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SUCCESSIVE APPROXIMATION METHOD FOR INVESTIGATING THREE POINT BOUNDARY VALUE PROBLEM WITH SINGULAR MATRICES

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Abstract: In this paper the existence of the solution of a three-point boundary value problem belonging to a system of nonlinear differential equations

$$\frac{dx}{dt} = f(t,x), \quad x, f \in \mathbb{R}^n, \quad Ax(0) + A_1x(t_1) + Cx(T) = d$$

is investigated by using a new version of the numerical-analytic methods. The approximate solution is determined and an estimation for the error is given.

1. Introduction

In the literature different numerical, analytic and functional-analytic methods are known to investigate both the two- and the n-point boundary value problems depending on the type of the equation and the boundary value condition ([1], [4], [5]).

When the existence of the solution can be supposed some numerical methods aim mainly at the approximate determination of the solution ([2], [6]).

The analytic methods (i.e. those of the continuous closed form) based mostly upon various series expansions are generally used for qualitative investigations (uniqueness, stability) ([3], [10], [11], [12]).

When using functional analytic methods the given boundary value problems are substituted by a suitably chosen equivalent operator equation ([4], [12]). For certain three-point boundary value problems — see papers ([7], [8]) — this operator equation is an integral equation, which is set up by using a suitably chosen Green-function. These integral equations are generally investigated by using contraction and fix-point theorems.

The so-called numerical-analytic methods which have been developed in the last some years [9] give the opportunity of investigating the two most important approach for solving the boundary value problems — the existence and the approximate determination of the solution — simultaneously.

These methods are fairly widely used (see monograph [9]), mostly for handling periodic or two-point nonlinear boundary value problems.

When the boundary value problems are of more general nature (three- or n-point b.v. problems) and, in addition, even degenerate matrices are contained the evaluation and the mathematical foundation of numerical analytic methods based on successive approximations are facing several difficulties. In this connection we mention the determination of the successive approximation satisfying the boundary conditions and the proof of the uniform convergence, the determination of the necessary and sufficient conditions ensuring the existence based upon the features of the approximate solutions.

In this paper both the existence of the solution and the approximate solution of a three-point boundary value problem belonging to a system of nonlinear differential equations are investigated by using a numerical-analytic method. It is worth mentioning that the earlier versions of the numerical-analytic methods are not suitable for solving our problem due to the singularity of the matrices in the boundary value conditions.

Let a nonlinear differential equation be given

$$\dot{x} = f(t, x), \qquad x, f \in \mathbb{R}^n, \qquad t \in [0, T], \tag{1}$$

with a three-point linear boundary value condition

$$Ax(0) + A_1 x(t_1) + Cx(T) = d \tag{2}$$

where x, f, d are points of the n-dimensional Euclidean space $\mathbb{R}^n$ while

A, A_1, C are constant matrices of type $n \times n$ and $t_1 \in (0, T)$. Matrices A, A_1, C are allowed to be singular, but it is supposed that there exist constants k_1 and k_2 ($k_1 \neq k_2$) satisfying

$$\det\Big[k_1 A + k_2 A_1 + \Big[k_1 + \frac{T}{t_1}(k_2 - k_1)\Big] C\Big] \neq 0. \tag{3}$$

It will be shown that a $\{x_m(t, x_0)\}$ sequence of functions depending on the parameter x_0 can be constructed on the set of continuous functions satisfying the boundary value conditions (2) such that for certain value of the parameter x_0 the sequence of functions uniformly converges and its limit is the solution of the nonlinear boundary value problem (1), (2). The existence of the solution for the underlying problem is proved by using the properties of the approximate solution. An estimation for the error of the approximate solution is given.

2. Construction of successive approximations

Let $\mathcal{D}$ denote a closed, connected domain in $\mathbb{R}^n$. Let us suppose that the domain of definition of the right hand side function $f(t, x)$ in Eq. (1) fulfills

$$(t, x) \in [0, T] \times \mathcal{D} \tag{4}$$

and the following conditions hold

(i) $f(t, x)$ is continuous in its domain of definition (4);

(ii) $f(t, x)$ is bounded by the vector M

$$|f(t, x)| \leq M, \quad (t, x) \in [0, T] \times \mathcal{D},$$

and $f(t, x)$ satisfies the Lipschitz-condition in the variable x with matrix K:

$$|f(t, x') - f(t, x'')| \leq K|x' - x''|, \tag{5}$$

where the vector $|f(t, x)|$ is

$$|f(t, x)| = \big(|f_1(t, x)|, \dots, |f_n(t, x)|\big),$$

and both the vector $M = (M_1, M_2, \dots, M_n)$ and the matrix $K = \{K_{ij},\ i, j = 1, \dots, n\}$ contain only non-negative constant elements.

