text{otherwise.} \end{cases}$$

and let $x := \frac{Q - l_1 q^1 - \ldots - l_m q^m}{q^m}$. We determine the set $M(q, x, m)$. Of course, $(l_1, \ldots, l_m) \in M(q, x, m)$. Now let $(l'_1, \ldots, l'_m) \in M(q, x, m)$. Then $l'_i > 0$, and the polynomial $(l'_m - l_m) z^m + \ldots + (l'_1 - l_1) z$ is a multiple of $a - bz^k$ in the ring $\mathbb{Z}[Z]$, i.e., there is a polynomial $s(z)$ with integer coefficients such that $(l'_m - l_m)z^n + \ldots + (l'_1 - l_1)z = (a - bz^k) \cdot$ $\cdot s(z)$. This fact implies that the first nonvanishing difference $l'_i - l_i$ is an integer multiple of a. Now suppose that $j < r$ and $l_1 = l'_1, \ldots, l_{j-1} =$ $= l'_{j-1}$, $l_j \neq l'_j$. Then $l'_j \geq l_j + a = 1 + a$. As the sums $l'_m q^m + \ldots + l'_1 q$ are equal for all $(l'_1, \ldots, l'_m)$ in $M(q, x, m)$, we have

$$1 > l'_m q^m + \ldots + l'_1 q \geq q^m + \ldots + q + aq^j = \frac{q - q^m}{1 - q} + q^j(a + q^{m-j}).$$

As $q \leq \frac{1}{3}$, we have $q(3a - 1) < q \cdot 3a < a + q^{m-j}$, and therefore,

$$1 > l'_m q^m + \ldots + l'_1 q \geq \frac{q - q^m}{1 - q} + q^{j+1}(3a - 1).$$

According to the minimality of r we must have $j \geq r - 1$. On the other hand, $\deg((a - bz^k) \cdot s(z)) = k + r$, which implies that $\deg(s) \leq r$. Thus the only possibilities are that $s(z) = z^{r-1} \cdot (\alpha + \beta z)$ with integer coefficients α, β.

Case 1: $k > 1$. Then $l'_{m-1} = 1 - \alpha b > 0$ and $l'_{r-1} = l_{r-1} + a\alpha \geq 0$, which implies that $\alpha = 0$.

Case 2: $k = 1$. If $r = 1$, then clearly $\alpha = 0$ (the left-hand-side polynomial has no constant term). If $r > 1$, then

$$\begin{aligned} l'_m &= b - \beta b > 0, \quad \text{which implies that } \beta \leq 0; \\ l'_{r-1} &= 1 + \alpha a > 0, \quad \text{which implies that } \alpha \geq 0; \\ l'_r &= 2a - \alpha b + \beta a > 0, \quad \text{i.e. } (2 + \beta)a > \alpha b. \end{aligned}$$

As $q = \frac{a}{b} \leq \frac{1}{3}$, we immediately get $\alpha = 0$.

Thus in any case we have $\alpha = 0$ and, therefore,

$$l'_i = \begin{cases} (2 + \beta)a, & \text{if } i = r \\ (1 - \beta)b, & \text{if } i = m \text{ for some integer } \beta \\ 1 & \text{otherwise.} \end{cases}$$

The condition $l'_i > 0$ implies that the only possible values for β are 0 and -1. Thus the set $M(q, x, m)$ contains exactly two elements, namely,

$$M = \{(1, \ldots, 1, 2a, 1, \ldots, 1, b), (1, \ldots, 1, a, 1, \ldots, 1, 2b)\}.$$

As $1. \ldots .1.2a.1. \ldots .1.b = 1. \ldots .1.a.1. \ldots .1.2b$ and $(1 + \ldots + 1 + 2a +$ $+ 1 + \ldots + 1 + b) - (1 + \ldots + 1 + a + 1 + \ldots + 1 + 2b) = a - b$ is odd, the sum $S(q, x, m)$ is equal to 0. As $(3a - 1)q^r < 1$, we have $(3a - 1)q^{r+1} < 1 + q^{m+1}$, which implies that

$$Q(1 - q) - q = 0 < 1 + (1 - 2a)q^{r+1} + (1 - b)q^{m+1} \quad \text{resp.}$$

$$Q - q < q^{m+1}x + q + \ldots + q^{m+1}.$$

As we had seen in the proof of Prop. 1(c), this condition guarantees that for any $n \geq (m+1) - 1$ we have $S(q, x, n+1) = -S(q, x, n)$. Thus $S(q, x, n) = 0$ for any $n \geq m$, which implies $x \in C(q)$. $\Diamond$

Theorem 9 (The case $S(f) \subseteq]-\infty, Q]$). *Let $h\colon]Q-1, qQ] \to \mathbb{R}$ be a function, and α a real number which fulfills $\alpha = 0$ in the case $q \neq \frac{1}{4}$. Then the unique solution f of* (1) *which coincides with h on $]Q-1, qQ]$ and fulfills $f(Q) = \alpha$ and $S(f) \subseteq]-\infty, Q]$ is continuous iff $\alpha = 0$ and h fulfills the following condition:*

(i) *case $q \leq \frac{1}{4}$*: $h \equiv 0$ (in other words: *in this case the zero function is the only continuous solution);*

(ii) *case $q > \frac{1}{4}$: h is continuous and* $\lim\limits_{x \searrow Q-1} h(x) = 4qh(qQ)$.

Proof. We use the notatios x_n, h_n, y_n, g_n, f_n of Th. 5.

First suppose that f is continuous. As $f(Q) = \lim\limits_{x\searrow Q} f(x) = \lim\limits_{x\searrow Q} 0 = 0$, we must have $\alpha = 0$, furthermore, h must be continuous. Also $\lim\limits_{x\searrow Q-1} h(x) = \lim\limits_{x\searrow Q-1} f(x) = f(Q-1) = 4qf(qQ) - f(Q+1) - -2f(Q) = 4qf(qQ) = 4qh(qQ)$. Now let $z \in]Q-1, qQ]$ be arbitrary, and define a sequence (z_n) by $z_0 := z$, $z_{n+1} := q(z_n+1)$. Then $\lim\limits_{n\to\infty} z_n = Q$, (z_n) is strictly increasing, and because $z_n + 2 > z_n + 1 > Q$, we have

$$f(z_{n+1}) = f(q(z_n+1)) = \frac{1}{4q}\big(f(z_n) + f(z_n+2) + 2f(z_n+1)\big) = \frac{1}{4q}f(z_n).$$

Thus, $f(z_n) = \left(\frac{1}{4q}\right)^n f(z)$, and therefore

$$0 = f(Q) = \lim_{n\to\infty} f(z_n) = \lim_{n\to\infty}\left(\frac{1}{4q}\right)^n f(z).$$

The right-hand-side limit exists and is equal to 0 iff $q > \frac{1}{4}$ or $f(z) = 0$.

For the reverse direction we may suppose that $q > \frac{1}{4}$, $\alpha = 0$, $\lim\limits_{s\searrow Q-1} h(x) = 4qh(qQ)$ and h continuous. We use the construction of the solution f of Th. 5. Let (x_n) be the sequence given by $x_0 := Q-1$, $x_{n+1} := q(x_n+1)$. Let $h_0 := h$, and h_n be defined on the interval $]x_0, x_{n+1}]$ by induction:

$$h_{n+1}(x) := \begin{cases} h_n(x) & \text{for } x \in]x_0, x_{n+1}] \\ \frac{1}{4q}h_n(y) & \text{for } x = q(y+1) \in]x_{n+1}, x_{n+2}]. \end{cases}$$

If h_n is continuous on $]x_0, x_{n+1}]$, then by definition it is evident that h_{n+1} is continuous in $]x_0, x_{n+2}] \setminus \{x_{n+1}\}$. To prove continuity at the point x_{n+1} it is only necessary to compute $\lim\limits_{x\searrow x_{n+1}} h_{n+1}(x)$:

$$\lim_{x\searrow x_{n+1}} h_{n+1}(x) = \lim_{x\searrow x_{n+1}} \frac{1}{4q} h_n\left(\frac{x}{q}-1\right) =$$
$$= \lim_{x\searrow x_n} \frac{1}{4q} h_n(x) = \frac{1}{4q} h_n(x_n) = h_{n+1}(x_{n+1}).$$

