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POLYNOMIAL IDENTITIES FOR TENSOR PRODUCTS OF GRASSMANN ALGEBRAS

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Abstract: Let E be the Grassmann (or exterior) algebra of an infinite-dimensional vector space over a field of characteristic 0 and let E_k be the Grassmann algebra of a k-dimensional vector space. We describe the S_n-cocharacters and the asymptotic behaviour of the codimensions for the T-ideals of the polynomial identities for the tensor products $E_k \otimes E_l$ and $E \otimes E_l$, $k, l \geq 2$. As a consequence, we obtain a necessary and sufficient condition for the inclusion of the T-ideals $T(E_k \otimes E_l) \subset T(E_{k'} \otimes E_{l'})$.

Introduction

Let $K\langle X\rangle$ be the free unitary associative algebra freely generated by a countable set of variables $X = \{x_1, x_2, \dots\}$ over a field K of characteristic 0. For any unitary PI-algebra R we denote by $T(R)$ the

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ideal of $K\langle X\rangle$ consisting of all polynomial identities for R; $T(R)$ is called a T-ideal. Kemer [7] has discovered the structure theory of T-ideals. It turns out that all T-prime ideals correspond to algebras obtained by constructions with the $n\times n$ matrix algebra $M_n(K)$ and the Grassmann (or exterior) algebra E. The set of T-prime ideals is closed under tensor products over K. If $T(R_1)$ and $T(R_2)$ are T-prime ideals, then $T(R_1\otimes\otimes R_2)$ is also T-prime. The largest T-prime ideals are $T(K)=T(M_1(K))$, $T(M_2(K))$, $T(E)$ and $T(E\otimes E)$ with inclusions $T(K)\supset T(M_2(K))$ and $T(K)\supset T(E)\supset T(E\otimes E)$. The structure of $T(K)$ is very simple, that of $T(E)$ is also well known [8]. Since $T(E\otimes E)$ is the minimal T-prime ideal which is not contained in $T(M_2(K))$, it is an important object in the investigation of the non-matrix polynomial identities. Popov [10] has obtained a generating set for $T(E\otimes E)$ and has computed its S_n-cocharacters. The T-ideals $T(E\otimes E)$ and $T(M_2(K))$ have some similar properties and can be treated with the same combinatorial techniques. The second author [5] has computed the codimensions of $T(E\otimes E)$ and jointly with Luisa Carini [1] the Hilbert (or Poincaré) series of $T(E\otimes E)$. Recently the first author [3] has described the $\mathbb{Z}_2$-graded polynomial identities for $E\otimes E$.

In this paper we describe the polynomial identities for the tensor product $E_k\otimes E_l$ of two finite-dimensional Grassmann algebras and, as a consequence, the polynomial identities of $E\otimes E_l$. The algebras E_{2k} and E_{2k+1} have the same polynomial identities and it is sufficient to consider the algebras $E_{2k}\otimes E_{2l}$ and $E\otimes E_{2l}$, $k\geq l\geq 1$. Since we work with unitary algebras only, we study the proper (or commutator) polynomial identities introduced by Specht [11]. Our main result is the computing of the proper S_n-cocharacter sequence of $E_{2k}\otimes E_{2l}$ and $E\otimes\otimes E_{2l}$. There exists a simple relationship between the proper and the ordinary S_n-cocharacters [4] and our result allows to obtain also the usual cocharacters. As a consequence we give a sufficient and necessary condition for the inclusion $T(E_{2k}\otimes E_{2l})\subset T(E_{2k'}\otimes E_{2l'})$. This holds if and only if $k+l\geq k'+l'$ and $l\geq l'$. We also determine the exact asymptotic behaviour of the codimension sequences of $E_{2k}\otimes E_{2l}$ and $E\otimes E_{2l}$.

1. Proper identities

We fix a field K of characteristic 0. All algebras which we consider are unitary K-algebras, all vector spaces and tensor products are also over K. We use the following notation: $K\langle X\rangle$ is the free associative algebra generated by $X = \{x_1, x_2, \dots\}$, $K\langle x_1, \dots, x_m\rangle$ is the free subalgebra of rank m, P_n is the space of the multilinear polynomials of degree n in $K\langle x_1, \dots, x_n\rangle$. For an algebra R we denote by $T(R)$ the set of all polynomial identities for R.

A self-contained background and references on the proper (or commutator) polynomial identities can be found in [6]. We follow the notation in [6]. We define commutators of length ≥ 2 by

$$[x_1, x_2] = x_1 \mathrm{ad} x_2 = x_1x_2 - x_2x_1,\ [x_1 \dots, x_{n-1}, x_n] = [[x_1 \dots, x_{n-1}], x_n].$$

An element $f(x_1, \dots, x_m) \in K\langle X\rangle$ is called *proper* if f is a linear combination of products of commutators $[x_{i_1}, \dots] \dots [\dots, x_{i_n}]$. We denote by Γ_n the space of the multilinear proper polynomials of degree n. For a PI-algebra R we denote

$$P_n(R) = P_n/(P_n \cap T(R)),\ \Gamma_n(R) = \Gamma_n/(\Gamma_n \cap T(R)).$$

The vector spaces $P_n(R)$ and $\Gamma_n(R)$ are S_n-modules, where S_n is the symmetric group of degree n. Their S_n-characters are called respectively the n-th *cocharacter* and the n-th *proper cocharacter* of $T(R)$ (or of R). The degrees of these characters, i.e. the dimensions

$$c_n(R) = \mathrm{dim} P_n(R), \gamma_n(R) = \mathrm{dim}\Gamma_n(R),$$

are called the n-th *codimension* and the n-th *proper codimension* of $T(R)$.

We fix a partition $\lambda = (\lambda_1, \dots, \lambda_r)$ of n (notation $\lambda \vdash n$). We denote by $M(\lambda)$ the irreducible S_n-module corresponding to λ and by $T_\lambda(\tau)$ the λ-tableau corresponding to $\tau \in S_n$.

$\tau(1)$	$\tau(r_1+1)$	$\dots$	$\dots$	$\tau(n-r_k+1)$
$\tau(2)$	$\tau(r_1+2)$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\dots$	$\dots$	$\tau(n)$
$\tau(r_2)$	$\tau(r_1+r_2)$			
$\tau(r_2+1)$				
$\vdots$				
$\tau(r_1)$				

$T_\lambda(\tau)$

Let P and Ξ be the row and the column stabilizers of $T_\lambda(\tau)$, respectively.

Up to a multiplicative constant the element

$$t_{\lambda\tau}=\sum_{\substack{\rho\in P\\ \xi\in\Xi}}(-1)^{\xi}\rho\xi\in KS_n,\ (-1)^{\xi}=\mathrm{sign}\xi$$

is a minimal idempotent of KS_n and generates an S_n-module $M(\lambda)\subset\subset KS_n$.

Let $d=[x_1,\dots]\dots[\dots,x_n]$ be a product of commutators of length ≥ 2 and let

$$V_d=KS_n(d)=\mathrm{sp}\{\pi d=[x_{\pi(1)},\dots]\dots[\dots,x_{\pi(n)}]\mid\pi\in S_n\}.$$

Then $\Gamma_n=\sum V_d$, where the sum is on all possible products d of length n. If the polynomial

$$\phi_\lambda=\phi_\lambda(x_1,\dots,x_n)=t_{\lambda\tau}d$$

is non-zero is V_d, then ϕ_λ generates an S_n-submodule $M(\lambda)$ of Γ_n. Replacing by the same variable x_p all the variables of $\phi_\lambda(x_1,\dots,x_n)$ whose indices are in the p-th row of $T_\lambda(\tau)$, $p=1,\dots,r$, we obtain a proper polynomial

$$f_\lambda=f_\lambda(x_1,\dots,x_r)$$

which is the highest weight vector of the polynomial representation of the general linear group corresponding to the partition λ and the linearization of f_λ equals ϕ_λ up to a multiplicative constant.

Lemma 1.1. *If $n\geq m$ and $\mu=(\mu_1,\dots,\mu_r)$ and $\lambda=(\mu_1+1,\dots,\nu_r++1,1^{n-m-r})$ are partitions of m and n, respectively, then*

$$\dim M(\lambda)=\frac{1}{m!}\dim M(\mu)\psi_\mu(n),$$

where $\psi_\mu(n)\in\mathbb{Q}[n]$ is a polynomial of degree m in n and the leading term of $\psi_\mu(n)$ is equal to 1.

Proof. The dimension of $M(\lambda)$, $\lambda\vdash n$, is given by the hook formula

$$\dim M(\lambda)=n!\prod h_{ij}^{-1}(\lambda),$$

where $h_{ij}(\lambda)$ is the length of the (i,j)-th hook of the Young diagram of λ, i.e. $h_{ij}(\lambda)=\lambda_i+\lambda'_j-(i+j)+1$, where λ'_j is the length of the j-th column of the diagram. The hooks of λ are equal to

$$h_{i1}(\lambda)=n-m+1-i+\mu_i,\ i=1,\dots,r,$$

$$h_{i1}(\lambda)=n-m+1-i,\ i=r+1,\dots,n-m,$$

$$h_{ij}(\lambda)=h_{i,j-1}(\mu),\ j>1,\ i=1,\dots,r.$$

Hence

$$\dim M(\lambda) = \frac{1}{m!}(m!\prod h_{ij}^{-1}(\mu))\frac{n!}{(n-m-r)!}\prod_{i=1}^{r}(n-m+1-i+\mu_i)^{-1} =$$

$$= \frac{1}{m!}\dim M(\mu)\psi_\mu(n)$$

and

$$\psi_\mu(n) = n(n-1)\dots(n-m-r+1)\prod_{i=1}^{r}(n-m+1-i+\mu_i)^{-1}$$

is a polynomial of degree m in n with leading term equal to 1. ◊

Proposition 1.2. [4, 5] *Let R be a PI-algebra.*

(i) *If $P_n(R)=\sum m(\lambda)M(\lambda)$, $\Gamma_n(R)=\sum m'(\mu)M(\mu)$, then $m(\lambda)= =\sum m'(\mu)$, where for $\lambda = (\lambda_1,\dots,\lambda_r)$ the summation runs over all partitions $\mu = (\mu_1,\dots,\mu_r)$ such that $\lambda_1 \geq \mu_1 \geq \dots \geq \lambda_r \geq \mu_r$.*