In relations $|f(t, x)| \leq M$ and (5) the inequalities are meant componentwise. Those boundary value problems (1), (2) will be investigated, for which the parameters M, K, A, A_1, C, d, k_1, k_2 and the domain of definition (4) satisfy condition (3) and the following conditions:

1. The set $\mathcal{D}_\beta$, the collection of those points $x_0 \in \mathbb{R}^n$ belonging — together with their β-neighbourhood — to the set $\mathcal{D}$, is non-empty

$$\mathcal{D}_\beta \neq \emptyset, \tag{6}$$

where $\beta(x_0) = \frac{T}{2}M + \beta_1(x_0)$,

$$\beta_1(x_0) = \Big[|k_1| + \Big|\frac{(k_2-k_1)T}{t_1}\Big|\Big]\Big[|H(d-(A+A_1+C)x_0)| + \frac{T}{2}|HA_1|M\Big],$$

where $H = D^{-1}$, $D = k_1A + k_2A_1 + \big[k_1 + \frac{T}{t_1}(k_2-k_1)\big]C$. (The β-neighbourhood of the point x_0 is the following $\{x\colon x \in \mathbb{R}^n, |x - x_0| \leq \leq \beta\}$.)

2. The highest eigenvalue $\lambda(Q)$ for the matrix $Q = \frac{T}{2}(K+G)$ is less than unity

$$\lambda(Q) < 1, \tag{7}$$

where

$$G = \Big[|k_1| + \Big|\frac{(k_2-k_1)T}{t_1}\Big|\Big]|HA_1|K.$$

A sequence of functions $\{x_m(t,x_0)\}$ whose elements satisfy the boundary value conditions (2) in every point $x_0 \in \mathcal{D}$ is constructed.

Let us consider the functions determined by the following formula:

$$\begin{aligned} x_m(t,x_0) = x_0 + &\int_0^t \Big[f(t, x_{m-1}(t,x_0)) - \frac{1}{T}\int_0^T f(s, x_{m-1}(s,x_0))ds\Big]dt + \\ &+\alpha\Big[k_1T + \frac{T}{t_1}(k_2-k_1)t\Big],\ m = 1,2,\dots\ ;\ x_0(t,x_0) = x_0, \end{aligned} \tag{8}$$

where $x_0 = (x_{01},\dots,x_{0n})$ is a parameter of dimension n, $\alpha = (\alpha_1,\dots,\alpha_n)$ is an unknown vector chosen such that the functions (8) satisfy the boundary value conditions (2) for every point $x_0 \in \mathcal{D}_\beta$. Substituting the functions (8) into the boundary value conditions (2) the following system of linear algebraic equations is obtained

$$D\alpha = \frac{1}{T}d(x_0, x_{m-1}), \tag{9}$$

where

$$\begin{aligned} d(x_0, x_{m-1}) &= d - (A + A_1 + C)x_0 - \\ -A_1\int_0^{t_1}&\Big[f\big(t, x_{m-1}(t,x_0)\big) - \frac{1}{T}\int_0^T f\big(s, x_{m-1}(s,x_0)\big)ds\Big]dt. \end{aligned} \tag{10}$$

From (9) we get $\alpha = \frac{1}{T} H d(x_0, x_{m-1})$, and from (8) we obtain the sequence of functions we wanted to get

$$
\begin{aligned}
& x_m(t, x_0) = z_0(x_0, x_{m-1}) + \\
(11) \quad & + \int_0^t \left[f(t, x_{m-1}(t, x_0)) - \frac{1}{T} \int_0^T f\big(s, x_{m-1}(s, x_0)\big) ds \right] dt + \\
& + \frac{t}{t_1}(k_2 - k_1) H d(x_0, x_{m-1}), \\
& x_0(t, x_0) = x_0, \quad m = 1, 2, \dots
\end{aligned}
$$

where $z_0(x_0, x_{m-1}) = x_0 + k_1 H d(x_0, x_{m-1})$, and $d(x_0, x_{m-1})$ can be expressed from (10).