Thus the function $h_\infty\colon]Q-1,Q[\to \mathbb{R}$ is continuous, and for the continuity of the function

$$g_0 :]Q-1,\infty[\to \mathbb{R}\colon x\to \begin{cases} h_\infty(x) & \text{for } x\in]Q-1,Q[\\ 0 & \text{for } x\geq Q \end{cases}$$

we only have to show that $\lim\limits_{x\nearrow Q} h_\infty(x) = 0$. By assumption h is bounded on $]Q-1,qQ]$, let us say, by a constant M. But then we have h_∞ bounded on $]x_n,x_{n+1}]$ by $\left(\frac{1}{4q}\right)^n M$, which immediately implies that g_0 is continuous at Q. The next extension is done via the sequence (y_n) of Th. 5, defined by $y_0 := Q-1$, $y_{n+1} := \frac{1}{q}y_n - 1$, and the nonnegative integer k such that $y_k < 0$, $y_{k-1}\geq 0$. For g_m defined on $]y_m,\infty[$, $0\leq m<k$, we define g_{m+1} on $]y_{m+1},\infty[$ by

$$g_{m+1}(x) :=$$
$$:= \begin{cases} g_m(x) & \text{for } x\in]y_m,\infty[\\ 4qg_m(q(x+1)) - g_m(x+2) - 2g_m(x+1) & \text{for } x\in]y_{m+1},y_m]. \end{cases}$$

Once more using an induction argument, we have to show that g_{m+1} is continuous under the assumption that g_m is continuous. According to the definition, the only critical point is the point y_m:

$m=0$: $\lim\limits_{x\searrow y_0} g_1(x) = \lim\limits_{x\searrow y_0} g_0(x) = \lim\limits_{x\searrow y_0} h(x) = 4qh(qQ) =$
$= 4qg_0(q(y_0+1)) - g_0(y_0+2) - 2g_0(y_0+1) = g_1(y_0)$.

$m>0$: We use the fact that g_m is continuous at the interior points of its domain:

$$\lim_{x\searrow y_m} g_{m+1}(x) = \lim_{x\searrow y_m}\left(4qg_m(q(x+1)) - g_m(x+2) - 2g_m(x+1)\right) =$$
$$= 4qg_m(q(y_m+1)) - g_m(y_m+2) - 2g_m(y_m+1) = g_{m+1}(y_m).$$

Thus the function $f_0\colon]y_k,\infty[\to \mathbb{R}$ is continuous. Now, if $f_n\colon]y_k-n,\infty[\to \mathbb{R}$ is continuous, f_{n+1} is defined on $]y_k-n-1,\infty[$ by the formula

$$f_{n+1}(x):=\begin{cases} f_n(x) & \text{for } x\in]y_k-n,\infty[\\ 4qf_n(q(x+1))-f_n(x+2)-2f_n(x+1) & \text{otherwise.}\end{cases}$$

Thus it is continuous at any point except possibly the point $y_k - n$:
$n = 0$, $k = 0$:

$$\lim_{x\searrow y_k} f_1(x) = \lim_{x\searrow y_k} f_0(x) = \lim_{x\searrow y_k} g_0(x) = \lim_{x\searrow y_0} h(x) = 4qh(qQ) =$$
$$= 4qf_0(q(y_0+1)) - f_0(y_0+2) - 2f_0(y_0+1) = f_1(y_0).$$

$n = 0$, $k > 0$:

$$\lim_{x\searrow y_k} f_1(x) = \lim_{x\searrow y_k} f_0(x) = \lim_{x\searrow y_k} g_k(x) =$$
$$= \lim_{x\searrow y_k} \big(4qg_{k-1}(q(x+1)) - g_{k-1}(x+2) - 2g_{k-1}(x+1)\big) =$$
$$= \lim_{x\searrow y_k} \big(4qg_k(q(x+1)) - g_k(x+2) - 2g_k(x+1)\big) =$$
$$= 4qg_k(q(y_k+1)) - g_k(y_k+2) - 2g_k(y_k+1) =$$
$$= 4qf_0(q(y_k+1)) - f_0(y_k+2) - 2f_0(y_k+1) = f_1(y_k).$$

$n > 0$:

$$\lim_{x\searrow y_k-n} f_{n+1}(x) + \lim_{x\searrow y_k-n} \big(4qf_n(q(x+1)) - f_n(x+2) - 2f_n(x+1)\big) =$$
$$= 4qf_n(q(y_k-n+1)) - f_n(y_k-n+2) - 2f_n(y_k-n+1) = f_{n+1}(y_k-n).$$

This fact proves that the resulting solution f of (1) is continuous everywhere. ◊

Of course, it was necessary to have h continuous in the preceding theorem in order to get a continuous solution. And — together with the boundary condition $\lim_{x\searrow Q-1} h(x) = 4qh(qQ)$ — this is also sufficient in the case $q > \frac{1}{4}$. The question arises: What can be said about solutions f in the case $q \le \frac{1}{4}$, if the defining function h is continuous and fulfills this boundary condition?

Theorem 10 (the case $S(f) \subseteq]-\infty, Q]$). *Let* $q \le \frac{1}{4}$, $\alpha \in \mathbb{R}$, $\alpha = 0$ *if* $q < \frac{1}{4}$, $h\colon]Q-1, qQ] \to \mathbb{R}$ *be continuous and nonvanishing,* $\lim_{x\searrow Q-1} h(x) = 4qh(qQ)$*, and let* f *be the unique solution of* (1) *which extends* h *and fulfills* $S(f) \subseteq]-\infty, Q]$ *and* $f(Q) = \alpha$*. Then the set of points where* f *is continuous coincides with* $C(q)$*.*

Proof. We use the notations of Ths. 5 and 9. The proof in Th. 9 shows that f is continuous on the set $]Q-1, Q[$. Furthermore, if one chooses a point $z \in]Q-1, qQ]$ such that $h(z) \neq 0$, then the sequence (z_n) defined by $z_0 := z$, $z_{n+1} := q(z_n + 1)$ tends to Q, and the values are given by $f(z_n) = \left(\frac{1}{4q}\right)^n f(z)$. This sequence tends to infinity in the case $q < \frac{1}{4}$, and it has a constant value, different from 0, in the case $q = \frac{1}{4}$. Thus the function $g_0 :]Q-1, \infty[\to \mathbf{R}$ has exactly one point of discontinuity, namely the point $x = Q$.

Now for the solution f the equation

$$f(x) = 4qf(q(x+1)) - f(x+2) - 2f(x+1)$$

holds. By usual induction argument from this equation we can derive the formula

$$f(x) = 4q\sum_{l=1}^{k}(-1)^{l+1}l.f(q(x+1)) + (-1)^k\big((k+1)f(x+k) + kf(x+k+1)\big)$$

for any natural number k: In the case $k = 1$ this formula is nothing else but equation (1), and using equation (1) for the expression $f(x+k)$ we get

$$f(x) =$$

$$= 4q\sum_{l=1}^{k}(-1)^{l+1}l.f(q(x+1)) + (-1)^k\big((k+1)f(x+k) + kf(x+k+1)\big) =$$

$$= 4q\sum_{l=1}^{k}(-1)^{l+1}l.f(q(x+1)) + (-1)^k kf(x+k+1) +$$

$$+ (-1)^k(k+1)\big(4q.f(q(x+k+1)) - f(x+k+2) - 2f(x+k+1)\big) =$$

$$= 4q\sum_{l=1}^{k+1}(-1)^{l+1}l.f(q(x+l)) +$$

$$+ (-1)^{k+1}((k+2)f(x+k+1) + (k+1)f(x+k+2)).$$

Now suppose that $x < Q$, and let k be an integer such that $x + k > Q$. As $S(f) \subseteq]-\infty, Q]$, we immediately get

$$f(x) = 4q\sum_{l=1}^{k}(-1)^{l+1}l.f(q(x+1)).$$

Repeating this formula for the arguments $q(x+l)$ we immediately get

$$f(x) = 4q\sum_{l_1=1}^{k}(-1)^{l_1+1}.l_1.4q\sum_{l_2=1}^{k}(-1)^{l_2+1}.l_2.f(q^2x + l_1q^2 + l_2q),$$

and by a usual induction argument, for any natural number m we get

$$f(x) = (-4q)^m \cdot \sum_{l_1,\dots,l_m=1}^{k} (-1)^{l_1+\dots+l_m}.l_1\dots l_m.f(q^m x + l_m q^m + \dots + l_1 q).$$

Now we may choose m large enough such that $q^m x + q^m + \dots + q > Q - 1$. As f is continuous in the interval $]Q-1,\infty[$, except at the point Q, f can be discontinuous at x only, if at least one of the values $q^m x + l_m q^m + \dots + l_1 q$ is equal to Q, because otherwise we can find a whole neighbourhood U of x such that $q^m y + l_m q^m + \dots + l_1 q \neq Q$, for any $y \in U$. Thus f is continuous on the set $\mathbb{R} \setminus E(q)$.