(ii) *The codimension sequence $c_n(R)$ and the proper codimension sequence $\gamma_n(R)$, $n = 0,1,2,\dots$, are related by the equality*

$$c_n(R) = \sum_{m=0}^{n}\binom{n}{m}\gamma_m(R);$$

(iii) *The codimension series $c(R,t) = \sum c_(R)t^n$ and the proper codimension series $\gamma(R,t) = \sum \gamma_n(R)t^n$ satisfy the equation*

$$c(R,t) = \frac{1}{1-t}\gamma(R,\frac{t}{1-t}).$$

Proposition 1.3. [10] $\Gamma_n(E\otimes E) = \sum M(a+2,2^b,1^c) + \varepsilon_n M(1^n)$, *where $(a+2,2^b,1^c)\vdash n$, $a\geq 0$, $b+c>0$; $\varepsilon_n = 0$ for n odd and $\varepsilon_n = 1$ for n even. Here $(a+2,2^b,1^c)$ is a short notation for the partition*

$$(a+2,\underbrace{2,\dots,2}_{b},\underbrace{1,\dots,1}_{c}).$$

Since $T(E_k\otimes E_l) \supset T(E\otimes E_l) \supset T(E\otimes E)$, we obtain that $\Gamma_n(E_k\otimes E_l)$ and $\Gamma_n(E\otimes E_l)$ are factor modules of $\Gamma_n(E\otimes E)$. In order to obtain the proper cocharacters of $E_k\otimes E_l$ it is sufficient to establish for which irreducible S_n-modules $M(\lambda)\subset\Gamma_n(E\otimes E)$, $\lambda=(\lambda_1,\dots,\lambda_r)$, the corresponding polynomial $f_\lambda(x_1,\dots,x_r)$ vanishes on $E_k\otimes E_l$. We fix the following polynomials $f_\lambda = f_\lambda(x_1,\dots,x_{c+1})$ for $\lambda = (a+2,1^c)$, $a\geq 0$, $c\geq 1$:

$$f_\lambda = \sum(-1)^\sigma x_{\sigma(1)}\dots x_{\sigma(2p)}(x_{\sigma(2p+1)}\mathrm{ad}^{r+1}x_1)$$

for $\lambda = (r+2,1^{2p})$;

$$f_\lambda=\sum(-1)^\sigma x_{\sigma(1)}\dots x_{\sigma(2p-2)}([x_{\sigma(2p-1)},x_{\sigma(2p)}]\mathrm{ad}^{r+1}x_1)$$

for $\lambda=(r+2,1^{2p-1})$; and $f_\lambda=f_\lambda(x_1,\dots,x_{b+c+1})$ for $\lambda=(a+2,2^b,1^c)$, $a,c\geq 0$, $b>0$:

$$f_\lambda=\sum(-1)^\sigma(-1)^\tau x_{\sigma(1)}\dots x_{\sigma(2p)}x_{\tau(1)}\dots$$
$$\dots x_{\tau(2q-2)}([x_{\tau(2q-1)},x_{\tau(2q)}]\mathrm{ad}^r x_1)$$

for $\lambda=(r+2,2^{2q-1},1^{2(p-q)})$;

$$f_\lambda=\sum(-1)^\sigma(-1)^\tau x_{\sigma(1)}\dots$$
$$\dots x_{\sigma(2p-2)}[([x_{\sigma(2p-1)},x_{\sigma(2p)}]\mathrm{ad}^{2r}x_1),x_{\tau(1)}]x_{\tau(2)}\dots x_{\tau(2q+1)}$$

for $\lambda=(2r+2,2^{2q},1^{2(p-q)-1})$ and $\lambda=(2r+2,2^{2p-1},1^{2(q-p)+1})$;

$$f_\lambda=\sum(-1)^\sigma(-1)^\tau x_{\sigma(1)}\dots$$
$$\dots x_{\sigma(2p-2)}([x_{\sigma(2p-1)},x_{\sigma(2p)},x_{\tau(1)}]\mathrm{ad}^{2r+1}x_1)x_{\tau(2)}\dots x_{\tau(2q+1)}$$

for $\lambda=(2r+3,2^{2q},1^{2(p-q)-1})$ and $\lambda=(2r+3,2^{2p-1},1^{2(q-p)+1})$;

$$f_\lambda=\sum(-1)^\sigma(-1)^\tau[([x_{\sigma(1)},x_{\sigma(2)}]\mathrm{ad}^{2r}x_1),x_{\tau(1)}][x_{\tau(2)},x_{\tau(3)},x_{\sigma(3)}]\times$$
$$\times\, x_{\sigma(4)}\dots x_{\sigma(2p+1)}x_{\tau(4)}\dots x_{\tau(2q+1)}$$

for $\lambda=(2r+2,2^{2q},1^{2(p-q)})$;

$$f_\lambda=\sum(-1)^\sigma(-1)^\tau x_{\sigma(1)}\dots x_{\sigma(2p)}x_{\tau(1)}\dots$$
$$\dots x_{\tau(2q)}[(x_{\tau(2q+1)}\mathrm{ad}^{2r+1}x_1),x_{\sigma(2p+1)}]$$

for $\lambda=(2r+3,2^{2q},1^{2(p-q)})$;

$$f_\lambda=s_{2p}(x_1,\dots,x_{2p}),$$

for $\lambda=(1^{2p})$, $p\geq 1$, where

$$s_m(x_1,\dots,x_m)=\sum(-1)^\sigma x_{\sigma(1)}\dots x_{\sigma(m)}$$

is the standard polynomial of degree m. Since the standard polynomial of even degree is proper, it is easy to see that all the polynomials f_λ are also proper. Up to a multiplicative constant the linearization of each f_λ is equal to $\phi_\lambda=t_{\lambda\tau}d$ for some $\tau\in S_n$ and a product of commutators d and generates a submodule $M(\lambda)$ of Γ_n, $\lambda\vdash n$. Some of the f_λ's are as in the paper by Popov [10] but some of them are replaced by more convenient polynomials.

2. Preliminary reductions

Let V be a countably dimensional vector space with basis $\{e_1, e_2, \dots\}$. The *Grassmann* (or *exterior*) *algebra* $E = E(V)$ of V is the algebra generated by $e_1, e_2, \dots$ with defining relations $e_i e_j = -e_j e_i$, $i, j = 1, 2, \dots$. The Grassmann algebra E_k of a k-dimensional vector space is generated by $e_1, \dots, e_k$. Let $E \otimes E$ be the tensor square of E. We fix generators $e_1 \otimes 1, e_2 \otimes 1, \dots$ of $E \otimes 1$ and $1 \otimes \tilde{e}_1, 1 \otimes \tilde{e}_2, \dots$ of $1 \otimes E$ and write the elements $u \otimes v \in E \otimes E$ as uv without the symbol $\otimes$ between u and v.

Lemma 2.1. *For* $\delta, \varepsilon = 0, 1$,

$$T(E_{2k+\delta} \otimes E_{2l+\varepsilon}) = T(E_{2k} \otimes E_{2l}),\ T(E \otimes E_{2l+\varepsilon}) = T(E \otimes E_{2l}).$$

Proof. By [9, Th. 1], if A_1, A_2, B_1 and B_2 are PI-algebras such that $T(A_1) = T(A_2)$, $T(B_1) = T(B_2)$, then $T(A_1 \otimes B_1) = T(A_2 \otimes B_2)$. Since the T-ideals $T(E_{2k+1})$ and $T(E_{2k})$ are equal (see e.g. [2]), this gives immediately the proof of the lemma. $\Diamond$

In the sequel we fix $k \geq l \geq 1$ and study the polynomial identities for $E_{2k} \otimes E_{2l}$ and $E \otimes E_{2l}$. We use an idea from [3]. The algebra E has a natural $\mathbb{Z}_2$-grading $E = E^{[0]} \oplus E^{[1]}$, where $E^{[0]}$ and $E^{[1]}$ are spanned on the products of even and odd length, respectively. We denote by y and $y_1, y_2, \dots$ arbitrary elements of $E^{[1]} \otimes E^{[1]}$ and by $z_1, z_2, \dots$ arbitrary elements of $E^{[1]} \otimes E^{[0]} \oplus E^{[0]} \otimes E^{[1]}$.

Lemma 2.2. *The elements* $y, y_1, y_2, \dots, z_1, z_2, \dots$ *satisfy:*

(i) yy_1, $[z_1, z_2]$ *and* $y(z_1 \circ z_2)$ *are central in* $E \otimes E$, *where* $z_1 \circ z_2 = z_1 z_2 + z_2 z_1$;

(ii) $yy_1 = y_1 y$, $z_1 y = -y z_1$, $z_1(\mathrm{ad}^r y) = (-2)^r y^r z_1$;

(iii) $[y^{2r} z_1, z_2] = y^{2r}[z_1, z_2]$, $[y^{2r+1} z_1, z_2] = y^{2r+1}(z_1 \circ z_2)$;

(iv) *If* $\bar{x}_1 = y$, $\bar{x}_i = z_i$, $i = 2, \dots, q$, *then*

$$\sum (-1)^\sigma \bar{x}_{\sigma(1)} \dots \bar{x}_{\sigma(q)} = qy \sum (-1)^\sigma z_{\sigma(2)} \dots z_{\sigma(q)};$$

(v) $z_2 \mathrm{ad}^r(y + z_1) = (-2)^{r-1} y^{r-1}(-2yz_2 + (-1)^r z_1 z_2 + z_2 z_1) \equiv (-2)^r y^r z_2$ *modulo the centre of* $E \otimes E$;

(vi) $z_3 z_2 z_1 = -z_1 z_2 z_3$;

(vii) $z_1 z_2 z_1 = 0$, $z_2 z_1^2 = -z_1^2 z_2$, $z_3 z_4 z_1 z_2 = z_1 z_2 z_3 z_4$, $z_2^2 z_1^2 = z_1^2 z_2^2$;

(viii) $yuy_1 = y_1 uy$, $z_1 u z_1 v z_1 = 0$ *for all* $u, v \in E \otimes E$.

Proof. The case (i) is obvious because $yy_1, [z_1, z_2], y(z_1 \circ z_2) \in E^{[0]} \otimes E^{[0]}$, the centre of $E \otimes E$. Since the identity $z_1 y = -y z_1$ from (ii) is multilinear in y and z_1, it suffices to consider only the cases $y = e_1 \tilde{e}_1$,

$z_1 = e_2$ and $y = e_1\tilde{e}_1$, $z_1 = \tilde{e}_2$, similarly for $yy_1 = y_1y$. The verification is trivial. This gives also (iii) and (iv) which are consequences of (i) and (ii). For example,

$$[y^{2r+1}z_1, z_2] =(y^2)^r[yz_1, z_2] = y^{2r}(yz_1z_2 - z_2yz_1) =$$
$$=y^{2r}(yz_1z_2 + yz_2z_1) = y^{2r+1}(z_1 \circ z_2).$$

(v) We use induction on r. For $r = 1$, $z_2\mathrm{ad}(y+z_1) = -2yz_2 + [z_2, z_1] \equiv$ $\equiv -2yz_2$ modulo the centre $E^{[0]} \otimes E^{[0]}$ of $E \otimes E$. Let $z_2\mathrm{ad}^r(y + z_1) \equiv$ $\equiv (-2)^r y^r z_2 \,(\mathrm{mod}\, E^{[0]} \otimes E^{[0]})$. Then

$$z_2\mathrm{ad}^{r+1}(y + z_1) =(-2)^r[y^r z_2, y + z_1] =$$
$$=(-2)^r(y^r[z_2, y] + y^r z_2 z_1 - z_1 y^r z_2) =$$
$$=(-2)^r(-2y^{r+1}z_2 + y^r((-1)^{r+1}z_1z_2 + z_2z_1)).$$

In both the cases r even and r odd, $y^r((-1)^{r+1}z_1z_2+z_2z_1) \in E^{[0]}\otimes E^{[0]}$. For (vi) it is sufficient to consider the cases $z_i \in \{e_i, \tilde{e}_i\}$, $i = 1, 2, 3$, and (6) can be also easily checked. The identities from (vii) are consequences of (vi); (viii) follows from (ii) and (vii). $\Diamond$

By the convention of Section 1, for $\lambda = (\lambda_1, \dots , \lambda_r) = (a + 2, 2^b, 1^c) \vdash n$, $r = b+c+1$, $a \geq 0$, $b+c > 0$, $\tau \in S_n$ and a product of commutators d we consider the polynomial $\phi_\lambda(x_1, \dots , x_n) = t_{\lambda\tau}d$ and its symmetrization $f_\lambda(x_1, \dots , x_r)$.