The convergence of the above constructed functions is stated in

Theorem 1. *Let the function* $f(t, x)$ *in* Eq. (1) *be continuous in the domain* (4) *and satisfy the conditions* (5). *Furthermore, if the parameters of the boundary value problem* (1), (2) *satisfy the conditions* (6), (7) *then*

(i) *the functions of sequence* (11) *satisfy the boundary value conditions* (2) *for each* $x_0 \in \mathcal{D}_\beta$;

(ii) $\lim\limits_{m \to \infty} x_m(t, x_0) = x^*(t, x_0)$, *where the limit function is a solution for the integral equation*

$$
\begin{aligned}
& x(t) = z_0(x_0, x) + \\
(12) \quad & + \int_0^t \left[f\big(t, x(t)\big) - \frac{1}{T} \int_0^T f\big(s, x(s)\big) ds + \frac{1}{t_1}(k_2 - k_1) H d(x_0, x) \right] dt,
\end{aligned}
$$

where

$$
\begin{aligned}
z_0(x_0, x) = x_0 + k_1 H \Bigg[& d - (A + A_1 + C) x_0 - \\
& - A_1 \int_0^{t_1} \left[f(t, x(t)) - \frac{1}{T} \int_0^T f(s, x(s)) ds \right] dt \Bigg];
\end{aligned}
$$

(iii) $x^*(0, x_0) = z_0\big(x_0, x^*(t, x_0)\big)$ *and the limit function* $x^*(t, x_0)$ *satisfies the boundary value conditions* (2) *i.e.* x^* *is a solution for the perturbed boundary value problem*

$$
(13) \quad \begin{aligned}
\dot{x} &= f(t, x) + \Delta(x_0) \\
A x(0) + A_1 x(t_1) + C x(T) &= d,
\end{aligned}
$$

where

$$\Delta(x_0)=\frac{1}{t_1}(k_2-k_1)Hd\big(x_0,x^*(t,x_0)\big)-\frac{1}{T}\int_0^T f\big(t,x^*(t,x_0)\big)dt,$$

$$d\big(x_0,x^*(t,x_0)\big)=d-(A+A_1+C)x_0-$$
$$-A_1\int_0^{t_1}\left[f\big(t,x^*(t,x_0)\big)-\frac{1}{T}\int_0^T f\big(s,x^*(s,x_0)\big)ds\right]dt;$$

(iv) *the deviation of functions* $x^*(t,x_0)$ *and* $x_m(t,x_0)$ *is governed by the inequality*

$$|x^*(t,x_0)-x_m(t,x_0)|\le Q^m(E-Q)^{-1}\beta(x_0). \tag{14}$$

Proof. It will be shown that in the space $C(0,T)$ of continuous vector functions the sequence of functions given by (11) is a Cauchy-sequence and therefore it is uniformly convergent. First we prove that $x_0\in\mathcal{D}_\beta$ implies $x_m(t,x_0)\in\mathcal{D}$ for each element of the sequence.

From (11) we get

$$|x_1(t,x_0)-x_0|\le\left|\int_0^t\left[f\big(t,x_0(t,x_0)\big)-\frac{1}{T}\int_0^T f\big(s,x_0(s,x_0)\big)ds\right]dt\right|+$$
$$+\left|\frac{T}{t_1}(k_2-k_1)\right||Hd(x_0,x_0)|+|k_1||Hd(x_0,x_0)|.$$

Using Lemma 2.1 in [9, p. 31] it is obvious that if $f(t)\in C[0,T]$, then

$$\left|\int_0^t\left[f(t)-\frac{1}{T}\int_0^T f(s)ds\right]dt\right|\le\left(1-\frac{t}{T}\right)\int_0^t|f(s)|ds+\frac{t}{T}\int_t^T|f(s)|ds\le$$
$$\le\alpha_1(t)\max_{t\in[0,T]}|f(t)|,$$

where $\alpha_1(t)=2t\big(1-\frac{t}{T}\big)$, $|\alpha_1(t)|\le\frac{T}{2}$. Thus

$$|x_1(t,x_0)-x_0|\le\alpha_1(t)M+\left[\left|\frac{T}{t_1}(k_2-k_1)\right|+|k_1|\right]|Hd(x_0,x_0)|.$$

Furthermore, from (10)

$$|Hd(x_0,x_0)|\le$$

$$\le\big|H\big(d-(A+A_1+C)x_0\big)\big|+\left|HA_1\int_0^{t_1}\left[f(t,x_0)-\frac{1}{T}\int_0^T f(s,x_0)ds\right]dt\right|\le$$

$$\le\big|H\big(d-(A+A_1+C)x_0\big)\big|+|HA_1|\left|\int_0^{t_1}\left[f(t,x_0)-\frac{1}{T}\int_0^T f(s,x_0)ds\right]dt\right|,$$

$$|Hd(x_0, x_0)| \leq \Big[\big|H\big(d - (A + A_1 + C)x_0\big)\big| + \frac{T}{2}|HA_1|M\Big]$$

therefore

$$|x_1(t, x_0) - x_0| \leq \frac{T}{2}M + \beta_1(x_0), \tag{15}$$

and $x_1(t, x_0) \in \mathcal{D}$ when $x_0 \in \mathcal{D}_\beta$.