For a detailed description of the points of continuity of f now let $x \in E(q)$, and let m be chosen large enough such that $q^m y + q^m + \dots + q > Q - 1$ in a neighbourhood U of x. Furthermore, we choose k large enough such that $y + k > Q$ for $y \in U$ and $M(q,x,m) \subseteq \{1,2,\dots,k\}^m$ (the last condition makes sense because $M(q,x,m)$ is a finite set). As m, k are fixed, let us abbreviate $M(q,x,m)$ by M and use the notation P for the set $P := \{1,2,\dots,k\}^m \setminus M$. Then $\{1,2,\dots,k\}^m$ is the disjoint union of M and P. Thus for $y \in U$

$$f(y) = (-4q)^m \cdot \sum_{l_1,\dots,l_m=1}^{k} (-1)^{l_1+\dots+l_m}.l_1\dots l_m.f(q^m y + l_m q^m + \dots + l_1 q) =$$

$$= (-4q)^m \cdot \sum_{(l_1,\dots,l_m)\in M} (-1)^{l_1+\dots+l_m}.l_1\dots l_m.f(q^m y + l_m q^m + \dots + l_1 q) +$$

$$+ (-4q)^m \cdot \sum_{(l_1,\dots,l_m)\in P} (-1)^{l_1+\dots+l_m}.l_1\dots l_m.f(q^m y + l_m q^m + \dots + l_1 q).$$

As P is a finite set, we can choose a neighbourhood V of x such that $x \in V \subseteq U$ and $q^m y + l_m q^m + \dots + l_1 q \neq Q$ for any $(l_1,\dots,l_m) \in P$, $y \in V$. Then the sum $\sum_{(l_1,\dots,l_m)\in P} (-1)^{l_1+\dots+l_m}.l_1\dots l_m.f(q^m y + l_m q^m + \dots + l_1 q)$ describes a continuous function on V, because f is a continuous on the set $]Q-1,\infty[\setminus\{Q\}$. On the other hand, for $(l_1,\dots,l_m) \in M$ we have $f(q^m y + l_m q^m + \dots + l_1 q) = f(q^m y - q^m x + q^m x + l_m q^m + \dots + l_1 q) = f(q^m(y-x) + Q)$ by the definition of $M(q,x,m)$. Thus we have

$$\sum_{(l_1,\dots,l_m)\in M} (-1)^{l_1+\dots+l_m}.l_1\dots l_m.f(q^m y + l_m q^m + \dots + l_1 q) =$$
$$= \sum_{(l_1,\dots,l_m)\in M} (-1)^{l_1+\dots+l_m}.l_1\dots l_m.f(q^m(y-x)+Q) =$$
$$= f(q^m(y-x)+Q)\cdot \sum_{(l_1,\dots,l_m)\in M} (-1)^{l_1+\dots+l_m}.l_1\dots l_m =$$
$$= f(q^m(y-x)+Q).S(q,x,m).$$

As f is discontinuous at the point Q, f is continuous at x (in the neighbourhood V), if and only if $S(q,x,m)=0$. These arguments hold for any m large enough, therefore, we may conclude that f is continuous at x iff $x \in C(q)$. $\Diamond$

In general, it is not easy to decide for a point $x \in E(q)$ whether it belongs to the set $C(q)$ or not. A special case is the case $q=\frac{1}{4}$. In this case a complete description of the set $C(q)$ can be given. As a consequence, in this case the points of (dis-)continuity of the solution f can be given explicitely. The following theorem will give this description, and an example will illustrate this fact.

Theorem 11. *Let $q=\frac{1}{4}$. For any integer $p \geq 0$ let the sequences $\alpha(p)=(\alpha_0,\alpha_1,\alpha_2,\dots)$ and $\beta(p)=(\beta_0,\beta_1,\dots)$ be defined as follows: $\alpha_0 := p(\bmod 8)$ (the remainder term of the division by 8), $\beta_0 := \frac{p-\alpha_0}{8}$, and the next terms are defined by induction $\alpha_{i+1} := \beta_i(\bmod 4)$, $\beta_{i+1} :=$ $:= \frac{\beta_i-\alpha_{i+1}}{4}$ for $i \geq 0$. (The sequence $\alpha(p)$ is constructed like the 4-adic expansion of p, except the first element α_0.) Now let L denote the set*

$$L := \left\{ p \in \mathbb{Z} \;\middle|\; \begin{matrix} p \geq 0, \text{ and the sequence } \alpha(p) \text{ fulfills the condition:} \\ \alpha_0 = 7, \text{ or there is an } i \geq 1 \text{ such that } \alpha_i = 3 \end{matrix} \right\}.$$

Then $E(\frac{1}{4}) = \{Q-p | p \in \mathbb{Z}, p \geq 0\}$ and $E(\frac{1}{4}) \cap C(\frac{1}{4}) = \{Q-p | p \in L\}$.

Proof.

$$E(q) = \left\{ \frac{Q-a_0-a_1q^1-\dots-a_mq^m}{q^m} \;\middle|\; m,a_i \in \mathbb{Z}, m \geq 0, a_i \geq 1 \right\} \cup \{Q\} =$$
$$= \left\{ Qq^{-m} - a_0q^{-m} - \dots - a_m \;\middle|\; m,a_i \in \mathbb{Z}, m \geq 0, a_i \geq 1 \right\} \cup \{Q\}.$$

Now $q=\frac{1}{4}$ and $Q=\frac{1}{3}$, thus $q^{-1}=4$ and, therefore,

$$E\Big(\frac{1}{4}\Big)\{Q-p | p \in \mathbb{Z}, p \geq 0\}.$$

In order to find $C(\frac{1}{4})$ we compute the values $S(\frac{1}{4},x,m)$ for $x \in E(\frac{1}{4})$. For the sake of simplicity let us denote $S(p,m) := S(\frac{1}{4}, Q-p, m)$

for integers $m, p \geq 1$. (From the proof of Prop. 1(c) we know that $S(q, Q, m) = (-1)^m$ for any m, any q.) Of course, we use the recursion formula from Prop. 1(f):

$$S(q, x, m+1) = \sum_{l=1}^{u} (-1)^l l.S\Big(q, x + \frac{l-1}{q^m}, m\Big),$$

where u denotes the largest integer with $u \leq Q+1-q^m x$. In the special case $q = \frac{1}{4}$ and $x = Q - p$ this formula reads as

$$S(p, m+1) = \sum_{l=1}^{u} (-1)^l l.S\Big(\frac{1}{4}, Q-p+\frac{l-1}{q^m}, m\Big) = \sum_{l=1}^{u} (-1)^l l.S(p-4^m(l-1), m).$$

What is the upper bound u? By definition, we have to look for all l such that there is an m-tuple $(l_2, \dots, l_{m+1})$ with the property

$$q^{m+1}x + l_{m+1}q^{m+1} + \ldots + l_2 q^2 + lq = Q$$

resp.

$$\begin{aligned} Qq^{-m-1} &= x + l_{m+1} + \ldots + l_2 q^{1-m} + lq^{-m} = \\ &= Q - p + l_{m+1} + \ldots + l_2 q^{1-m} + lq^{-m}. \end{aligned}$$

Thus the equation

$$\frac{1}{3}(4^{m+1} - 1) + p = l_{m+1} + \ldots + l_2 4^{m-1} + l.4^m$$

should have a solution, which is possible if

$$l.4^m \leq \frac{4^{m+1}-1}{3} + p - \frac{4^m - 1}{3} = p + 4^m.$$

Therefore, $\frac{p}{4^m} + 1$ is an upper bound for l — let us denote by $u(p, m)$ the greatest integer less or equal to $\frac{p}{4^m} + 1$.

Now we start computing the values $S(p, m)$:

$m = 1$: We have to find all the solutions for the equation $qx + l_1 q = Q = q(Q+1)$, which is equivalent to $Q - p + l_1 = Q + 1$. The only possible choice is $l_1 = p + 1$, therefore

$$S(p, 1) = (-1)^{p+1}(p+1).$$

$m = 2$: Suppose that $p = 8r + \alpha$, where $r \in \mathbb{Z}$, $r \geq 0$, and $\alpha \in \{0, 1, 2, \dots, 7\}$. Then the upper bound $u(p, 1)$ is $2r + 1$ for $\alpha \in \{0, 1, 2, 3)$ and $2r + 2$ for $\alpha \in (4, 5, 6, 7)$.