Lemma 2.3. *If $f_\lambda(x_1, \dots , x_r)$ is not a polynomial identity for $E_{2k} \otimes E_{2l}$, then*

$$2a + 2b + c + 2 \leq 2(k + l),\ a + b + 1 \leq 2l.$$

Proof. Every variable of $f_\lambda(x_1, \dots , x_r)$ is in a commutator. Since $E^{[0]} \otimes E^{[0]}$ is the centre of $E \otimes E$, there exist elements $y_1 + z_1, \dots , y_r + z_r$ such that

$$\bar{f}_\lambda = f_\lambda(y_1 + z_1, \dots , y_r + z_r) \neq 0,$$
$$y_i \in E_{2k}^{[1]} \otimes E_{2l}^{[1]},\ z_i \in E_{2k}^{[1]} \otimes E_{2l}^{[0]} \oplus E_{2k}^{[0]} \otimes E_{2l}^{[1]}.$$

Let $\alpha = (\alpha_1, \dots , \alpha_r)$ and let $f_\lambda^{(\alpha)}$ be the homogeneous component of $\bar{f}_\lambda$ of degree α_i in y_i, $i = 1, \dots , r$. By Lemma 2.2 (viii), $\bar{f}_\lambda = \sum f_\lambda^{(\alpha)}$, where the summation runs over all α with $\alpha_i \leq 2$. Using Lemma 2.2 (ii) and (vii) we can write every non-zero $f_\lambda^{(\alpha)}$ in the form

$$f_\lambda^{(\alpha)} = y_1^{\beta_1} \dots y_r^{\beta_r} z_{i_1}^2 \dots z_{i_s}^2 g_\alpha(z_{j_1}, \dots , z_{j_t}),$$

where $\beta_i = \lambda_i - \alpha_i$ and $\alpha_{i_1} = \dots = \alpha_{i_s} = 2$, $\alpha_{j_1} = \dots = \alpha_{j_t} = 1$. The non-zero element $y_1^{\beta_1} \dots y_r^{\beta_r}$ is a linear combination of products $e_{m_1} \dots e_{m_\beta} \tilde{e}_{n_1} \dots \tilde{e}_{n_{\beta'}}$, where $\beta, \beta' \geq \beta_1 + \dots + \beta_r$ and $e_{m_i}, \tilde{e}_{n_j}$ are pairwise different generators of $E_{2k} \otimes 1$ and $1 \otimes E_{2l}$, respectively. Similarly, we need at least $2s + t = \alpha_1 + \dots \alpha_r$ generators for $z_{i_1}^2 \dots z_{i_s}^2 g_\alpha(z_{j_1}, \dots, z_{j_t})$. Therefore

$$2(k+l) \geq 2(\beta_1 + \dots + \beta_r) + \alpha_1 + \dots \alpha_r \geq$$
$$\geq \beta_1 + (\alpha_1 + \beta_1) + \dots (\alpha_r + \beta_r) = \beta_1 + \lambda_1 + \dots + \lambda_r.$$

Since $\beta_1 = \lambda_1 - \alpha_1 \geq a$ and $\lambda_1 + \dots + \lambda_r = a + 2b + c + 2$, we obtain $2(k+l) \geq 2a + 2b + c + 2$. If $z_i \in E_{2k}^{[1]} \otimes E_{2l}^{[0]}$, then $z_i^2 = 0$. Hence we need at least one generator $\tilde{e}_{p_i}$ for each product z_i^2. Since $f_\lambda^{(\alpha)}$ is equal to $y_i^{\lambda_i} h_0$, $y_i^{\lambda_i - 1} z_i h_1$ or $y_i^{\lambda_i - 2} z_i^2 h_2$ for some $h_0, h_1, h_2 \in E_{2k} \otimes E_{2l}$, we need at least $\lambda_i - 1$ generators of $1 \otimes E_{2l}$ for each $i = 1, \dots, r$, i.e. $2l \geq (\lambda_1 - 1) + \dots + (\lambda_r - 1) = = a + b + 1$. ◊

For a polynomial $f(x_1, x_2, \dots, x_r) \in K\langle X \rangle$ we denote by $f^{(j)}$ the homogeneous component of degree j in z_1 of the element $f(y + + z_1, z_2, \dots, z_r)$, $j = 0, 1, 2$. In virtue of Lemma 2.2 (viii), $f(y + + z_1, z_2, \dots, z_r) = f^{(0)} + f^{(1)} + f^{(2)}$.

Lemma 2.4. *If $f_\lambda(x_1, \dots, x_r)$ is not a polynomial identity for $E_{2k} \otimes \otimes E_{2l}$,*

$$\lambda = (a+2, 2^b, 1^c) \quad \textit{and} \quad 2a + 2b + c + 2 = 2(k+l),$$

then $f^{(2)}(y + z_1, z_2, \dots, z_r) \neq 0$ for some $y \in E_{2k}^{[1]} \otimes E_{2l}^{[1]}$, $z_i \in E_{2k}^{[1]} \otimes \otimes E_{2l}^{[0]} \oplus E_{2k}^{[0]} \otimes E_{2l}^{[1]}$.

Proof. As in the proof of Lemma 2.3, let

$$f_\lambda^{(\alpha)} = y_1^{\beta_1} \dots y_r^{\beta_r} z_{i_1}^2 \dots z_{i_s}^2 g_\alpha(z_{j_1}, \dots, z_{j_t})$$

be a non-zero homogeneous component of $f_\lambda(y_1 + z_1, \dots, y_r + z_r) \in \in E_{2k} \otimes E_{2l}$. Since $2(k+l) = 2a + 2b + c + 2 = a + \lambda_1 + \dots + \lambda_r$, we obtain from the inequalities

$$2(k+l) \geq (\beta_1 + \dots + \beta_r) + (\lambda_1 + \dots + \lambda_r) \geq$$
$$\geq \beta_1 + \lambda_1 + \dots + \lambda_r \geq a + \lambda_1 + \dots + \lambda_r$$

that $\beta_1 = a$, $\beta_2 = \dots = \beta_r = 0$, i.e. $f_\lambda^{(\alpha)}(y_1 + z_1, \dots, y_r + z_r) = = f_\lambda^{(2)}(y_1 + z_1, z_2, \dots, z_r) \neq 0$. ◊

All the sums in the sequel are on $\sigma \in S_m$, where the symmetric group S_m acts on the set of symbols $\{d+1, \dots, d+m\}$ and the values of d and t are clear from the context.

Lemma 2.5. *The elements $z_1, z_2, \dots$ satisfy the following identities:*

(i) $\sum(-1)^\sigma z_{\sigma(1)}\dots z_{\sigma(2p)}[z_{\sigma(2p+1)},z_1]=0$;

(ii) $\sum_{\sigma(2p+1)\neq 1}(-1)^\sigma z_{\sigma(1)}\dots z_{\sigma(2p+1)}z_1=pz_1^2s_{2p}(z_2,\dots,z_{2p+1})$;

(iii) $\sum(-1)^\sigma z_{\sigma(2)}\dots z_{\sigma(2p-1)}[z_{\sigma(2p)},z_1]=-p^{-1}s_{2p}(z_1,\dots,z_{2p})$;

(iv) $s_{2p}(z_1,\dots,z_{2p})s_{2q}(z_1,\dots,z_{2q})=$
$=(2q)!(p!)^2((p-q)!)^{-2}z_1^2\dots z_{2q}^2 s_{2(p-q)}(z_{2q+1},\dots,z_{2p})$, $p\geq q$;

(v) $s_{2p}(z_1,\dots,z_{2p})\sum(-1)^\tau z_{\tau(2)}\dots z_{\tau(2q-1)}(z_{\tau(2q)}\circ z_1)=0$, $p\geq q$;

(vi) $s_{2p-1}(z_1,\dots,z_{2p-1})s_{2q-1}(z_1,\dots,z_{2q-1})=$
$=s_{2q-1}(z_1,\dots,z_{2q-1})s_{2p-1}(z_1,\dots,z_{2p-1})=$
$=(2q-1)!p!(p-1)!((p-q)!)^{-2}z_1^2\dots z_{2q-1}^2 s_{2(p-q)}(z_{2q},\dots,z_{2p-1})$, $p\geq q$;

(vii) $s_{2p-1}(z_1,\dots,z_{2p-1})s_{2q}(z_1,\dots,z_{2q})=$
$=(2p-1)!(q!)^2((q-p)!(q-p+1)!)^{-1}z_1^2\dots$
$\dots z_{2p-1}^2 s_{2(q-p)+1}(z_{2p},\dots,z_{2q})$, $p\leq q$;

(viii) $s_{2p-1}(z_2,\dots,z_{2p})\circ s_{2q+1}(z_1,\dots,z_{2q+1})=$
$=(2q+1)!p!(p-1)!((p-q)!(p-q-1)!)^{-1}z_2^2\dots$
$\dots z_{2q+1}^2(s_{2(p-q)-1}(z_{2q+2},\dots,z_{2p})\circ z_1)$, $p>q$.

Proof. (i) Let $h=\sum(-1)^\sigma z_{\sigma(1)}\dots z_{\sigma(2p)}[z_{\sigma(2p+1)},z_1]$. Since $[z_i,z_j]$ are central elements,

$$h=2^{-p}\sum(-1)^\sigma[z_{\sigma(1)},z_{\sigma(2)}]\dots[z_{\sigma(2p-1)},z_{\sigma(2p)}][z_{\sigma(2p+1)},z_1]=$$
$$=2^{-(p+1)}(p+1)^{-1}\sum(-1)^\sigma[z_{\sigma(1)},z_{\sigma(2)}]\dots$$
$$\dots[z_{\sigma(2p-1)},z_{\sigma(2p)}][z_{\sigma(2p+1)},z_{\sigma(2p+2)}]$$

for $z_{2p+2}=z_1$ and $h=(p+1)^{-1}s_{2p+2}(z_1,\dots,z_{2p+1},z_1)=0$.