In a similar way, using induction we obtain

$$|x_m(t, x_0) - x_0| \leq \frac{T}{2}M + \beta_1(x_0),$$

that is $x_m(t, x_0) \in \mathcal{D}$, when $x_0 \in \mathcal{D}_\beta$.

We prove that $\{x_m(t, x_0)\}$ is really a Cauchy-sequence. Let us consider the following difference:

$$x_2(t,x_0) - x_1(t,x_0) = \int_0^t \Big[f\big(t, x_1(t,x_0)\big) - \frac{1}{T}\int_0^T f\big(s, x_1(s,x_0)\big)ds\Big]dt-$$

$$-\int_0^t \Big[f\big(t, x_0(t,x_0)\big) - \frac{1}{T}\int_0^T f\big(s, x_0(s,x_0)\big)ds\Big]dt+$$

$$+\frac{t}{t_1}(k_2 - k_1)H\Big[-A_1\int_0^{t_1}\Big[f\big(t, x_1(t,x_0)\big) - \frac{1}{T}\int_0^T f\big(s, x_1(s,x_0)\big)ds\Big]dt+$$

$$+A_1\int_0^{t_1}\Big[f\big(t, x_0(t,x_0)\big) - \frac{1}{T}\int_0^T f\big(s, x_0(s,x_0)\big)ds\Big]dt\Big]+$$

$$+k_1H\big[d(x_0, x_1) - d(x_0, x_0)\big].$$

Rearranging and using Lemma 2.1 of [9, p. 31]

$$|x_2(t,x_0) - x_1(t,x_0)| \leq \Big(1 - \frac{t}{T}\Big)\int_0^t \big|f\big(t, x_1(t,x_0)\big) - f\big(t, x_0(t,x_0)\big)\big|dt+$$

$$+\frac{t}{T}\int_t^T \big|f\big(t, x_1(t,x_0)\big) - f\big(t, x_0(t,x_0)\big)\big|dt + \Big[|k_1| + \Big|\frac{(k_2 - k_1)T}{t_1}\Big|\Big]|HA_1|\cdot$$

$$\cdot\Big[\Big(1 - \frac{t_1}{T}\Big)\int_0^{t_1}\big|f\big(t, x_1(t,x_0)\big) - f\big(t, x_0(t,x_0)\big)\big|dt+$$

$$+\frac{t_1}{T}\int_{t_1}^T \big|f\big(t, x_1(t,x_0)\big) - f\big(t, x_0(t,x_0)\big)\big|dt\Big] \leq \Big[|k_1| + \Big|\frac{(k_2 - k_1)T}{t_1}\Big|\Big]|HA_1|K\cdot$$

$$\cdot\Big[\Big(1-\frac{t_1}{T}\Big)\int_0^{t_1}(\alpha_1(t)M+\beta_1(x_0))dt+\frac{t_1}{T}\int_{t_1}^{T}(\alpha_1(t)M+\beta_1(x_0))dt\Big]+$$

$$+K\Big[\Big(1-\frac{t}{T}\Big)\int_0^{t}(\alpha_1(t)M+\beta_1(x_0))dt+\frac{t}{T}\int_{t}^{T}(\alpha_1(t)M+\beta_1(x_0))dt\Big].$$

Applying Lemma 2.2 of [9, p. 31] we get

$$|x_2(t,x_0)-x_1(t,x_0)|\leq K\big[\alpha_2(t)M+\alpha_1(t)\beta_1(x_0)\big]+$$

$$+G\big[\alpha_2(t_1)M+\alpha_1(t_1)\beta_1(x_0)\big],$$

where

$$\alpha_2(t)\leq\frac{T}{3}\alpha_1(t)\quad\text{and}\quad\alpha_1(t)\leq\frac{T}{2},$$

consequently

$$|x_2(t,x_0)-x_1(t,x_0)|\leq K\Big[\frac{T}{3}M+\beta_1(x_0)\Big]\alpha_1(t)+G\Big[\frac{T}{3}M+\beta_1(x_0)\Big]\alpha_1(t_1),$$