$\alpha \in \{0,1,2,3\}$:

$$S(p,2) = \sum_{l=1}^{2r+1} (-1)^l . l . (8r + \alpha - 4(l-1) + 1) = (-\alpha - 1)(r+1).$$

$\alpha \in \{4,5,6,7\}$:

$$S(p,2) = \sum_{l=1}^{2r+2} (-1)^l . l . (8r + \alpha - 4(l-1) + 1) = (\alpha - 7)(r+1).$$

Thus

$$S(p,2) = (r+1)\varphi(\alpha),$$

where

$$\varphi(\alpha) = \begin{cases} -\alpha - 1 & \text{for } \alpha \in \{0,1,2,3\} \\ \alpha - 7 & \text{for } \alpha \in \{4,5,6,7\}. \end{cases}$$

$m = 3$: As $p = 8r + \alpha$, we now suppose that $r = 4s + \delta$ ($s \in \in \mathbb{Z}$, $s \geq 0$, $\delta \in \{0,1,2,3\}$). Then $u(p,2) = 2s + 1$ for $\delta \in \{0,1\}$, and $u(p,2) = 2s + 2$ for $\delta \in \{2,3\}$.
$\delta \in \{0,1\}$:

$$\begin{aligned} S(p,3) &= \sum_{l=1}^{2s+1} (-1)^l . l . S(p - 16(l-1), 2) = \\ &= \sum_{l=1}^{2s+1} (-1)^l . l . S(8r + \alpha - 8.2(l-1), 2) = \\ &= \sum_{l=1}^{2s+1} (-1)^l . l . (r - 2(l-1) + 1)\varphi(\alpha) = \\ &= \sum_{l=1}^{2s+1} (-1)^l . l . (4s + \delta + 3 - 2l)\varphi(\alpha) = -(\delta + 1)(s+1)\varphi(\alpha). \end{aligned}$$

$\delta \in \{2,3\}$:

$$\begin{aligned} S(p,3) &= \sum_{l=1}^{2s+2} (-1)^l . l . S(p - 16(l-1), 2) = \\ &= \sum_{l=1}^{2s+2} (-1)^l . l . (4s + \delta + 3 - 2l)\varphi(\alpha) = (\delta - 3)(s+1)\varphi(\alpha). \end{aligned}$$

Thus

$$S(m,3)=(s+1)\varphi(\alpha)\psi(\delta),$$

where ψ is given by

$$\psi(\delta)=\begin{cases}-\delta-1 & \text{for } \delta\in\{0,1\}\\ \delta-3 & \text{for } \delta\in\{2,3\}\end{cases}.$$

Using the sequences $\alpha(p)$ and $\beta(p)$ defined in the statement of this theorem, we have

$$S(p,1)=(-1)^{m+1}(m+1),\quad S(p,2)=(\beta_0+1)\varphi(\alpha_0)$$

$$S(p,3)=(\beta_1+1)\psi(\alpha_1)\varphi(\alpha_0).$$

Now we may proceed by induction:

$$S(p,k+2)=(\beta_k+1)\psi(\alpha_k)\ldots\psi(\alpha_2)\psi(\alpha_1)\varphi(\alpha_0)\quad \text{for } k\geq 1.$$

This formula is true for $k=1$, and $k\to k+1$: By definition of the sequences $\alpha(p)$ and $\beta(p)$ the bound $u=u(p,k+2)$ is given by $2\beta_{k+1}+1$ in the case $\alpha_{k+1}\in\{0,1\}$ and by $2\beta_{k+1}+2$ in the case $\alpha_{k+1}\in\{2,3\}$. Thus we have

$$S(p,k+3)=\sum_{l=1}^{u}(-1)^l.l.S(p-4^{k+2}(l-1),k+2)=$$

$$=\sum_{l=1}^{u}(-1)^l.l.(4\beta_{k+1}+3+\alpha_{k+1}-2l)\psi(\alpha_k)\ldots\psi(\alpha_2)\psi(\alpha_1)\varphi(\alpha_0)=$$

(by the same computation as before)

$$=(\beta_{k+1}+1)\psi(\alpha_{k+1})\psi(\alpha_k)\ldots\psi(\alpha_2)\psi(\alpha_1)\varphi(\alpha_0).$$

From the last formula it follows that $S(p,m)=0$, if and only if $\varphi(\alpha_0)=$ $=0$ or $\psi(\alpha_i)=0$ for some $i\geq 1$. The first is fulfilled iff $\alpha_0=7$, the latter is fulfilled iff $\alpha_i=3$ for some $i\geq 1$, which proves the theorem. ◊

Remark 3. The first elements of the set L in the preceding theorem are given by $L=\{7,15,23,24,25,26,27,28,29,30,31,39,47,55,56,57,58,59,60,61,62,63,71,79,87,88,89,90,91,92,93,94,95,96,97,98,99,\ldots\}$.

The following example is very easy to construct and shows in the case $q=\frac{1}{4}$ that the point $Q-7$ really is a point of continuity, though the defining function h (and the value α) at the beginning do not look as if this were true.

Example. Let $q=\frac{1}{4}$, then $Q=\frac{1}{3}$, and let h be defined on $]Q-1,qQ]=$ $=\left]-\frac{2}{3},\frac{1}{12}\right]$ to be constant equal to 1, and let $\alpha\in\mathbb{R}$ be arbitrary. It is evident that h fulfills the conditions of Th. 11. By the construction of Th. 5 f is given by

$$f(x) = \begin{cases} 0 & \text{for } x \in]Q, \infty[\\ \alpha & \text{for } x = Q \\ 1 & \text{for } x \in]Q-1, Q[\\ 1-2\alpha & \text{for } x = Q-1 \\ -1 & \text{for } x \in]Q-2, Q-1[\\ -1+3\alpha & \text{for } x = Q-2 \\ 2 & \text{for } x \in]Q-3, Q-2[\\ 2-4\alpha & \text{for } x = Q-3 \\ -2 & \text{for } x \in]Q-4, Q-3[\\ -2+3\alpha & \text{for } x = Q-4 \\ 1 & \text{for } x \in]Q-5, Q-4[\\ 1-2\alpha & \text{for } x = Q-5 \\ -1 & \text{for } x \in]Q-6, Q-5[\\ -1+\alpha & \text{for } x = Q-6 \\ 0 & \text{for } x \in]Q-7, Q-6[\\ 0 & \text{for } x = Q-7 \\ 0 & \text{for } x \in]Q-8, Q-7[\\ 2\alpha & \text{for } x = Q-8 \\ 2 & \text{for } x \in]Q-9, Q-8[\\ \ldots\ldots & \ldots \\ \ldots\ldots & \ldots \end{cases}$$

It is clear that f is continuous at the point $Q-7$, but not continuous at the points Q, $Q-1$, $Q-2$, $Q-3$, $Q-4$, $Q-5$, $Q-6$, $Q-8$.

After this example we close this section on continuous solutions and turn over to

c) Differentiable solutions

Like in the continuous case, the solutions without any boundary conditions are much easier to handle, and for the proofs we again use the constructions of the solutions given in Ths. 3 and 5.

Theorem 12 (The case $S(f) \subseteq \mathbb{R}$). *Let p be a positive integer,*

$h\colon [-1,1[\to \mathbb{R}$ *an arbitrary function, and let* f *be the unique solution of* (1) *which coincides with* h *on* $[-1,1[$. *Then* f *is* p*-times differentiable (resp.* p*-times continuously differentiable) iff* h *is continuous, continuously extendable to* $H\colon [-1,1] \to \mathbb{R}$ *and* H *is* p*-times differentiable (resp.* p*-times continuously differentiable) and fulfills the following system of equations:*

$$\begin{cases} 4q.H(0) = H(-1) + H(1) + 2H(0) \\ 4q.q.H'(0) = H'(-1) + H'(1) + 2H'(0) \\ 4q.q^2.H''(0) = H''(-1) + H''(1) + 2H''(0) \\ \dots\dots\dots\dots \\ 4q.q^p.H^{(p)}(0) = H^{(p)}(-1) + H^{(p)}(1) + 2H^{(p)}(0). \end{cases}$$

Remark: As H is defined on the closed interval $[-1,1]$, differentiability is to be understood in the following sense: At any point in the open interval $]-1,1[$ the derivative $H'(x)$ exists, at the point 1 the left derivative of H exists, and at the point -1 the right derivative of H exists. In the case of continuous differentiability this function H' has to be continuous on the whole interval $[-1,1]$ and similarly for derivatives of higher order.