(ii) By Lemma 2.2 (viii) $z_1z_{i_1}\dots z_{i_{2q-1}}z_1=0$, $q\geq 1$, and the only non-zero summands of

$$h=\sum_{\sigma(2p+1)\neq 1}(-1)^\sigma z_{\sigma(1)}\dots z_{\sigma(2p+1)}z_1$$

are for $1\in\{\sigma(1),\sigma(3),\dots,\sigma(2p-1)\}$. Using the identity (vi) from Lemma 2.2 we obtain

$$h=pz_1\sum(-1)^\sigma z_{\sigma(2)}\dots z_{\sigma(2p+1)}z_1=pz_1^2s_{2p}(z_2,\dots,z_{2p+1}).$$

(iii) $\sum(-1)^\sigma z_{\sigma(2)}\dots z_{\sigma(2p-1)}[z_{\sigma(2p)},z_1]=$

$$=2^{-(p-1)}\sum(-1)^\sigma[z_{\sigma(2)},z_{\sigma(3)}]\dots[z_{\sigma(2p-2)},z_{\sigma(2p-1)}][z_{\sigma(2p)},z_1]=$$
$$=2^{-p}p^{-1}\sum[z_{\sigma(2)},z_{\sigma(3)}]\dots[z_{\sigma(2p-2)},z_{\sigma(2p-1)}][z_{\sigma(2p)},z_{\sigma(1)}]=$$
$$=-p^{-1}s_{2p}(z_1,\dots,z_{2p}).$$

(iv) Let

$$h = h(z_1, \dots, z_{2p}) = s_{2p}(z_1, \dots, z_{2p})z_1 \dots z_{2q} =$$
$$= \sum (-1)^\sigma z_{\sigma(1)} \dots z_{\sigma(2p)} z_1 \dots z_{2q}.$$

In virtue of the first identity from Lemma 2.2 (vii), the only non-zero summands of h are for $1, 3, \dots, 2q-1 \in \{\sigma(2), \sigma(4), \dots, \sigma(2p)\}$, $2, 4, \dots, 2q \in \{\sigma(1), \sigma(4), \dots, \sigma(2p-1)\}$ and by Lemma 2.2 (vi) and (vii)

$$h = (-1)^q (p(p-1)\dots(p-q+1))^2 \sum (-1)^\sigma (z_2 z_1)(z_4 z_3)\dots$$
$$\dots (z_{2q} z_{2q-1}) \times z_{\sigma(2q+1)} \dots z_{\sigma(2p)} (z_1 z_2) \dots (z_{2q-1} z_{2q}) =$$
$$= (-1)^q (p!)^2 ((p-q)!)^{-2} (z_2 z_1^2 z_2)(z_4 z_3^2 z_4)\dots$$
$$\dots (z_{2q} z_{2q-1}^2 z_{2q}) s_{2(p-q)}(z_{2q+1}, \dots, z_{2p}) =$$
$$= (p!)^2 ((p-q)!)^{-2} z_1^2 \dots z_{2q}^2 s_{2(p-q)}(z_{2q+1}, \dots, z_{2p}).$$

Now we extend the action of S_{2q} trivially on $\{2q+1, \dots, 2p\}$. For a fixed $\tau \in S_{2q}$, $S_{2p}\tau = S_{2p}$. Hence

$$s_{2p}(z_1, \dots, z_{2p}) s_{2q}(z_1, \dots, z_{2q}) =$$
$$= \sum (-1)^{\sigma\tau} (-1)^\tau z_{\sigma\tau(1)} \dots z_{\sigma\tau(2p)} z_{\tau(1)} \dots z_{\tau(2q)} =$$
$$= \sum (-1)^\sigma z_{\sigma(\tau(1))} \dots z_{\sigma(\tau(2p))} z_{\tau(1)} \dots z_{\tau(2q)} =$$
$$= \sum s_{2p}(z_{\tau(1)}, \dots, z_{\tau(2p)}) z_{\tau(1)} \dots z_{\tau(2q)} =$$
$$= \sum h(z_{\tau(1)}, \dots, z_{\tau(2p)}) = (2q)! h(z_1, \dots, z_{2p}).$$

(v) Using the polynomial h defined in the proof of (iv), we obtain

$$s_{2p}(z_1, \dots, z_{2p}) \sum (-1)^\tau z_{\tau(2)} \dots z_{\tau(2q-1)} (z_{\tau(2q)} \circ z_1) =$$
$$= \sum (-s_{2p}(z_{\tau(2)}, \dots, z_{\tau(2q-1)}, z_{\tau(2q)}, z_1, z_{2q+1}, \dots, z_{2p}) z_{\tau(2)} \dots$$
$$\dots z_{\tau(2q-1)} z_{\tau(2q)} z_1 +$$
$$+ s_{2p}(z_{\tau(2)}, \dots, z_{\tau(2q-1)}, z_1, z_{\tau(2q)}, z_{2q+1}, \dots, z_{2p}) z_{\tau(2)} \dots$$
$$\dots z_{\tau(2q-1)} z_1 z_{\tau(2q)}) =$$
$$= \sum (2q-1)! (p!)^2 ((p-q)!)^{-2} z_{\tau(2)}^2 \dots z_{\tau(2q-1)}^2 \times$$

$$\times(-z^2_{\tau(2q)}z^2_1 + z^2_1 z^2_{\tau(2q)})s_{2(p-q)}(z_{2q+1},\dots,z_{2p}) = 0.$$

(vi) The non-zero summands of $\sum(-1)^\sigma z_{\sigma(1)}\dots z_{\sigma(2p-1)}z_1\dots z_{2q-1}$ are for

$$1,3,\dots,2q-1 \in \{\sigma(1),\sigma(3),\dots,\sigma(2p-1)\},$$
$$2,4,\dots,2q-2 \in \{\sigma(2),\sigma(4),\dots,\sigma(2p-2)\}$$

and

$$s_{2p-1}(z_1,\dots,z_{2p-1})z_1\dots z_{2q-1} =$$
$$= p!(p-1)!((p-q)!)^{-2}z_1\dots z_{2q-1}s_{2(p-q)}(z_{2q},\dots,z_{2p-1})z_1\dots z_{2q-1} =$$
$$= p!(p-1)!((p-q)!)^{-2}(z_1\dots z_{2q-1})^2 s_{2(p-q)}(z_{2q},\dots,z_{2p-1}),$$
$$(z_1\dots z_{2q-1})^2 = z_1(z_2z_3)\dots(z_{2q-2}z_{2q-1})(z_1z_2)\dots(z_{2q-3}z_{2q-2})z_{2q-1} =$$
$$= z_1(z_1z_2)(z_2z_3)\dots(z_{2q-3}z_{2q-2})(z_{2q-2}z_{2q-1})z_{2q-1} = z^2_1\dots z^2_{2q-1}.$$

As in (iv)

$$s_{2p-1}(z_1,\dots,z_{2p-1})s_{2q-1}(z_1,\dots,z_{2q-1}) =$$
$$= (2q-1)!s_{2p-1}(z_1,\dots,z_{2p-1})z_1\dots z_{2q-1}.$$

The calculations for $s_{2q-1}(z_1,\dots,z_{2q-1})s_{2p-1}(z_1,\dots,z_{2p-1})$ are similar.

(vii) is similar to (vi).

(viii) $s_{2q+1}(z_1,\dots,z_{2q+1}) =$

$$(q+1)z_1\sum(-1)^\tau z_{\tau(2)}\dots z_{\tau(2q+1)} - q\sum(-1)^\tau z_{\tau(2)}z_1z_{\tau(3)}\dots z_{\tau(2q+1)},$$
$$h_1 = s_{2p-1}(z_2,\dots,z_{2p})\circ(((q+1)z_1z_2 - qz_2z_1)z_3\dots z_{2q+1}) =$$
$$= (q+1)(s_{2p-1}(z_2,\dots,z_{2p})z_1)(z_2z_3\dots z_{2q+1})-$$
$$-q(z_2z_1)(s_{2p-1}(z_2,\dots,z_{2p})z_3\dots z_{2q+1})+$$
$$+(q+1)(z_1z_2)(z_3\dots z_{2q+1}s_{2p-1}(z_2,\dots,z_{2p}))-$$
$$-q(s_{2p-1}(z_2,\dots,z_{2p})z_2)(z_1z_3\dots z_{2q+1}).$$

Since all elements in the parenticies are of even length, Lemma 2.2 (vii) gives

$$h_1 = (q+1)z_2h_2z_1 - qh_2z_2z_1 + (q+1)h_2z_1z_2 - qz_1h_2z_2,$$

where $h_2 = z_3\dots z_{2q+1}s_{2p-1}(z_2,\dots,z_{2p})$. As in the proof of (vi)

$$h_2 = -z_3\dots z_{2q+1}s_{2p-1}(z_3,\dots,z_{2q+1},z_2,z_{2q+2},\dots,z_{2p}) =$$
$$= -p!(p-1)!((p-q)!)^{-2}s_{2(p-q)}(z_2,z_{2q+2},\dots,z_{2p})z^2_3\dots z^2_{2q+1}.$$

Since $z^2_i z_j = -z_j z^2_i$, $i = 3,\dots,2q+1$, $j = 1,2$, and the standard polynomial of even length is central,

$$h_1 = p!(p-1)!((p-q)!)^{-2}(2q+1) \times (z_2 s_{2(p-q)}(z_2, z_{2q+2}, \dots, z_{2p})z_1 -$$
$$-z_1 z_2 s_{2(p-q)}(z_2, z_{2q+2}, \dots, z_{2p}))z_3^2 \dots z_{2q+1}^2 =$$
$$= p!(p-1)!((p-q)!)^{-2}(2q+1)(p-q)\times$$
$$\times(z_2^2 s_{2(p-q)-1}(z_{2q+2}, \dots, z_{2p})z_1 -$$
$$- z_1 z_2^2 s_{2(p-q)-1}(z_{2q+2}, \dots, z_{2p}))z_3^2 \dots z_{2q+1}^2 =$$
$$= p!(p-1)!((p-q)!)^{-2}(2q+1)(p-q)z_2^2 \dots$$
$$\dots z_{2q+1}^2(s_{2(p-q)-1}(z_{2q+2}, \dots, z_{2p}) \circ z_1).$$

Hence

$$s_{2p-1}(z_2, \dots, z_{2p}) \circ s_{2q+1}(z_1, \dots, z_{2q+1}) =$$
$$= (2q+1)!p!(p-1)!((p-q)!(p-q-1)!)^{-1} z_2^2 \dots$$
$$\dots z_{2q+1}^2(s_{2(p-q)-1}(z_{2q+2}, \dots, z_{2p}) \circ z_1). \quad \Diamond$$

Lemma 2.6. *Let* $\lambda = (a+2, 2^b, 1^c)$, $a \geq 0$, $b+c > 0$ *and let* $f_\lambda(x_1, \dots, x_{b+c+1})$ *be the polynomials from Section 1. If* $f_\lambda^{(i)} = f_\lambda^{(i)}(y+z_1, z_2, \dots, z_{b+c+1})$, $y \in E_{2k}^{[1]} \otimes E_{2l}^{[1]}$, $z_i \in E_{2k}^{[1]} \otimes E_{2l}^{[0]} \oplus E_{2k}^{[0]} \otimes E_{2l}^{[1]}$, *then there exist non-zero constants* α_λ *from* $\mathbb{Q}$ *such that:*