thus

$$|x_2(t,x_0)-x_1(t,x_0)|\leq Q\beta(x_0), \tag{16}$$

with

$$Q=\frac{T}{2}[K+G],\quad \beta(x_0)=\frac{T}{3}M+\beta_1(x_0).$$

Equality (11) immediately gives

$$x_{m+1}(t,x_0)-x_m(t,x_0)=-\Big[k_1+\frac{t}{t_1}(k_2-k_1)\Big]HA_1\int_0^{t_1}\Big[\big[f\big(t,x_m(t,x_0)\big)-$$

$$-f\big(t,x_{m-1}(t,x_0)\big)\big]-\frac{1}{T}\int_0^T\big[f\big(s,x_m(s,x_0)\big)-f\big(s,x_{m-1}(s,x_0)\big)\big]ds\Big]dt+$$

$$+\int_0^t\Big[\big[f\big(t,x_m(t,x_0)\big)-f\big(t,x_{m-1}(t,x_0)\big)\big]-\frac{1}{T}\int_0^T\big[f\big(s,x_m(s,x_0)\big)-$$

$$-f\big(s,x_{m-1}(s,x_0)\big)\big]ds\Big]dt$$

and

$$\big|x_{m+1}(t,x_0)-x_m(t,x_0)\big|\leq G\Big[\Big(1-\frac{t_1}{T}\Big)\int_0^{t_1}\big|x_m(t,x_0)-x_{m-1}(t,x_0)\big|dt+$$

$$
(17)\qquad \begin{aligned}
&+\frac{t_1}{T}\int_{t_1}^{T}\left|x_m(t,x_0)-x_{m-1}(t,x_0)\right|dt\Bigg]+\\
&+K\left[\left(1-\frac{t}{T}\right)\int_0^t\left|x_m(t,x_0)-x_{m-1}(t,x_0)\right|dt+\right.\\
&\left.+\frac{t}{T}\int_t^T\left|x_m(t,x_0)-x_{m-1}(t,x_0)\right|dt\right].
\end{aligned}
$$

Using induction and (15) and (16) it can be shown that

$$
(18)\qquad \left|x_{m+1}(t,x_0)-x_m(t,x_0)\right|\le Q^m\beta(x_0),
$$

supposing the validity of inequality

$$
(19)\qquad \left|x_m(t,x_0)-x_{m-1}(t,x_0)\right|\le Q^{m-1}\beta(x_0).
$$

In fact, using inequalities (17) and (19) we get

$$
\begin{aligned}
&\left|x_{m+1}(t,x_0)-x_m(t,x_0)\right|\le G\left[\left(1-\frac{t_1}{T}\right)\int_0^{t_1}Q^{m-1}\beta(x_0)dt+\right.\\
&\left.+\frac{t_1}{T}\int_{t_1}^{T}Q^{m-1}\beta(x_0)dt\right]+K\left[\left(1-\frac{t}{T}\right)\int_0^{t}Q^{m-1}\beta(x_0)dt+\right.\\
&\left.+\frac{t}{T}\int_t^T Q^{m-1}\beta(x_0)dt\right]=GQ^{m-1}\beta(x_0)\left[\left(1-\frac{t_1}{T}\right)\int_0^{t_1}dt+\frac{t_1}{T}\int_{t_1}^{T}dt\right]+\\
&+KQ^{m-1}\beta(x_0)\left[\left(1-\frac{t}{T}\right)\int_0^{t}dt+\frac{t}{T}\int_t^T dt\right]=\\
&=Q^{m-1}\beta(x_0)\left[G\alpha_1(t_1)+K\alpha_1(t)\right]\le Q^{m-1}\beta(x_0)\left[\frac{T}{2}(G+K)\right]=Q^m\beta(x_0).
\end{aligned}
$$

Introducing the notation

$$
r_{m+1}(t)=\left|x_{m+1}(t,x_0)-x_m(t,x_0)\right|
$$

and using (18),

$$
(20)\qquad \left|x_{m+j}(t,x_0)-x_m(t,x_0)\right|\le\sum_{i=1}^{j}r_{m+i}(t)\le Q^m\sum_{i=0}^{j-1}Q^i\beta(x_0).
$$

From (7) we get

$$
\sum_{i=0}^{j-1}Q^i\le\sum_{i=0}^{\infty}Q^i=(E-Q)^{-1},\qquad \lim_{m\to\infty}Q^m\to 0,
$$

where E is the unit matrix. Hence from (20) one can easily get that

$\{x_m(t,x_0)\}$ is a Cauchy-sequence, therefore it uniformly converges to a continuous limit function $x^*(t,x_0)$:

$$\lim_{m\to\infty} x_m(t,x_0) = x^*(t,x_0).$$

It is evident that taking the limit $(m \to \infty)$ in (11) the limit function $x^*(t,x_0)$ is a solution of the integral equation (12). Since all the elements of sequences (11) satisfy the boundary-value conditions (2), therefore so does the limit function too.