Proof. We use the notations f_n and F_n of Th. 3.

"only if": f is a continuous solution, which coincides with h on $[-1,1[$. Therefore h has to be continuously extendable to the function $H\colon [-1,1] \to \mathbb{R}$. Furthermore, f is p-times differentiable (resp. p-times continuously differentiable) and fulfills the equation

$$4q.f(qx) = f(x-1) + f(x+1) + 2f(x) \quad \text{for all} \quad x \in \mathbb{R}.$$

Differentiating this equation with respect to x up to p times and putting $x = 0$ gives the system of equations for h. As H is the restriction of f to the interval $[-1,1]$ it is clear that H has to be p-times differentiable (resp. p-times continuously differentiable).

"if": Suppose that h fulfills the conditions of the theorem. Th. 8 guarantees that the unique solution f is continuous (from the first equation of the system for H). Thus we only have to show that this solution is p-times differentiable (resp. p-times continuously differentiable). Now let $f_1 := h\colon [-1,1[\to \mathbb{R}$. If f_n is given on $[-1,n[$ for some nonnegative integer n, then f_{n+1} is defined on $[-1,n+1[$ by

$$f_{n+1}(x) := \begin{cases} f_n(x) & \text{for } x \in [-1, n[\\ 4qf_n(q(x-1)) - f_n(x-2) - 2f_n(x-1) & \text{otherwise.} \end{cases}$$

As f_n is supposed to be p-times differentiable (resp. p-times continuously differentiable) by induction hypothesis, we only have to show that f_{n+1} is p-times differentiable (resp. p-times continuously differentiable) at the point n (in the neighbourhood of any other point f_{n+1} is given as a composition of p-times differentiable (resp. p-times continuously differentiable) functions):

Case $n = 1$. Left side: Here we have $f_2'(1) = H'(1), \dots, f_2^{(k)}(1) = H^{(k)}(1)$, because f_2 is continuous and coincides with h on $[-1, 1[$ and therefore with H on $[-1, 1]$. Right side: We have to take the right derivatives of the defining expression:

$$f_2^{(k)}(1) = 4q.q^k.f_1^{(k)}(0) - f_1^{(k)}(-1) - 2f_1^{(k)}(0) =$$

$$= 4q.q^k.H^{(k)}(0) - H^{(k)}(-1) - 2H^{(k)}(0) = H^{(k)}(1).$$

Thus in this case the right and left derivatives are identical. In the case of p-times continuous differentiability we have to show that $f_2^{(p)}$ is continuous at the point 1. From the left: $H^{(p)}$ is continuous on the left at 1, and therefore also $f_2^{(p)}$. From the right: As $H^{(p)}$ is continuous on the right at 0 and -1, the definition of f_2 shows that $f_2^{(p)}$ is also continuous on the right at 1.

Case $n \geq 2$. The crucial point is that for $x < n$ and $x \geq n$ we have two different expressions defining the value of $f_{n+1}(x)$, but we can very well use the fact that any two of the functions (f_n) coincide on the intersection of their domains:

$n \leq x < n+1$: $f_{n+1}(x) = 4qf_n(q(x-1)) - f_n(x-2) - 2f_n(x-1)$;
$n-1 < x < n$: $f_{n+1}(x) = f_n(x) = 4qf_{n-1}(q(x-1)) - f_{n-1}(x-2) - 2f_{n-1}(x-1) = 4qf_n(q(x-1)) - f_n(x-2) - 2f_n(x-1)$.

As these two expressions are identical and f_n is p-times differentiable (resp. p-times continuously differentiable) on its domain, we immediately get that f_{n+1} is also p-times differentiable (resp. p-times continuously differentiable) at the point n.

Thus each of the functions f_n is p-times differentiable (resp. p-times continuously differentiable) and, therefore, the resulting function $F_1 : [-1, \infty[\to \mathbb{R}$ is p-times differentiable (resp. p-times continuously differentiable).

The next step is dealing with the functions F_n defined on $[-n, \infty[$ which are given by

$$F_{n+1}(x) := \begin{cases} F_n(x) & \text{for } x \in [-n, \infty[\\ 4qF_n(q(x+1)) - F_n(x+2) - 2F_n(x+1) & \text{otherwise.} \end{cases}$$

Like in the case of the functions f_n here we have to verify that F_{n+1} is p-times differentiable (resp. p-times continuously differentiable) at the point $-n$.

Case $n = 1$. Right side: $F_2^{(k)}(-1) = F_1^{(k)}(-1) = H^{(k)}(-1)$. Left side: $F_1(-1) = H(-1) = 4qH(0) - H(1) - 2H(0)$. For $-2 < x < -1$ we have

$$F_2(x) = 4qF_1(q(x+1)) - F_1(x+2) - 2F_1(x+1) = \\ = 4qH(q(x+1)) - H(x+2) - 2H(x+1).$$

Thus it is possible to compute the left derivatives

$$F_2^{(k)}(-1) = 4q.q^k.H^{(k)}(0) - H^{(k)}(1) - 2H^{(k)}(0) = H^{(k)}(-1).$$

As in the case of f_2 the continuity of $F_2^{(p)}$ at the point -1 in the case of p-times continuous differentiability immediately follows by the same arguments.

Case $n \geq 2$.

$-n-1 < x < -n$: $F_{n+1}(x) = 4qF_n(q(x+1)) - F_n(x+2) - 2F_n(x+1)$.

$-n \leq x < -n+1$: $F_{n+1}(x) = F_n(x) = 4qF_{n-1}(q(x+1)) - F_{n-1}(x+2) - 2F_{n-1}(x+1) = 4qF_n(q(x+1)) - F_n(x+2) - 2F_n(x+1)$.

Thus we have one expression for all arguments x such that $-n-1 < x < -n+1$, which is p-times differentiable (resp. p-times continuously differentiable) because F_n is supposed to be p-times differentiable (resp. p-times continuously differentiable).

Thus the solution defined by h is p-times differentiable (resp. p-times continuously differentiable). $\Diamond$

The question of differentiable solutions in the case that $S(f) \subseteq]-\infty, Q]$ is similar to handle. An answer in this case can be given, the proofs are very similar to the case of continuous solutions.

Theorem 13. *Let $h\colon]Q-1, qQ] \to \mathbb{R}$ be an arbitrary function, and α a real number which is arbitrary in the case $q = \frac{1}{4}$ and 0 otherwise, and let f be the unique solution which extends h and fulfills $f(Q) = \alpha$, $S(f) \subseteq]-\infty, Q]$. Furthermore, let r be a natural number. Then f is r-times differentiable (resp. r-times continuously differentiable) on the set $\mathbb{R} \setminus E(q)$ if and only if the function h fulfills the following conditions:*

(a) *$h\colon]Q-1, qQ] \to \mathbb{R}$ is continuous;*

(b) *h is continuously extendable to a function $H\colon [Q-1, qQ] \to \mathbb{R}$, where* (b1) *$H$ is r-times differentiable (resp. r-times continuously differentiable),*

(b2) *H fulfills the following conditions:*

$$\begin{cases} H(Q-1) = 4q.H(qQ) \\ H'(Q-1) = 4q^2.H'(qQ) \\ \quad \ldots\ldots \\ H^{(r)}(Q-1) = 4q^{r+1}.H^{(r)}(qQ). \end{cases}$$

Remark. As H is defined on the closed interval $[Q-1, qQ]$, differentiability is to be understood in the following way: At any point in the open interval $]Q-1, qQ[$ the derivative $H'(x)$ exists, at the point qQ the left derivative of H exists, and at the point $Q-1$ the right derivative of H exists. In the case of continuous differentiability this function H' has to be continuous on the whole interval $[Q-1, qQ]$ — and similarly for derivatives of higher order.

Proof. "only if": Suppose that f is r-times differentiable (resp. r-times continuously differentiable) on the set $\mathbb{R} \setminus E(q)$. As this set contains the interval $]Q-1, Q[$, we immediately get:

(a) $h = f\,|]Q-1, qQ]$ is continuous.

(b) $\lim\limits_{x \searrow Q-1} h(x) = \lim\limits_{x \searrow Q-1} f(x) = \lim\limits_{x \searrow Q-1} 4q.f(q(x+1)) = \lim\limits_{x \searrow qQ} 4q.f(x) = 4q.f(qQ) = 4q.h(qQ)$. Thus h can be extended continuously to a function h on $[Q-1, qQ]$ which fulfills $H(Q-1) = 4qH(qQ)$.