(i) $f_\lambda^{(2)} = 0$, $f_\lambda^{(0)} = \alpha_\lambda y^{2r+2} s_{2p}(z_2, \dots, z_{2p+1})$, $\lambda = (2r+2, 1^{2p})$;

(ii) $f_\lambda^{(2)} = \alpha_\lambda y^{2r+1} z_1^2 s_{2p}(z_2, \dots, z_{2p+1})$, $\lambda = (2r+3, 1^{2p})$;

(iii) $f_\lambda^{(1)} = \alpha_\lambda y^{2r+1}(s_{2p-1}(z_2, \dots, z_{2p}) \circ z_1)$, $\lambda = (2r+2, 1^{2p-1})$;

(iv) $f_\lambda^{(1)} = \alpha_\lambda y^{2r+2} s_{2p}(z_1, \dots, z_{2p})$, $\lambda = (2r+3, 1^{2p-1})$;

(v) $f_\lambda^{(2)} = \alpha_\lambda y^{2r} z_1^2 \dots z_{2q}^2 s_{2(p-q)}(z_{2q+1}, \dots, z_{2p})$,
$\lambda = (2r+2, 2^{2q-1}, 1^{2(p-q)})$;

(vi) $f_\lambda^{(2)} = 0$, $f_\lambda^{(0)} = \alpha_\lambda y^{2r+3} z_2^2 \dots z_{2q}^2 s_{2(p-q)}(z_{2q+1}, \dots, z_{2p})$,
$\lambda = (2r+3, 2^{2q-1}, 1^{2(p-q)})$;

(vii) $f_\lambda^{(1)} = \alpha_\lambda y^{2r+1} z_2^2 \dots z_{2q+1}^2(s_{2(p-q)-1}(z_{2q+2}, \dots, z_{2p}) \circ z_1)$ *for*
$\lambda = (2r+2, 2^{2q}, 1^{2(p-q)-1})$ *and*
$f_\lambda^{(1)} = \alpha_\lambda y^{2r+1} z_2^2 \dots z_{2p}^2 s_{2(q-p)+2}(z_1, z_{2p+1}, \dots, z_{2q+1})$ *for*
$\lambda = (2r+2, 2^{2p-1}, 1^{2(q-p)+1})$;

(viii) $f_\lambda^{(1)} = \alpha_\lambda y^{2r+2} z_2^2 \dots z_{2q+1}^2 s_{2(p-q)}(z_1, z_{2q+2}, \dots, z_{2p})$ *for*
$\lambda = (2r+3, 2^{2q}, 1^{2(p-q)-1})$ *and*
$f_\lambda^{(1)} = \alpha_\lambda y^{2r+2} z_2^2 \dots z_{2p}^2(s_{2(q-p)+1}(z_{2p+1}, \dots, z_{2q+1}) \circ z_1)$ *for*
$\lambda = (2r+3, 2^{2p-1}, 1^{2(q-p)+1})$;

(ix) $f_\lambda^{(2)} = 0$, $f_\lambda^{(0)} = \alpha_\lambda y^{2r+2} z_2^2 \dots z_{2q+1}^2 s_{2(p-q)}(z_{2q+2}, \dots, z_{2p+1})$,
$\lambda = (2r+2, 2^{2q}, 1^{2(p-q)})$;

(x) $f_\lambda^{(2)} = \alpha_\lambda y^{2r+1} z_1^2 \dots z_{2q+1}^2 s_{2(p-q)}(z_{2q+2}, \dots, z_{2p+1})$,
$\lambda = (2r+3, 2^{2q}, 1^{2(p-q)})$.

Proof. Let $\bar{x}_1 = y + z_1$, $\bar{x}_i = z_i$, $i > 1$, and let $\bar{f}_\lambda = f_\lambda(\bar{x}_1, \dots, \bar{x}_{b+c+1})$ for $\lambda = (a+2, 2^b, 1^c)$, $a, b, c \geq 0$.

(i) $\bar{f}_\lambda = \sum(-1)^\sigma \bar{x}_{\sigma(1)} \dots \bar{x}_{\sigma(2p)}(\bar{x}_{\sigma(2p+1)} \mathrm{ad}^{2r+1} \bar{x}_1) =$

$$= 2^{2r} y^{2r} \sum(-1)^\sigma \bar{x}_{\sigma(1)} \dots \bar{x}_{\sigma(2p)} [z_{\sigma(2p+1)}, y + z_1]$$

and $f_\lambda^{(2)} = 2^{2r} y^{2r} \sum(-1)^\sigma z_{\sigma(1)} \dots z_{\sigma(2p)} [z_{\sigma(2p+1)}, z_1] = 0$ by Lemma 2.5 (i);

$$f_\lambda^{(0)} = -2^{2r+1} y^{2r} \sum_{\sigma(2p+1) \neq 1} (-1)^\sigma (\bar{x}_{\sigma(1)} \dots \bar{x}_{\sigma(2p)})^{(0)} y z_{\sigma(2p+1)}$$

and by Lemma 2.2 (iv)

$$f_\lambda^{(0)} = -2^{2r+2} p y^{2r+1} \sum(-1)^\sigma z_{\sigma(2)} \dots z_{\sigma(2p)} y z_{\sigma(2p+1)} =$$

$$= 2^{2r+2} p y^{2r+2} s_{2p}(z_2, \dots, z_{2p+1}).$$

(ii) Since $z_2 \mathrm{ad}^{2r+2}(y + z_1) = -2^{2r+1} y^{2r+1}(-2yz_2 + z_2 \circ z_1)$, we obtain that

$$f_\lambda^{(2)} = -2^{2r+1} y^{2r+1} \sum_{\sigma(2p+1) \neq 1} (-1)^\sigma z_{\sigma(1)} \dots z_{\sigma(2p)} (z_{\sigma(2p+1)} \circ z_1) =$$

$$= -2^{2r+1} y^{2r+1} \sum_{\sigma(2p+1) \neq 1} (-1)^\sigma z_{\sigma(1)} \dots$$

$$\dots z_{\sigma(2p)} (2 z_{\sigma(2p+1)} z_1 - [z_{\sigma(2p+1)}, z_1])$$

and the identity follows from Lemma 2.5 (i) and (ii).

(iii) $\bar{f}_\lambda = \sum(-1)^\sigma \bar{x}_{\sigma(1)} \dots \bar{x}_{\sigma(2p-2)}([\bar{x}_{\sigma(2p-1)}, \bar{x}_{\sigma(2p)}] \mathrm{ad}^{2r+1} \bar{x}_1) =$

$$= -2 \sum(-1)^\sigma z_{\sigma(2)} \dots z_{\sigma(2p-1)} [[z_{\sigma(2p)}, y] \mathrm{ad}^{2r} y, y + z_1],$$

$$f_\lambda^{(1)} = 2^{2r+2} y^{2r+1} \sum(-1)^\sigma z_{\sigma(2)} \dots z_{\sigma(2p-1)} (z_{\sigma(2p)} \circ z_1).$$

Since $\sum(-1)^\sigma z_{\sigma(2)} \dots z_{\sigma(2p-1)}$ is central, we obtain for some $\alpha_\lambda \neq 0$, $\alpha_\lambda \in \mathbb{Q}$,

$$f_\lambda^{(1)} = \alpha_\lambda y^{2r+1} \sum(-1)^\sigma (z_{\sigma(2)} \dots z_{\sigma(2p-1)} z_{\sigma(2p)} z_1 +$$

$$+ z_1 z_{\sigma(2p)} z_{\sigma(2)} \dots z_{\sigma(2p-1)}) = \alpha_\lambda (s_{2p-1}(z_2, \dots, z_{2p}) \circ z_1)$$

(iv) As in (iii)

$$f_\lambda^{(1)} = -2\sum(-1)^\sigma z_{\sigma(2)}\dots z_{\sigma(2p-1)}[[z_{\sigma(2p)},y]\mathrm{ad}^{2r+1}y,z_1] =$$
$$= -2^{2r+3}y^{2r+2}\sum(-1)^\sigma z_{\sigma(2)}\dots z_{\sigma(2p-1)}[z_{\sigma(2p)},z_1] =$$
$$= \alpha_\lambda y^{2r+2}s_{2p}(z_1,\dots,z_{2p})$$

by Lemma 2.5 (iii).

(v) For $r=0$, $f_\lambda^{(2)} = 2s_{2p}(z_1,\dots,z_{2p})s_{2q}(z_1,\dots,z_{2q})$. Let $r>0$. The non-zero summands of $\bar{f}_\lambda$ are for $\tau(2q-1)=1$ or $\tau(2q)=1$ and

$$f_\lambda^{(2)} = -2\sum(-1)^\sigma(-1)^\tau z_{\sigma(1)}\dots z_{\sigma(2p)}z_{\tau(2)}\dots$$
$$\dots z_{\tau(2q-1)}[[z_{\tau(2q)},y]\mathrm{ad}^{2r-1}y,z_1] =$$
$$= -2^{2r+1}y^{2r}\sum(-1)^\sigma(-1)^\tau z_{\sigma(1)}\dots z_{\sigma(2p)}z_{\tau(2)}\dots z_{\tau(2q-1)}[z_{\tau(2q)},z_1].$$

By Lemma 2.5 (iii) and (iv),

$$f_\lambda^{(2)} = 2^{2r+1}q^{-1}y^{2r}s_{2p}(z_1,\dots,z_{2p})s_{2q}(z_1,\dots,z_{2q}) =$$
$$= \alpha_\lambda y^{2r}z_1^2\dots z_{2q}^2 s_{2(p-q)}(z_{2q+1},\dots,z_{2p}).$$

(vi) $f_\lambda^{(2)} = -2s_{2p}(z_1,\dots,z_{2p})\sum(-1)^\tau z_{\tau(2)}\dots$

$$\dots z_{\tau(2q-1)}[z_{\tau(2q)}\mathrm{ad}^{2r+1}y,z_1] =$$
$$= 2^{2r+2}y^{2r+1}s_{2p}(z_1,\dots,z_{2p})\sum(-1)^\tau z_{\tau(2)}\dots z_{\tau(2q-1)}(z_{\tau(2q)}\circ z_1)$$

and $f_\lambda^{(2)}=0$ by Lemma 2.5 (v).

$$f_\lambda^{(0)} = -2^2pys_{2p-1}(z_2,\dots,z_{2p})z_{\tau(2)}\dots z_{\tau(2q-1)}(z_{\tau(2q)}\mathrm{ad}^{2r+2}y) =$$
$$= 2^{2r+4}py^{2r+3}s_{2p-1}(z_2,\dots,z_{2p})s_{2q-1}(z_2,\dots,z_{2q})$$

and we apply Lemma 2.5 (vi).