From (12) it is easily seen that $x^*(0,x_0) = z_0(x_0, x^*(t,x_0))$ and $x^*(t,x_0)$ is a solution of the perturbed boundary-value problem (13), which is equivalent to the integral equation (12).

It is easy to see that taking the $(j \to \infty)$ limit in (20) the inequality (14) holds for the deviation of the limit function from its m^{th} iteration. $\Diamond$

3. Some properties of the limit function

It is demonstrated how the right-hand side of the system of differential equations can be modified in such a way that the solution of the Cauchy-problem belonging to the newly constructed equation satisfies the given boundary value condition.

Theorem 2. *If the conditions of* Th. 1 *are satisfied then in an arbitrary point* $x_0 \in \mathcal{D}_\beta$ *a unique regulating parameter* $\mu = (\mu_1, \dots, \mu_n)$ *of the form*

$$(21) \qquad \mu = \frac{1}{t_1}(k_2 - k_1)Hd(x_0, x^*(t,x_0)) - \frac{1}{T}\int_0^T f(t, x^*(t,x_0))dt,$$

can be constructed, where $x^*(t,x_0)$ *is the limit function of the sequence of functions* $\{x_m(t,x_0)\}$ *given by* (11). *Under these conditions the solution* $x = x(t) = x^*(t,x_0)$ *of the Cauchy-problem*

$$(22) \qquad \dot{x} = f(t,x) + \mu, \qquad x(0) = z_0(x_0)$$

$$z_0(x_0) = z_0\big(x_0, x^*(t,x_0)\big) = x_0 + k_1 H\Bigg[d - (A + A_1 + C)x_0 -$$

$$-A_1 \int_0^{t_1} \left[f\big(t, x^*(t,x_0)\big) - \frac{1}{T}\int_0^T f\big(s, x^*(s,x_0)\big)ds \right] dt\Bigg]$$

satisfies the boundary value conditions (2) *i.e. it is a solution of the perturbed boundary value problem* (13) *with* $\Delta(x_0) = \mu$.

Proof. Th. 1 implies that the function $x(t) = x^*(t, x_0)$ is a solution for both the integral equation (12) and the Cauchy-problem

$$\dot{x} = f(t,x) + \frac{1}{t_1}(k_2 - k_1)Hd\big(x, x^*(t,x_0)\big) - \\ -\frac{1}{T}\int_0^T f\big(t, x^*(t,x_0)\big)dt,$$

(23)

$$x(0) = z_0(x_0) = z_0\big(x_0, x^*(t,x_0)\big) = x_0 + k_1 H\Big[d - (A + A_1 + C)x_0 - \\ -A_1\int_0^{t_1}\Big[f\big(t, x^*(t,x_0)\big) - \frac{1}{T}\int_0^T f\big(s, x^*(s,x_0)\big)ds\Big]dt\Big]$$

and, in addition, $x^*(t, x_0)$ satisfies the boundary value conditions (2). This means that we have found the parameter μ of the form (21) for which the function $x(t) = x^*(t, x_0)$ is a solution of the initial value problem (23). It can be shown that this parameter value is unique, since for any other value of the parameter μ (not of the form (21)) x^* is a solution of the Cauchy-problem (22) but does not satisfy the boundary conditions (2).

Let us suppose that the statement above is not true. Then there exist two such values μ', μ'', $\mu' \neq \mu''$ that the solutions of the Cauchy-problem (22) $x(t, x_0, \mu')$ and $x(t, x_0, \mu'')$ with $\mu = \mu'$ and $\mu = \mu''$ satisfy even the boundary value conditions (2). Then using (12) the following identity for the difference of these solutions is obtained

$$x(t,x_0,\mu'') - x(t,x_0,\mu') = \int_0^t\Big[\big[f\big(t, x(t,x_0,\mu'')\big) - f\big(t, x(t,x_0,\mu')\big)\big] - \\ -\frac{1}{T}\int_0^T\big[f\big(s, x(s,x_0,\mu'')\big) - f\big(s, x(s,x_0,\mu')\big)\big]ds\Big]dt + \\ +\frac{T}{t_1}(k_2 - k_1)Hd(x_0, x(t,\mu'')) - \frac{T}{t_1}(k_2 - k_1)Hd(x_0, x(t,\mu')) - \\ -k_1 H\Big[A_1\int_0^{t_1}\Big[\big(f(t, x(t,x_0,\mu'')) - f(t, x(t,x_0,\mu'))\big) - \\ -\frac{1}{T}\int_0^T\big(f(s, x(s,x_0,\mu'')) - f(s, x(s,x_0,\mu'))\big)ds\Big]dt\Big].$$