(b1) We have $H(x) = 4q.f(q(x+1))$ and $q(x+1) \in [qQ, q(qQ+1)] \subseteq]Q-1, Q[$ for each $x \in [Q-1, qQ]$. As f is r-times differentiable (resp. r-times continuously differentiable) on the interval $]Q-1, Q[$, it is evident that the same holds for H.

(b2) From the formula $H(x) = 4q.f(q(x+1))$ we immediately get $H'(Q-1) = 4q^2.f'(qQ) = 4q^2.H'(qQ)$, and by induction for any integer k, $1 \leq k \leq r$: $H^{(k)}(Q-1) = 4q^{k+1}.f^{(k)}(qQ) = 4q^{k+1}.H^{(k)}(qQ)$.

"if": Suppose that h fulfills conditions (a) and (b). Like in Th. 9 first we show that the solution f is r-times differentiable (resp. r-times continuously differentiable) on the interval $]Q-1, Q[$. Let (as in Th. 5 resp. Th. 9) $x_0 := Q-1$, $x_{n+1} := q(x_n+1)$. From Th. 9 it follows that f is continuous on the interval $]Q-1, Q[$, and from the construction it is evident that f is r-times differentiable (resp. r-times continuously differentiable) on each of the open intervals $]x_n, x_{n+1}[$ for $n \geq 0$. Furthermore, the conditions (b2) immediately show that the left derivatives (of order k, $1 \leq k \leq r$) at the point qQ coincide with the right derivatives at this point. If H is r-times continuously differentiable, the left and the right limits of $f^{(r)}(x)$ coincide with $H^{(r)}(qQ) = f^{(r)}(qQ)$ when x tends to qQ. Thus f has the desired properties on the interval $]x_0, x_2[$. Now we may proceed by induction: Suppose that f is r-times differentiable (resp. r-times continuously differentiable) on the interval $]x_0, x_n[$ $(n \geq 2)$. Then f is given on the interval $]x_1, x_{n+1}[$ by the formula $f(x) = \frac{1}{4q}.f\left(\frac{x}{q} - 1\right)$, where the right hand side uses arguments of the interval $]x_0, x_n[$. Thus f is r-times differentiable (resp. r-times continuously differentiable) on the interval $]x_1, x_{n+1}[$, and as the intersection of $]x_0, x_n[$ and $]x_1, x_{n+1}[$ is nonvoid, f has this property on the interval $]x_0, x_{n+1}[$. We may conclude that f is r-times differentiable (resp. r-times continuously differentiable) on the interval $]Q-1, Q[$ and, therefore, on the set $]Q-1, \infty[\backslash\{Q\}$. For further investigations on f we use the formula derived in Th. 10: Let be $x < 0$, let $k \in \mathbb{N}$ be such that $x + k > Q$, and let $m \in \mathbb{N}$ be such that $q^m x + q^m + \ldots + q > Q - 1$. Then

$$f(x) = (-4q)^m \cdot \sum_{l_1, \ldots, l_m = 1}^{k} (-1)^{l_1 + \ldots + l_m} . l_1 \ldots l_m . f(q^m x + l_m q^m + \ldots + l_1 q).$$

The right-hand-side expression is a finite sum of terms, where each argument depends continuously on x and is contained in the interval $]Q-1, \infty[$. Thus if none of these arguments is equal to Q, this property holds in a whole neighbourhood of x, and f is given in this neighbourhood as a finite sum of r-times differentiable (resp. r-times continuously differentiable) expressions. Therefore, f is r-times differentiable (resp. r-times continuously differentiable) in this neighborhood. On the other hand, from Prop. (1) we know that the set where at least one of these arguments in the right-hand-side expression is equal to Q is the set $E(q)$, which proves the statement that the solution f is r-times differentiable (resp. r-times continuously differentiable) on the set $\mathbb{R} \backslash E(q)$. $\Diamond$

The preceding theorem gives two possibilities to make the set of points where f is not r-times differentiable "small":

— f is r-times differentiable at Q or

— the sum of coefficients at $f(Q)$ (used in the proof of Th. 13 in the sum expression for $f(x)$) is equal to 0.

A precise answer will be given in the following two theorems.

Theorem 14. *Let h and α be as in* Th. 13 *and suppose that h fulfills conditions* (a) *and* (b). *Then the solution f is r-times differentiable (resp. r-times continuously differentiable) at the point Q, if and only if h is identically* 0 *and $\alpha = 0$, or q fulfills the condition $4q^{r+1} > 1$.*

Proof. We may assume that h is nonvanishing, and we give the proof by induction.

$r = 1$: First suppose that f is differentiable at Q. Then f is continuous at Q, and we have $\alpha = 0$ and $4q > 1$ by Th. 9. Let $z \in]Q-1, qQ]$ be arbitrary such that $h(z) \neq 0$. Then the sequence $z_0 := z$, $z_{n+1} := q(z_n + 1)$ tends to Q, and from the right derivative at Q we have $f'(Q) = 0$. Thus

$$0 = f'(Q) = \lim_{n\to\infty} \frac{f(z_n) - f(Q)}{z_n - Q}.$$

Now

$$f(z_{n+1}) - f(Q) = f(z_{n+1}) = f(q(z_n+1)) = \frac{1}{4q} f(z_n) = \frac{1}{4q}(f(z_n) - f(Q))$$

and

$$z_{n+1} - Q = q(z_n + 1) - q(Q+1) = q(z_n - Q),$$

which implies

$$\frac{f(z_n) - f(Q)}{z_n - Q} = \left(\frac{1}{4q^2}\right)^n \cdot \frac{f(z) - f(Q)}{z - Q}.$$

This sequence tends to 0 iff $4q^2 > 1$.

Now suppose that $4q^2 > 1$. Then $4q > 1$, which implies that f is continuous at Q. We have to show that the left derivative of f at Q is equal to 0 and — in the case of continuous differentiability — that $\lim_{x \nearrow Q} f'(x) = 0$. As in Th. 5, let $x_0 = Q-1$, $x_{n+1} = q(x_n+1)$, and let M be a bound for the function h on $]Q-1, qQ] =]x_0, x_1]$. (Such a bound exists because h is continuously extendable to the compact interval $[x_0, x_1]$.) Then f is bounded by $\left(\frac{1}{4q}\right)^n.M$ on the interval $]x_n, x_{n+1}]$. Now suppose that $(z_n)_{n\in\mathbb{N}}$ is an arbitrary, strictly increasing sequence tending to Q. Without loss of generality we may assume that $z_0 > Q - 1$. Then for each $n \in \mathbb{N}$ there is a unique $m \in \mathbb{N}$ such that $z_n \in]x_m, x_{m+1}]$. Let us denote this m by $m(n)$. Then $\lim_{n\to\infty} m(n) = \infty$. Furthermore,

$$|z_n - Q| \geq |x_{m(n)+1} - Q| = q^{m(n)}|x_1 - Q|,$$

and

$$|f(z_n) - f(Q)| = |f(z_n)| \leq \left(\frac{1}{4q}\right)^{m(n)} M.$$

Thus

$$\left|\frac{f(z_n) - f(Q)}{z_n - Q}\right| \leq \frac{\left(\frac{1}{4q}\right)^{m(n)} M}{q^{m(n)}|x_1 - Q|} = \left(\frac{1}{4q^2}\right)^{m(n)} \cdot \frac{M}{Q - x_1},$$

which goes to 0 when n tends to infinity. Further, if H is continuously differentiable then H' is bounded by some constant N on the interval $[x_0, x_1]$. As f fulfills the equation $f(x) = 4qf(q(x+1))$ in the interval $]x_0, Q[$ we immediately get $f'(x) = 4q^2 f'(q(x+1))$ in this interval. Thus f' is bounded by $\left(\frac{1}{4q^2}\right)^n N$ on the interval $[x_n, x_{n+1}]$. By the same arguments as before we can conclude that $\lim_{x \nearrow Q} f'(x) = 0$.