(vii) $f_\lambda^{(1)} = -2\sum(-1)^\sigma(-1)^\tau z_{\sigma(2)}\dots$

$$\dots z_{\sigma(2p-1)}[(z_{\sigma(2p)}\mathrm{ad}^{2r+1}y),z_{\tau(1)}]z_{\tau(2)}\dots z_{\tau(2q+1)} =$$
$$= 2^{2r+2}y^{2r+1}\sum(-1)^\sigma(-1)^\tau z_{\sigma(2)}\dots$$
$$\dots z_{\sigma(2p-1)}(z_{\sigma(2p)}\circ z_{\tau(1)})z_{\tau(2)}\dots z_{\tau(2q+1)} =$$
$$= 2^{2r+2}y^{2r+1}(s_{2p-1}(z_2,\dots,z_{2p})\circ s_{2q+1}(z_1,\dots,z_{2q+1}))$$

and we apply Lemma 2.5 (viii). For $p\le q$,

$$s_{2q+1}(z_1,\dots,z_{2q+1}) = -s_{2q+1}(z_2,\dots,z_{2p},z_1,z_{2p+1},\dots,z_{2q+1})$$

and we apply Lemma 2.5 (vi).

(viii) $f_\lambda^{(1)} = -2\sum(-1)^\sigma(-1)^\tau z_{\sigma(2)}\cdots$

$$\cdots z_{\sigma(2p-1)}[(z_{\sigma(2p)}\mathrm{ad}^{2r+2}y), z_1]z_{\tau(2)}\cdots z_{\tau(2q+1)} =$$

$$= -2^{2r+3}y^{2r+2}\sum(-1)^\sigma z_{\sigma(2)}\cdots z_{\sigma(2p-1)}[z_{\sigma(2p)}, z_1]s_{2q}(z_2,\dots,z_{2q+1}).$$

For $p > q$ we apply Lemma 2.5 (iii) and (iv). Let $p \leq q$.

$$f_\lambda^{(1)} = -2^{2r+3}y^{2r+2}[s_{2p-1}(z_2,\dots,z_{2p}), z_1]s_{2q}(z_2,\dots,z_{2q+1}) =$$

$$= -2^{2r+3}y^{2r+2}[s_{2p-1}(z_2,\dots,z_{2p})s_{2q}(z_2,\dots,z_{2q+1}), z_1].$$

By Lemma 2.5 (vii)

$$s_{2p-1}(z_2,\dots,z_{2p})s_{2q}(z_2,\dots,z_{2q+1}) =$$

$$= (2p-1)!(q!)^2((q-p)!(q-p+1)!)^{-1}z_2^2\cdots$$

$$\cdots z_{2p}^2 s_{2(q-p)+1}(z_{2p+1},\dots,z_{2q+1}).$$

Bearing in mind that $z_1z_2^2\dots z_{2p}^2 = -z_2^2\dots z_{2p}^2z_1$, we obtain

$$[s_{2p-1}(z_2,\dots,z_{2p})s_{2q}(z_2,\dots,z_{2q+1}), z_1] =$$

$$= \alpha_\lambda z_2^2\dots z_{2p}^2(s_{2(q-p)+1}(z_{2p+1},\dots,z_{2q+1})\circ z_1).$$

(ix) $\bar{f}_\lambda = -4\sum(-1)^\sigma(-1)^\tau[(z_{\sigma(2)}\mathrm{ad}^{2r+1}y), z_{\tau(2)}][z_{\tau(3)}, y, z_{\sigma(3)}]\times$

$$\times z_{\sigma(4)}\cdots z_{\sigma(2p+1)}z_{\tau(4)}\cdots z_{\tau(2q+1)} = f_\lambda^{(0)}.$$

Hence $f_\lambda^{(2)} = 0$.

$$f_\lambda^{(0)} = -2^{2r+4}y^{2r+2}\sum(-1)^\sigma(-1)^\tau(z_{\sigma(2)}\circ z_{\tau(2)})(z_{\sigma(3)}\circ z_{\tau(3)})\times$$

$$\times z_{\sigma(4)}\cdots z_{\sigma(2p+1)}z_{\tau(4)}\cdots z_{\tau(2q+1)} =$$

$$= -2^{2r+4}y^{2r+2}\sum(-1)^\sigma(-1)^\tau(z_{\sigma(2)}z_{\tau(2)}z_{\tau(3)}z_{\sigma(3)}+$$

$$+z_{\tau(2)}z_{\sigma(2)}z_{\sigma(3)}z_{\tau(3)} + z_{\tau(2)}z_{\sigma(2)}z_{\tau(3)}z_{\sigma(3)}+$$

$$+z_{\sigma(2)}z_{\tau(2)}z_{\sigma(3)}z_{\tau(3)})z_{\sigma(4)}\cdots z_{\sigma(2p+1)}z_{\tau(4)}\cdots z_{\tau(2q+1)}.$$

Since $\sum(-1)^\tau z_{\tau(2)}z_{\tau(3)}$ and $\sum(-1)^\sigma z_{\sigma(2)}z_{\sigma(3)}$ are central elements, we obtain

$$\sum(-1)^\sigma(-1)^\tau(z_{\sigma(2)}z_{\tau(2)}z_{\tau(3)}z_{\sigma(3)} + z_{\tau(2)}z_{\sigma(2)}z_{\sigma(3)}z_{\tau(3)})\times$$

$$\times z_{\sigma(4)}\cdots z_{\sigma(2p+1)}z_{\tau(4)}\cdots z_{\tau(2q+1)} =$$

$$= 2s_{2p}(z_2,\dots,z_{2p+1})s_{2q}(z_2,\dots,z_{2q+1}) =$$

$$= 2(2q)!(p!)^2((p-q)!)^{-2}z_2^2\dots z_{2q+1}^2 s_{2(p-q)}(z_{2q+2},\dots,z_{2p+1}).$$

$$\sum(-1)^{\sigma}(-1)^{\tau}z_{\sigma(2)}z_{\tau(2)}z_{\sigma(3)}z_{\tau(3)}z_{\sigma(4)}\dots z_{\sigma(2p+1)}z_{\tau(4)}\dots z_{\tau(2q+1)}=$$

$$=\sum(-1)^{\sigma\tau}(z_{\sigma\tau(2)}z_{\tau(2)}z_{\sigma\tau(3)}\dots z_{\sigma\tau(2p+1)})z_{\tau(3)}\dots z_{\tau(2q+1)}=$$

$$=(p+1)\sum z_{\tau(2)}^2 s_{2p-1}(z_{\tau(3)},\dots,z_{\tau(2p+1)})z_{\tau(3)}\dots z_{\tau(2q+1)}=$$

$$=(p+1)p!(p-1)!((p-q)!)^{-2}\sum z_{\tau(2)}^2\dots$$

$$\dots z_{\tau(2q+1)}^2 s_{2(p-q)}(z_{2q+2},\dots,z_{2p+1})=$$

$$=(2q)!(p+1)!(p-1)!((p-q)!)^{-2}z_2^2\dots z_{2q+1}^2 s_{2(p-q)}(z_{2q+2},\dots,z_{2p+1});$$

$$\sum(-1)^{\sigma}(-1)^{\tau}z_{\tau(2)}z_{\sigma(2)}z_{\tau(3)}z_{\sigma(3)}\dots z_{\sigma(2p+1)}z_{\tau(4)}\dots z_{\tau(2q+1)}=$$

$$=-\sum(-1)^{\sigma\tau}z_{\tau(3)}(z_{\sigma\tau(2)}z_{\tau(2)}z_{\sigma\tau(3)}\dots z_{\sigma\tau(2p+1)})z_{\tau(4)}\dots z_{\tau(2q+1)}=$$

$$=-(p+1)\sum z_{\tau(3)}z_{\tau(2)}^2 s_{2p-1}(z_{\tau(3)},\dots,z_{\tau(2p+1)})z_{\tau(4)}\dots z_{\tau(2q+1)}=$$

$$=(p+1)\sum z_{\tau(2)}^2 z_{\tau(3)}\dots z_{\tau(2q+1)}s_{2p-1}(z_{\tau(3)},\dots,z_{\tau(2p+1)})=$$

$$=(2q)!(p+1)!(p-1)!((p-q)!)^{-2}z_2^2\dots z_{2q+1}^2 s_{2(p-q)}(z_{2q+2},\dots,z_{2p+1});$$

$$f_\lambda^{(0)}=-2^{2r+5}y^{2r+2}(2q)!((p!)^2+(p+1)!(p-1)!)((p-q)!)^{-2}\times$$

$$\times z_2^2\dots z_{2q+1}^2 s_{2(p-q)}(z_{2q+2},\dots,z_{2p+1}).$$

(x) $f_\lambda^{(2)}=\sum_{\tau(2q+1)\neq 1}(-1)^{\sigma}(-1)^{\tau}z_{\sigma(1)}\dots z_{\sigma(2p)}\times$

$$\times z_{\tau(1)}\dots z_{\tau(2q)}[(z_{\tau(2q+1)}\mathrm{ad}^{2r+1}y),z_{\sigma(2p+1)}]=$$

$$=-2^{2r+1}y^{2r+1}\sum_{\tau(2q+1)\neq 1}(-1)^{\sigma}(-1)^{\tau}z_{\sigma(1)}\dots$$

$$\dots z_{\sigma(2p)}z_{\tau(1)}\dots z_{\tau(2q)}(z_{\tau(2q+1)}\circ z_{\sigma(2p+1)})=$$

$$=-2^{2r+1}y^{2r+1}\sum_{\tau(2q+1)\neq 1}(-1)^{\tau}(z_{\tau(1)}\dots z_{\tau(2q+1)}\circ s_{2p+1}(z_1,\dots,z_{2p+1}));$$

$$\sum_{\tau(2q+1)\neq 1}(-1)^{\tau}z_{\tau(1)}\dots z_{\tau(2q+1)}=$$

$$=s_{2q+1}(z_1,\dots,z_{2q+1})-s_{2q}(z_2,\dots,z_{2q})z_1;$$

$$s_{2p+1}(z_1,\dots,z_{2p+1})\circ s_{2q+1}(z_1,\dots,z_{2q+1})=$$

$$= 2(2q+1)!(p+1)!p!((p-q)!)^{-2}z_1^2\dots z_{2q+1}^2 s_{2(p-q)}(z_{2q+2},\dots,z_{2p+1});$$

$$s_{2p+1}(z_1,\dots,z_{2p+1})\circ(s_{2q}(z_2,\dots,z_{2q+1})z_1)=$$

$$=(s_{2p+1}(z_1,\dots,z_{2p+1})\circ z_1)s_{2q}(z_2,\dots,z_{2q+1})=$$

$$=2(p+1)z_1^2 s_{2p}(z_2,\dots,z_{2p+1})s_{2q}(z_2,\dots,z_{2q+1})=$$

$$=2(2q)!(p+1)!p!((p-q)!)^{-2}z_1^2\dots z_{2q+1}^2 s_{2(p-q)}(z_{2q+2},\dots,z_{2p+1}).$$

Hence

$$f_\lambda^{(2)} = -2^{2r+2}y^{2r+1}((2q+1)!(p+1)!p!((p-q)!)^{-2}-$$

$$-(2q)!(p+1)!p!((p-q)!)^{-2})z_1^2\dots z_{2q+1}^2 s_{2(p-q)}(z_{2q+2},\dots,z_{2p+1})=$$

$$=-2^{2r+3}y^{2r+1}q(2q)!(p+1)!p!((p-q)!)^{-2}z_1^2\dots$$

$$\dots z_{2q+1}^2 s_{2(p-q)}(z_{2q+2},\dots,z_{2p+1}).\quad\Diamond$$

The results of Lemma 2.6 can be summarized in the following way.