Supposing $|x(t,x_0,\mu'') - x(t,x_0,\mu')| = r(t)$ and using Lemma 2.1 of [9,

p. 31],

$$r(t) \le K\left[\left(1-\frac{t}{T}\right)\int_0^t r(s)ds + \frac{t}{T}\int_t^T r(s)ds\right] + \left|\frac{T}{t_1}(k_2-k_1)H\right||A_1|\cdot$$

$$\cdot\left|\int_0^{t_1}\left[\left(f(t,x(t,x_0,\mu''))-f(t,x(t,x_0,\mu'))\right)-\frac{1}{T}\int_0^T\left(f(s,x(s,x_0,\mu''))-\right.\right.\right.$$

$$\left.\left.\left.-f(s,x(s,x_0,\mu'))\right)ds\right]dt\right| + |k_1H||A_1|\left|\int_0^{t_1}\left[\left(f(t,x(t,x_0,\mu''))-\right.\right.\right.$$

$$\left.\left.\left.-f(t,x(t,x_0,\mu'))\right)-\frac{1}{T}\int_0^T\left(f(s,x(s,x_0,\mu''))-f(s,x(s,x_0,\mu'))\right)ds\right]dt\right| \le$$

$$\le\left[|k_1|+\left|\frac{(k_2-k_1)T}{t_1}\right|\right]|HA_1|K\left[\left(1-\frac{t_1}{T}\right)\int_0^{t_1} r(t)dt + \frac{t_1}{T}\int_{t_1}^T r(t)dt\right]+$$

$$+K\left[\left(1-\frac{t}{T}\right)\int_0^t r(t)dt + \frac{t}{T}\int_t^T r(t)dt\right],$$

$$r(t) \le G\left[\left(1-\frac{t_1}{T}\right)\int_0^{t_1} r(t)dt + \frac{t_1}{T}\int_{t_1}^T r(t)dt\right]+$$

$$+K\left[\left(1-\frac{t}{T}\right)\int_0^t r(t)dt + \frac{t}{T}\int_t^T r(t)dt\right].$$

Let $|r(t)|_0 = \left(\sup_t |r_1(t)|, \dots, \sup_t |r_n(t)|\right)$. We have

$$r(t) \le \left[G\alpha_1(t_1) + K\alpha_1(t)\right]|r(t)|_0 \le Q|r(t)|_0,$$

$$r(t) \le G\left[\left(1-\frac{t_1}{T}\right)\int_0^{t_1} Q|r(t)|_0 dt + \frac{t_1}{T}\int_{t_1}^T Q|r(t)|_0 dt\right]+$$

$$+K\left[\left(1-\frac{t}{T}\right)\int_0^t Q|r(t)|_0 dt + \frac{t}{T}\int_t^T Q|r(t)|_0 dt\right] \le$$

$$\le Q\left[G\alpha_1(t_1) + K\alpha_1(t)\right]|r(t)|_0 \le Q^2|r(t)|_0, \dots$$

$$r(t) \le Q^m|r(t)|_0, \quad \text{i.e.} \quad |r(t)|_0 \le Q^m|r(t)|_0.$$

Since all the eigenvalues of the matrix Q are within the circle of unit radius, therefore the last inequality holds only if $|r(t)|_0 = 0$, i.e.

$\mu' = \mu''$. This is a contradiction, thus there exists only one parameter value μ. $\Diamond$

The following statement gives a necessary and sufficient condition for the existence of the solution of the boundary value problem (1), (2).