Now the step $r \to r+1$: Suppose that f is $(r+1)$-times differentiable in $]Q-1, \infty[$. Then f is r-times differentiable at Q, which implies $4q > 4q^2 > \ldots > 4q^{r+1} > 1$. As the function H fulfills $H^{(k)}(Q-1) = 4q^{k+1}H^{(k)}(qQ)$ for any k, $0 \leq k \leq r$, each of these functions $H^{(k)}$ is either identically 0 or nonconstant. Nonconstant differentiable functions have a nonvanishing derivative, and, therefore, if h is nonvanishing, the only possibility is that $H^{(r)}$ is nonconstant. Thus there is a point $z \in]x_0, x_1]$ such that $f^{(r)}(z) = H^{(r)}(z) \neq 0$. Once more we use the sequence $z_0 := z$, $z_{n+1} := q(z_n + 1)$. From $f(x) = 4qf(q(x+1))$ we derive $f^{(r)}(x) = 4q^{r+1}f^{(r)}(q(x+1))$, especially $f^{(r)}(z_n) = 4q^{r+1}f^{(r)}(z_{n+1})$, which implies

$$f^{(r)}(z_n) = \left(\frac{1}{4q^{r+1}}\right)^n \cdot f^{(r)}(z).$$

Comparing the right and left derivative of $f^{(r)}$ at the point Q, we get

$$0 = f^{(r+1)}(Q) = \lim_{n\to\infty} \frac{f^{(r)}(z_n) - f^{(r)}(Q)}{z_n - Q} = \lim_{n\to\infty} \left(\frac{1}{4q^{r+2}}\right)^n \cdot \frac{f^{(r)}(z)}{z - Q}.$$

As $f^{(r)}(z) \neq 0$, this limit is equal to 0 iff $4q^{r+2} > 1$. On the other hand, suppose that $4q^{r+2} > 1$. Then $4q^{r+1} > 1$, and as H is $(r+1)$-times differentiable in $[Q-1, qQ]$, the r-th derivative $H^{(r)}$ is continuous on $[Q-1, qQ]$ and therefore bounded by a constant M. With the same arguments as before we can conclude that $f^{(r)}$ is bounded by $\left(\frac{1}{4q^{r+1}}\right)^n M$ on the interval $]x_n, x_{n+1}]$. Now suppose that $(z_n)_{n\in\mathbb{N}}$ is an arbitrary strictly increasing sequence tending to Q. Like before, we may assume that $z_0 > Q - 1$, and denote by $m(n)$ the unique $m \in \mathbb{N}$ such that $z_n \in]x_m, x_{m+1}]$. Then

$$\left|\frac{f^{(r)}(z_n) - f^{(r)}(Q)}{z_n - Q}\right| \leq \frac{\left(\frac{1}{4q^{r+1}}\right)^{m(n)} M}{q^{m(n)}|x_1 - Q|} = \left(\frac{1}{4q^{r+2}}\right)^{m(n)} \cdot \frac{M}{Q - x_1}.$$

As $4q^{r+2} > 1$ we immediately get that $f^{(r+1)}(Q)$ exists and is equal to 0.

Further, if H is $(r+1)$-times continuously differentiable then $H^{(r+1)}$ is bounded by some constant N on the interval $[x_0, x_1]$. Like before we get the equation $f^{(r+1)}(x) = 4q^{r+2} f^{(r+1)}(q(x+1))$ in the interval $]x_0, Q[$. Thus $f^{(r+1)}$ is bounded by $\left(\frac{1}{4q^{r+2}}\right)^n N$ on the interval $[x_n, x_{n+1}]$. From this we can conclude that $\lim_{x \nearrow Q} f^{(r+1)}(x) = 0$. ◊

Corollary 1. *The only solution f of equation* (1) *which fulfills $S(f) \subseteq$ $\subseteq]-\infty, Q]$ and which is C^∞ on $\mathbb{R}$ is the zero function.*

Proof. The preceding theorem shows that for a nonvanishing C^∞-solution the inequality $4q^{r+1} > 1$ has to be fulfilled for any natural number r. But this is impossible because $0 < q < 1$. ◊

Corollary 2. *Let h and α be as in* Th. 13 *and suppose that h fulfills conditions* (a) *and* (b). *Then the solution f is r-times differentiable (resp. r-times continuously differentiable) on the whole real line, if and only if h is identically* 0 *and $\alpha = 0$, or q fulfills the condition $4q^{r+1} >$ > 1.*

Proof. First suppose that f is r-times differentiable on $\mathbb{R}$. Then f is r-times differentiable at Q, which implies (by Th. 14) that $4q^{r+1} > 1$. On the other hand, suppose that $4q^{r+1} > 1$. By Th. 14, f is r-times differentiable (resp. r-times continuously differentiable) in the interval $]Q - 1, \infty[$. Thus from the formula

$$f(x) = (-4q)^m . \sum_{l_1, \ldots, l_m = 1}^{k} (-1)^{l_1 + \ldots + l_m} . l_1 \ldots l_m . f(q^m x + l_m q^m + \ldots + l_1 q)$$

of Th. 13, where $x < Q$, $x + k > Q$, and m is chosen large enough such that the arguments on the right side are greater than $Q - 1$, we immediately get that f is r-times differentiable (resp. r-times continuously differentiable) on the whole real line. ◊

Theorem 15. *Let h and α be as in* Th. 13, *and suppose that h fulfills conditions* (a) *and* (b) *and $4q^{r+1} \leq 1$. Then the solution f is r-times* *differentiable (resp. r-times continuously differentiable) on the set $C(q)$, and not r-times differentiable at the points of the set $E(q) \setminus C(q)$.*

Proof. In Th. 13 it was proved that f is r-times differentiable (resp. r-times continuously differentiable) in the set $\mathbb{R} \setminus E(q)$. Now let $x < Q$, $x \in E(q)$, and choose m large enough such that $q^m x + q^m + \ldots + q > Q - 1$. Furthermore, choose an integer k such that $x + k > Q$ and $M(q,x,m) \subseteq \{1, \ldots, k\}^m$ as in Th. 10. As in the stated theorem, let $M := M(q,x,m)$, $P := \{1,2,\ldots,k\}^m \setminus M$. As in Th. 10, in a neighbourhood of x the formula

$$f(y) = (-4q)^m \sum_{l_1,\ldots,l_m=1}^{k} (-1)^{l_1+\ldots+l_m}.l_1\ldots l_m.f(q^m y + l_m q^m + \ldots + l_1 q) =$$

$$= (-4q)^m. \sum_{(l_1,\ldots,l_m)\in M} (-1)^{l_1+\ldots+l_m}.l_1\ldots l_m.f(q^m y + l_m q^m + \ldots + l_1 q) +$$

$$+(-4q)^m. \sum_{(l_1,\ldots,l_m)\in P} (-1)^{l_1+\ldots+l_m}.l_1\ldots l_m.f(q^m y + l_m q^m + \ldots + l_1 q) =$$

$$= (-4q)^m. \sum_{(l_1,\ldots,l_m)\in P} (-1)^{l_1+\ldots+l_m}.l_1\ldots l_m.f(q^m y + l_m q^m + \ldots + l_1 q) +$$

$$+(-4q)^m.f\big(q^m(y-x) + Q\big).S(q,x,m)$$

holds. The first sum gives f in this neighbourhood of x as a finite sum of r-times differentiable (resp. r-times continuously differentiable) terms, thus the second summand makes the decision whether f is r-times differentiable (resp. r-times continuously differentiable) at the point x.

Case 1: $x \in C(q)$. In this case $S(q,x,m) = 0$, thus f is r-times differentiable (resp. r-times continuously differentiable) at x.

Case 2: $x \notin C(q)$. In this case $S(q,x,m) \neq 0$. As f is not r-times differentiable at Q, it cannot be r-times differentiable at x. $\Diamond$

After these investigations on differentiable solutions we turn over to measurable and integrable solutions. However, one question has not been discussed because it is still unsolved: Do there exist analytic solutions (of course, only in the case $S(f) \subseteq \mathbb{R}$)?

d) Measurable solutions

In this section we deal with measurability in the sense of Borel or Lebesgue (i.e., the term "measurable" is to be understood in this sense). The results are very simple:

Theorem 16 (the case $S(f) \subseteq \mathbb{R}$). *Let $h\colon [-1,1[\to \mathbb{R}$ be an arbitrary function, and let f be the unique solution of equation* (1) *which coincides with h on $[-1,1[$. Then f is measurable if and only if h is measurable.*

Proof. "only if" is obvious. "if": Suppose that h is measurable. Then from the construction given in Th. 3 and from the σ-additivity of the measure it follows immediately that f is measurable. $\Diamond$

A similar result holds for the case $S(f) \subseteq]-\infty, Q]$:

Theorem 17 (the case $S(f) \subseteq]-\infty, Q]$). *Let $h\colon]Q-1, qQ] \to \mathbb{R}$ be an arbitrary function and α a real number which is arbitrary in the case $q = \frac{1}{4}$ and* 0 *otherwise, and let f be the unique solution of equation* (1) *which coincides with h on $]Q-1, qQ]$ and fulfills $f(Q) = \alpha$, $S(f) \subseteq]-\infty, Q]$. Then f is measurable if and only if h is measurable.*