Lemma 2.7. (i) *If* $a+b+1\equiv c\equiv 0\,(\mathrm{mod}\,2)$, *then*

$$f_\lambda^{(2)} = \alpha_\lambda y^a z_1^2\dots z_{b+1}^2 s_c(z_{b+2},\dots,z_{b+c+1});$$

(ii) *If* $a+b+1\equiv 1,\ c\equiv 0\,(\mathrm{mod}\,2)$, *then*

$$f_\lambda^{(2)}=0,\ f_\lambda^{(0)} = \alpha_\lambda y^{a+2} z_2^2\dots z_{b+1}^2 s_c(z_{b+2},\dots,z_{b+c+1});$$

(iii) *If* $a+b+1\equiv 0,\ c\equiv 1\,(\mathrm{mod}\,2)$, *then*

$$f_\lambda^{(1)} = \alpha_\lambda y^{a+1} z_2^2\dots z_{b+1}^2 s_{c+1}(z_1,z_{b+2},\dots,z_{b+c+1});$$

(iv) *If* $a+b+1\equiv c\equiv 1\,(\mathrm{mod}\,2)$,

$$f_\lambda^{(1)} = \alpha_\lambda y^{a+1} z_2^2\dots z_{b+1}^2 (s_c(z_{b+2},\dots,z_{b+c+1})\circ z_1).$$

Proof. The assertion (i) follows from Lemma 2.6 (ii), (v) and (x); (ii) is a consequence of Lemma 2.6 (i), (vi) and (ix); (iii) is derived from Lemma 2.6 (iv), (vii) and (viii); (iv) from Lemma 2.6 (iii), (vii) and (viii). $\Diamond$

3. Cocharacters and codimensions

In this section we prove the main results of the paper.

Theorem 3.1. *Let* $h_{ij}(\lambda)$ *denote the* (i,j)*-th hook of the Young diagram of the partition* λ. *If* $k\geq l\geq 1$, *then*

$$\Gamma_n(E_{2k}\otimes E_{2l})=\sum M(\lambda)+\varepsilon_n M(1^n),$$

where $\varepsilon_n=1$ *for* n *even and* $n\le 2(k+l)$ *and* $\varepsilon_n=0$ *otherwise and the summation is over all partitions* $\lambda=(a+2,2^b,1^c)$ *of* n *such that* $a\ge 0$, $b+c>0$, $h_{12}(\lambda)=a+b+1\le 2l$ *and one of the following conditions holds:*

(i) $h_{11}(\lambda)+h_{12}(\lambda)-1=2a+2b+c+2<2(k+l)$;

(ii) $h_{11}(\lambda)+h_{12}(\lambda)-1=2(k+l)$ *and* $h_{12}(\lambda)\equiv 0\,(\mathrm{mod}\,2)$.

Proof. Let $M(1^n)\subset\Gamma_n(E_{2k}\otimes E_{2l})$. Then n is even, for example $n=2p$,

$$s_{2p}(x_1,\dots,x_{2p})=2^{-p}\sum[x_{\sigma(1)},x_{\sigma(2)}]\dots[x_{\sigma(2p-1)},x_{\sigma(2p)}]$$

generates $M(1^n)$ and $s_n(u_1,\dots,u_n)\ne 0$ for some $u_i\in E_{2k}\otimes E_{2l}$. As in the proof of Lemma 2.3 we need at least n different generators e_i and $\tilde e_j$ for the elements $u_1,\dots,u_n$, i.e. $n\le 2(k+l)$.

If $n\le 2(k+l)$, then it is easy to see that

$$s_{2p}(e_1,\dots,e_{2k},\tilde e_1,\dots,\tilde e_{2(p-k)})=$$
$$=\binom{p}{k}(2k)!(2(p-k))!e_1\dots e_{2k}\tilde e_1\dots\tilde e_{2(p-k)}\ne 0.$$

Let $M(a+2,2^b,1^c)\subset\Gamma_n(E_{2k}\otimes E_{2l})$, $a\ge 0$, $b+c>0$. In virtue of Lemma 2.3, $2a+2b+c+2\le 2(k+l)$ and $a+b+1\le 2l$.

First, let $a+b+1\le 2l$ and $2a+2b+c+2=2(k+l)$. Hence $c\equiv 0\,(\mathrm{mod}\,2)$. By Lemma 2.4 $f_\lambda=0$ is a polynomial identity for $E_{2k}\otimes E_{2l}$ if and only if $f_\lambda^{(2)}(y+z_1,z_2,\dots,z_{b+c+1})=0$ for all $y\in E_{2k}^{[1]}\otimes E_{2l}^{[1]}$, $z_i\in E_{2k}^{[1]}\otimes E_{2l}^{[0]}\oplus E_{2k}^{[0]}\otimes E_{2l}^{[1]}$. If $h_{12}(\lambda)=a+b+1\equiv 0\,(\mathrm{mod}\,2)$, then Lemma 2.7 (i) gives

$$f_\lambda^{(2)}=\alpha_\lambda y^a z_1^2\dots z_{b+1}^2 s_c(z_{b+2},\dots,z_{b+c+1}).$$

Let $y=e_1\tilde e_1+\dots+e_a\tilde e_a$, $z_1=e_{a+1}+\tilde e_{a+1},\dots,z_{b+1}=e_{a+b+1}+\tilde e_{a+b+1}$. We use even number of generators from each set $\{e_1,\dots,e_{2k}\}$ and $\{\tilde e_1,\dots,\tilde e_{2l}\}$. Hence we still have available even numbers of elements in each set and $s_c(e_{a+b+2},\dots,e_{2k},\tilde e_{a+b+2},\dots,e_{2l})\ne 0$. If $h_{12}(\lambda)\equiv 1\,(\mathrm{mod}\,2)$, then by Lemma 2.7 (ii) $f_\lambda^{(2)}=0$, i.e. $f_\lambda=0$ is a polynomial identity for $E_{2k}\otimes E_{2l}$.

Now, let $a+b+1\le 2l$ and $2a+2b+c+2<2(k+l)$. Depending on the parity of $a+b+1$ and c we consider four different cases.

(1) $a+b+1\equiv c\equiv 0\,(\mathrm{mod}\,2)$. The proof in this case is similar to the case $2a+2b+c+2=2(k+l)$ and $f_\lambda^{(2)}\ne 0$ for suitable $y,z_i\in E_{2k}\otimes E_{2l}$.

(2) $a+b+1\equiv 1$, $c\equiv 0\,(\mathrm{mod}\,2)$. By Lemma 2.7 (ii),

$$f_\lambda^{(0)}=\alpha_\lambda y^{a+2}z_2^2\dots z_{b+1}^2 s_c(z_{b+2},\dots,z_{b+c+1}).$$

Clearly $a+b+1\leq 2l-1$ and $2a+2b+c+2\leq 2(k+l-1)$. Hence we use $a+b+2$ generators of both $E_{2k}\otimes 1$ and $1\otimes E_{2l}$ for $y=e_1\tilde{e}_1+ +\dots+e_{a+2}\tilde{e}_{a+2}$, $z_2=e_{a+3}+\tilde{e}_{a+3},\dots,z_{b+1}=e_{a+b+2}+\tilde{e}_{a+b+2}$ and we still have even sets of generators $\{e_{a+b+3},\dots,e_{2k}\}$, $\{\tilde{e}_{a+b+3},\dots,\tilde{e}_{2l}\}$ in order to obtain $s_c(z_{b+2},\dots,z_{b+c+1})\neq 0$.

(3) $a+b+1\equiv 0$, $c\equiv 1\,(\mathrm{mod}\,2)$. By Lemma 2.7 (iii)

$$f_\lambda^{(1)}=\alpha_\lambda y^{a+1}z_2^2\dots z_{b+1}^2 s_{c+1}(z_1,z_{b+2},\dots,z_{b+c+1}).$$

Let $y=e_1\tilde{e}_1+\dots+e_{a+1}\tilde{e}_{a+1}$, $z_2=e_{a+2}+\tilde{e}_{a+2},\dots,z_{b+1}=e_{a+b+1}+ +\tilde{e}_{a+b+1}$. Then we have left $2k-(a+b+1)\equiv 0\,(\mathrm{mod}\,2)$ elements $e_{a+b+2},\dots,e_{2k}$ and $2l-(a+b+1)\equiv 0\,(\mathrm{mod}\,2)$ elements $\tilde{e}_{a+b+2},\dots,\tilde{e}_{2l}$. Since $2a+2b+c+2\leq 2(k+l)-1$, we can choose $z_1,z_{b+2},\dots,z_{b+c+1}$ in such a way that $s_{c+1}(z_1,z_{b+2},\dots,z_{b+c+1})\neq 0$ and $f_\lambda^{(1)}\neq 0$.

(4) $a+b+1\equiv c\equiv 1\,(\mathrm{mod}\,2)$. By Lemma 2.7 (iv)

$$f_\lambda^{(1)}=\alpha_\lambda y^{a+1}z_2^2\dots z_{b+1}^2 (s_c(z_{b+2},\dots,z_{b+c+1})\circ z_1).$$

Again $y=e_1\tilde{e}_1+\dots+e_{a+1}\tilde{e}_{a+1}$, $z_2=e_{a+2}+\tilde{e}_{a+2},\dots,z_{b+1}=e_{a+b+1}+ +\tilde{e}_{a+b+1}$ and we have on disposal odd number of elements $e_{a+b+2},\dots, e_{2k}\in E_{2k}\otimes 1$ and $\tilde{e}_{a+b+2},\dots,\tilde{e}_{2l}\in 1\otimes E_{2l}$. Since $e_i\circ e_j=0$, $e_i\circ\tilde{e}_j= =2e_i\tilde{e}_j$ and

$$s_{2m+1}(x_1,\dots,x_{2m+1})=\sum(-)^{i-1}s_{2m}(x_1,\dots,\hat{x}_i,\dots,x_{2m+1})x_i,$$

it is easy to see that

$$s_{2(p+q)+1}(e_{i_1},\dots,e_{i_{2p}},\tilde{e}_{j_1},\dots,\tilde{e}_{i_{2q+1}})\circ e_{i_{2p+1}}=$$
$$=\binom{p+q}{p}s_{2p}(e_{i_1},\dots,e_{i_{2p}})(s_{2q+1}(\tilde{e}_{j_1},\dots,\tilde{e}_{i_{2q+1}})\circ e_{i_{2p+1}})=$$
$$=2\binom{p+q}{p}(2p)!(2q+1)!e_{i_1}\dots e_{i_{2p+1}}\tilde{e}_{j_1}\dots\tilde{e}_{i_{2q+1}}$$

and $f_\lambda^{(1)}\neq 0.\Diamond$

Remark 3.2. Using the proper cocharacters of $E_{2k}\otimes E_{2l}$ we can obtain the ordinary cocharacter sequence. For example

$$\Gamma_0(E_2\otimes E_2)=M(0),\ \Gamma_2(E_2\otimes E_2)=M(1^2),$$
$$\Gamma_3(E_2\otimes E_2)=M(2,1),\ \Gamma_4(E_2\otimes E_2)=M(2^2)+M(1^4)$$

and $\Gamma_n(E_2\otimes E_2)=0$ for all other n. Applying Prop. 1.2 (i) we obtain for $n\geq 6$

$$P_n(E_2 \otimes E_2) = M(n) + 2M(n-1,1) + 2M(n-2,2) + 2M(n-2,1^2) + \\ +2M(n-3,2,1) + M(n-3,1^3) + M(n-4,2^2) + M(n-4,1^4).$$

Corollary 3.3. *Let $k \geq l \geq 1$, $k' \geq l' \geq 1$. Then $T(E_{2k} \otimes E_{2l}) \subset \subset T(E_{2k'} \otimes E_{2l'})$ if and only if $k + l \geq k' + l'$ and $l \geq l'$.*

Proof. Since the S_n-module $\Gamma_n(E \otimes E)$ is a sum of pairwise non-isomorphic irreducible submodules, $T(E_{2k} \otimes E_{2l}) \subset T(E_{2k'} \otimes E_{2l'})$ if and only if $\Gamma_n(E_{2k'} \otimes E_{2l'})$ is isomorphic to a submodule of $\Gamma_n(E_{2k} \otimes E_{2l})$ for all n.