Theorem 3. *Let us consider the initial value problem*

$$(24)\qquad \begin{cases} \dot{x} = f(t,x) \\ x(0) = x_0^* + k_1 H\Big[d - (A + A_1 + C)x_0^* - \\ -A_1 \int_0^{t_1} \Big[f\big(t, x^*(t,x_0^*)\big) - \frac{1}{T}\int_0^T f\big(s, x^*(s,x_0^*)\big)ds\Big]dt\Big] \end{cases}$$

connected to the given differential equation. If the conditions of Th. 1 *are fulfilled, then a solution of* (24) $x = x^*(t)$ *is a solution of the original boundary value problem* (1), (2) *if and only if the determining function* $\Delta(x_0)$ *assumes the value zero at point* x_0^**:*

$$(25)\quad \Delta(x_0^*) = \frac{1}{t_1}(k_2 - k_1)H d(x_0^*, x^*(t,x_0^*)) - \frac{1}{T}\int_0^T f(t, x^*(t,x_0^*))dt = 0,$$

where $x^*(t,x_0^*)$ *is the limit function of the sequence of function* $x_m(t,x_0^*)$*. In this case* $x^*(t) = x^*(t,x_0^*)$ *and the deviation of* $x^*(t)$ *from its* $x_m(t,x_0^*)$ m^{th} *approximation is determined by inequality* (14).

Proof. Since the function $x^*(t,x_0)$ is a solution of the Cauchy-initial value problem (23) and satisfies the boundary-value conditions (2), therefore if inequality (25) holds, then the problems (24) and (23) are equivalent at value $x_0 = x_0^*$. In such a way we proved that (25) is a sufficient condition.

The necessity of the condition (25) is a consequence of the fact that if $x = x^*(t)$ is a solution of the boundary value problem (1), (2) with the initial value

$$x^*(0) = x_0^* + k_1 H\left[d - (A + A_1 + C)x_0^* - A_1\int_0^{t_1}\left[f(t, x^*(t,x_0^*)) - \right.\right.$$

$$\left.\left. -\frac{1}{T}\int_0^T f(s, x^*(s,x_0^*))ds\right]dt\right],$$

then the solution $x = x(t, x_0^*, \mu)$ of the Cauchy-initial value problem satisfies the initial value conditions (2) exactly at $\mu = \Delta(x_0^*) = 0$. Then equality $x(t,x_0^*,\mu) = x^*(t)$ holds and according to Th. 2 the parameter $\mu = \Delta(x_0^*) = 0$ is unique. Thus $x^*(t) = x^*(t,x_0^*)$ and the following inequality holds

$$|x^*(t) - x_m(t, x_0^*)| \leq Q^m(E - Q)^{-1}\beta(x_0^*). \quad \Diamond$$

References

[1] AKTAS, Z. and STETTER, H. J.: A classification and survey of numerical methods for boundary value problems for ordinary differential equations, *Int. J. Numer. Math. Eng.* **11/5** (1977), 771–796.

[2] KELLER, H. B.: Numerical solution of boundary value problems for ordinary differential equations: survey and some recent results on difference methods. In: Numerical solution of boundary value problems for ordinary differential equations, Acad. Press, New York, 1975, 27–88.

[3] KIGURADZE, I. T.: Boundary value problems for the system of ordinary differential equations. Recent problems of mathematics. The new results, Nauka, Moscow, 1987, 3–103 (in Russian).

[4] KRASNOSELSKIJ, M. A.: Functional-analytic methods in the theory of non-linear oscillations, Fifth Internat. Conference on Nonlinear Oscillat., 1970, Vol. 1, 322–331.

[5] MAWHIN, J.: Recent trends in nonlinear boundary value problems, *Abh. Wiss. DDR Abt. Math. Naturwiss. Techn.* **4** (1977), 51–70.

[6] MEYER, G. H.:Initial value methods for boundary value problems, Acad. Press, New York – London, 1973, p. 220.

[7] MOORTI, V. R. G. and GARNER, J. B.: Existence-uniqueness theorems for three-point boundary value problems for nth order nonlinear differential equations, *Journal of Differential Equations* **29** (1978), 205–213.

[8] RAO, D. R. K. S., MURTHY, K. N. and MURTHY, M. S. N.: Three-point boundary value problem for nth order differential equations, *Journal Math. Phys. Sci.* **18** (3), (1984), 323–327.

[9] SAMOILENKO, A. M. and RONTO, N. I.: Numerical-analytic methods on the theory of boundary value problems for ordinary differential equations, Nauka dumka, Kiev (1992) (in Russian), p. 279.

[10] SCHWABIK, S., TVRDY, M. and VEJVODA, O.: Differential and integral equations. Boundary value problems and adjoints, Academia, Praha, 1979, p. 248.

[11] URABE, M.: An existence theorem for multi-point boundary value problems, *Funkc. ekvacioj* **9** (1/3) (1966), 43–60.

[12] VAINIKKO, G.: Approximative methods for nonlinear equations (two approaches to the convergence problem), *Nonlinear Analysis: Theory, Methods and Appl.* **2/6** (1978), 647–687.