Proof. Like that of Th. 16; the construction of f from h has been given in Th. 5. $\Diamond$

More interesting than these "trivial results" on measurable solutions are the following about

e) Integrable solutions

It will be shown that the vector space of integrable solutions for a given number q is at most of dimension 1 over the field of reals. Furthermore, the very interesting result is that any integrable solution has bounded support, in other words, for any integrable solution $S(f) \subseteq [-Q, Q]$ holds. Thus the result on the dimension of this space of solutions follows immediately from the theorem of Baron and Volkmann [1]. We prepare the results by a lemma:

Lemma 3. (α) *Let f be a Lebesgue (resp. Borel)-integrable solution of equation* (1). *Then the function $F(x) := \int_{]-\infty,x]} f\,d\lambda$ (λ represents the usual Borel resp. Lebesgue measure) is well defined and has the properties:*

(i) *F is continuous,*

(ii) $\lim_{x\to-\infty} F(x) = 0$,

(iii) $\lim_{x\to\infty} F(x) = \int_{\mathbb{R}} f\,d\lambda \in \mathbb{R}$,

(iv) $F(qx) = \frac{1}{4}(F(x+1) + F(x-1) + 2F(x))$ *for any $x \in \mathbb{R}$.*

(β) *Let f be a solution of equation* (1) *whose improper Riemann integral over* $\mathbb{R}$ *exists. Then the function* $F(x) := \int_{-\infty}^{x} f(t)\,dt$ *is well defined and has the properties:*

(i) *F is continuous,*

(ii) $\lim_{x\to-\infty} F(x) = 0$,

(iii) $\lim_{x\to\infty} F(x) = \int_{-\infty}^{\infty} f(t)\,dt \in \mathbb{R}$,

(iv) $F(qx) = \frac{1}{4}(F(x+1) + F(x-1) + 2F(x))$ *for any* $x \in \mathbb{R}$.

Proof. (α): (i), (ii), (iii) are well-known from elementary integration theory (e.g., cf. Hewitt–Stromberg [3]). (iv) can be derived as follows:

$$\frac{1}{4}(F(x+1) + F(x-1) + 2F(x)) =$$

$$= \frac{1}{4}\Big(\int_{]-\infty,x+1]} f\,d\lambda + \int_{]-\infty,x-1]} f\,d\lambda + 2\int_{]-\infty,x]} f\,d\lambda\Big) =$$

$$= \frac{1}{4}\Big(\int_{]-\infty,x]} f(\xi+1)\,d\lambda(\xi) + \int_{]-\infty,x]} f(\xi-1)\,d\lambda(\xi) + 2\int_{]-\infty,x]} f(\xi)\,d\lambda(\xi)\Big) =$$

$$= \frac{1}{4}\int_{]-\infty,x]} (f(\xi+1) + f(\xi-1) + 2f(\xi))d\lambda(\xi) =$$

$$= \frac{1}{4}\int_{]-\infty,x]} 4qf(q\xi)\,d\lambda(\xi) = \int_{]-\infty,qx]} f(\xi)\,d\lambda(\xi) = F(qx).$$

(β): (i) is well-known from elementary analysis, (ii) and (iii) are immediate consequences of the definition of the Riemann integral from $-\infty$ to ∞. (iv) can be computed similarly to the case (α):

$$F(qx) = \int_{-\infty}^{qx} f(t)\,dt = \int_{-\infty}^{x} f(qt)\,q dt =$$

$$= q\int_{-\infty}^{x} \frac{1}{4q}(f(t+1) + f(t-1) + 2f(t))dt =$$

$$= \frac{1}{4}\Big(\int_{-\infty}^{x+1} f(t)\,dt + \int_{-\infty}^{x-1} f(t)\,dt + 2\int_{-\infty}^{x} f(t)\,dt\Big) =$$

$$= \frac{1}{4}(F(x+1) + F(x-1) + 2F(x)). \quad \Diamond$$

Theorem 18. *Let f be a solution of equation* (1) *which is Riemann (resp. Borel- resp. Lebesgue-) integrable and let the value of this integral* *be* 0. *Then f vanishes almost everywhere (in the Borel resp. Lebesgue case) resp. f is equal to* 0 *except on a zero set (in the Riemann case).*

Proof. We use the function F defined in Lemma 3. Then it follows that F fulfills the following conditions:

(a) F is continuous,

(b) $\lim_{x\to-\infty} F(x) = 0 = \lim_{x\to\infty} F(x)$,

(c) $F(qx) = \frac{1}{4}(F(x+1) + F(x-1) + 2F(x))$ for any $x \in \mathbb{R}$.

As F is continuous and tends to 0 when x tends to $\pm\infty$, it has a maximum value M at some point x_0. Now let $x_0 = qy_0$. Then

$$M = F(x_0) = F(qy_0) = \frac{1}{4}(F(y_0+1) + F(y_0-1) + 2F(y_0)).$$

As M is maximal, this equality can only hold if

$$F(y_0+1) = F(y_0-1) = F(y_0) = F(qy_0) = F(x_0) = M.$$

Case 1: $x_0 \neq 0$. Define a sequence (x_n) by $x_n =: qx_{n+1}$. Then $y_0 = x_1$, and repeating the above argument by induction, we get that the sequence $(F(x_n))$ is constant with value M. As $\lim_{n\to\infty} x_n = \pm\infty$ we immediately get that $M = 0$.

Case 2: $x_0 = 0$. The computation given above shows that $F(1) = M$, and we may proceed with the value $x_0 = 1$ like in Case 1.

Thus in any case we get $M = 0$. The same arguments show that also the minimum of F must be equal to 0, and therefore F vanishes identically. As a trivial consequence the assertion of the theorem holds. ◊

Corollary 3. *The set of Riemann integrable solutions of equation* (1) *as well as the set of Lebesgue (resp. Borel) integrable solutions is at most of dimension* 1.

Proof. Integration is a linear mapping from the set of all integrable solutions into the one-dimensional space $\mathbb{R}$. By the preceding theorem, the kernel of this mapping contains only the zero function. Thus the dimension of the space of integrable solutions cannot exceed 1. ◊

Theorem 19. *Let f be an integrable solution of equation* (1) *(in the Riemann or Borel resp. Lebesgue sense). Then f vanishes almost everywhere outside the interval* $[-Q, Q]$.

Proof. We use the function F of Lemma 3: Let G be the value of the integral of f, i.e., $\lim_{x\to\infty} F(x) = G \in \mathbb{R}$. Now let $\varepsilon > 0$ be arbitrarily chosen. There exists a number z such that $|F(x) - G| < \varepsilon$ for any $x > z$. Without loss of generality we may assume that $z > Q$. Thus for any $x > z + 1$ we have that $F(x-1), F(x), F(x+1) < G + \varepsilon$ and, similarly, $F(x-1), F(x), F(x+1) > G - \varepsilon$, which implies that $G - \varepsilon < F(qx) < G + \varepsilon$. In other words, the inequality $|f(x) - G| < \varepsilon$ holds for any $x > q(z+1)$. Repeating this argument, we immediately get $|F(x) - G| < \varepsilon$ for any $x > Q$, because the sequence z, $q(z+1)$, $q(q(z+1)+1)$, $\ldots$ tends to Q. As ε was chosen arbitrarily, we may conclude that $F(x) = G$ for any $x > Q$. Similarly, using the same arguments in the other direction, we conclude that $F(x) = 0$ for any $x < -Q$. As a trivial consequence, f must be 0 a.e. (resp. except on a zero set outside the interval $[-Q, Q]$. $\Diamond$

After these results on solutions with unbounded support we make a short break. A paper on solutions with bounded support will follow.

References

[1] BARON, K. and VOLKMANN, P.: Unicité pour une équation fonctionelle, *Rocnik Naukowo-Dydaktyczny WSP w Krakowie* **159** (1993), 53–56.

[2] GRZASLEWICZ, A.: The general solution of the generalized Schilling's equation, *Aequ. Math.* **44** (1992), 317–326.

[3] HEWITT, E. and STROMBERG, K.: Real and Abstract Analysis, Springer Verlag, Berlin – Heidelberg – New York, 1975.

[4] SCHILLING, R.: Spatially-chaotic Structures, in: H. Thomas (ed.), Nonlinear Dynamics in Solids, Springer Verlag, Berlin – Heidelberg – New York, 1992.