Let $k + l \geq k' + l'$ and $l \geq l'$. Applying Th. 3.1 we obtain that every irreducible submodule of $\Gamma_n(E_{2k'} \otimes E_{2l'})$ participates in the decomposition of $\Gamma_n(E_{2k} \otimes E_{2l})$, i.e. $\Gamma_n(E_{2k'} \otimes E_{2l'}) \subset \Gamma_n(E_{2k} \otimes E_{2l})$ and $T(E_{2k} \otimes E_{2l}) \subset T(E_{2k'} \otimes E_{2l'})$.

Let $T(E_{2k} \otimes E_{2l}) \subset T(E_{2k'} \otimes E_{2l'})$. Hence $\Gamma_n(E_{2k'} \otimes E_{2l'}) \subset \subset \Gamma_n(E_{2k} \otimes E_{2l})$ for all n. Since

$$M(2^{b+1}, 1^c) = M(2^{2l'}, 1^{2(k'-l')}) \subset \Gamma_{2(k'+l')}(E_{2k'} \otimes E_{2l'}),$$

Th. 3.1 gives $2b + c + 2 = 2(k' + l') \leq 2(k + l)$, $b + 1 = 2l' \leq 2l$, i.e. $k + l \geq k' + l'$, $l \geq l'$. $\Diamond$

Theorem 3.4.. *Let $l \geq 1$. Then*

$$\Gamma_n(E \otimes E_{2l}) = \sum M(a+2, 2^b, 1^c) + \varepsilon_n M(1^n),$$

where the sum is over all partitions $(a+2, 2^b, 1^c)$ of n, such that $a \geq 0$, $b + c > 0$ and $a + b + 1 \leq 2l$; $\varepsilon_n = 1$ for n even and $\varepsilon_n = 0$ for n odd.

Proof. Considering $\Gamma_n(E \otimes E_{2l})$ and $\Gamma_n(E_{2k} \otimes E_{2l})$ as S_n-submodules of $\Gamma_n(E \otimes E)$ we obtain

$$\Gamma_n(E \otimes E_{2l}) = \cup_{k \geq l} \Gamma_n(E_{2k} \otimes E_{2l}).$$

Hence by Th. 3.1 $M(1^n) \subset \Gamma_n(E \otimes E_{2l})$ for n even. Let $\lambda = (a + +2, 2^b, 1^c) \vdash n$ and let k be large enough. Then the condition $h_{11}(\lambda) + + h_{12}(\lambda) - 1 < 2(k + l)$ from Th. 3.1 is satisfied automatically and $M(\lambda) \subset \Gamma_n(E_{2k} \otimes E_{2l})$ if and only if $h_{12}(\lambda) = a + b + 1 \leq 2l$. $\Diamond$

Theorem 3.5. *Let $k \geq l \geq 1$.*

(i) *The codimension sequence $c_n(E_{2k} \otimes E_{2l})$ is a polynomial with rational coefficients of degree $2(k + l)$ in n.*

(ii) *For $n > 0$*

$$c_n(E \otimes E_{2l}) = 2^{n-1}\xi_l(n) + \eta_l(n),$$

where $\xi_l(n)$ and $\eta_l(n)$ are polynomials with rational coefficients in n, $\deg\xi_l(n) = 2l$, $\deg\eta_l(n) \leq 4l - 1$ *and the leading term of $\xi(n)$ is equal to $((2l)!)^{-1}$.*

Proof. (i) Let $\lambda = (a+2, 2^b, 1^c) \vdash n$ and let $M(\lambda) \subset \Gamma_n(E_{2k} \otimes E_{2l})$. By Th. 3.1, $h_{11}(\lambda) + h_{12}(\lambda) - 1 \leq 2(k+l)$. Since $n \leq h_{11}(\lambda) + h_{12}(\lambda) - 1$, we obtain that $n \leq 2(k+l)$. Similarly, $M(1^n) \subset \Gamma_n(E_{2k} \otimes E_{2l})$ if and only if n is even and $n \leq 2(k+l)$. Hence $\Gamma_{2(k+l)}(E_{2k} \otimes E_{2l}) \neq 0$ and $\Gamma_n(E_{2k} \otimes E_{2l}) = 0$ for $n > 2(k+l)$. Equivalently, $\gamma_{2(k+l)}(E_{2k} \otimes E_{2l}) > 0$ and $\gamma_n(E_{2k} \otimes E_{2l}) = 0$ for $n > 2(k+l)$ and the assertion follows from Prop. 1.2 (ii).

(ii) By Th. 3.4, $\Gamma_n(E \otimes E_{2l}) = \sum M(\lambda) + \varepsilon_n M(1^n)$, where $\lambda = (a+2, 2^b, 1^c) \vdash n$, $a \geq 0$, $b+c > 0$, $a+b+1 \leq 2l$ and $\varepsilon_n = 0, 1$. The dimension of $M(1^n)$ is equal to 1. By Lemma 1.1, for fixed $a+b+1$

$$\dim M(a+2, 2^b, 1^c) = \psi_{ab}(n) = \frac{1}{(a+b+1)!} \dim M(a+1, 1^b) n^{a+b+1} + \dots,$$

where $\psi_{ab}(n) \in \mathbb{Q}[n]$ and $\deg \psi_{ab}(n) = a+b+1$. Hence for $n \geq 4l$

$$\gamma_n = \gamma_n(E \otimes E_{2l}) = \varepsilon_n + \sum_{a+b+1 \leq 4l} \psi_{ab}(n),$$

$\psi_l(n) = \sum \psi_{ab}(n)$ is a polynomial of degree $2l$ and with leading term

$$\tilde{\gamma}_n = \frac{1}{(2l)!} \sum_{p=0}^{2l} \dim M(2l-p, 1^p).$$

The polynomials $\binom{n+m}{m}$, $m = 0, 1, 2, \dots$, form a basis of $\mathbb{Q}[n]$ and we rewrite $\psi_l(n)$ in the form

$$\psi_l(n) = \sum_{m=0}^{2l} \gamma'_m \binom{n+m}{m}$$

for some $\gamma'_m \in \mathbb{Q}$ and

$$\gamma'_{2l} = (2l)! \tilde{\gamma}_n = \sum_{p=0}^{2l} \dim M(2l-p, 1^p).$$

Therefore

$$\gamma_n = \sum_{m=0}^{2l} \gamma'_m \binom{n+m}{m} + \varepsilon_n, \; n \geq 4l,$$

$$\gamma_n = \nu_n + \sum_{m=0}^{2l} \gamma'_m \binom{n+m}{m} + \varepsilon_n, \; \nu_n \in \mathbb{Q}, \; n < 4l,$$

$$\gamma(t)=\gamma(E\otimes E_{2l},t)=\sum_{n\geq 0}\sum_{m=0}^{2l}\gamma'_m\binom{n+m}{m}t^n+\sum_{m=0}^{4l-1}\nu_n t^n+\sum_{n\geq 0}t^{2n}=$$

$$=\sum_{m=0}^{2l}\frac{\gamma'_m}{(1-t)^{m+1}}+\theta_l(t)+\frac{1}{1-t^2},$$

where $\theta_l(t)\in \mathbb{Q}[t]$ and $\deg\theta_l(t)\leq 4l-1$. Applying Prop. 1.2 (iii) we obtain

$$c(t)=c(E\otimes E_{2l},t)=\sum c_n(E\otimes E_{2l})t^n=$$

$$\sum_{m=0}^{2l}\frac{\gamma'_m(1-t)^m}{(1-2t)^{m+1}}+\frac{1}{1-t}\theta_l\left(\frac{t}{1-t}\right)+\frac{1}{2(1-2t)}+\frac{1}{2},$$

$$\frac{(1-t)^m}{(1-2t)^{m+1}}=\frac{(1+(1-2t))^m}{2^m(1-2t)^{m+1}}=\frac{1}{2^m(1-2t)^{m+1}}=\rho_m\left(\frac{1}{1-2t}\right),$$

where $\rho_m(t)\in\mathbb{Q}[t]$, $\deg\rho_m(t)<m$. Similarly

$$\frac{1}{1-t}\theta_l\left(\frac{t}{1-t}\right)=\tau_l\left(\frac{1}{1-t}\right),\ \tau_l(t)=\sum\tau_{lm}t^m\in\mathbb{Q}[t],$$

$$\deg\tau_l(t)\leq 4l-1.$$

Hence

$$c(t)=\frac{\gamma'_{2l}}{2^{2l}(1-2t)^{2l+1}}+\sum_{m=0}^{2l-1}\frac{\gamma''_m}{(1-2t)^{m+1}}+\sum_{m=0}^{4l-1}\frac{\tau_{lm}}{(1-t)^{m+1}}+\frac{1}{2}=$$

$$=\sum_{n\geq 0}\left(\left(\frac{\gamma'_{2l}}{2^{2l}}\binom{n+2l}{2l}+\sum_{m=0}^{2l-1}\gamma''_m\binom{n+m}{m}\right)2^n+\right.$$

$$\left.+\sum_{m=0}^{4l-1}\tau_{lm}\binom{n+m}{m}\right)t^n+\frac{1}{2}$$

and $c_n=\xi_l(n)2^n+\eta_l(n)$, $n>0$, where $\xi_l(n),\eta_l(n)\in\mathbb{Q}[n]$, $\deg\xi_l(n)=$ $=2l$, $\deg\eta_l(n)\leq 4l-1$ and the leading term of $\xi_l(n)$ is equal to

$$\frac{\gamma'_{2l}}{2^{2l}(2l)!}=\frac{1}{2^{2l}(2l)!}\sum_{p=0}^{2l}\dim M(2l-p,1^p).$$

Using the hook formula it is easy to see that

$$\dim M(2l-p,1^p) = \binom{2p-1}{p}, \sum_{p=0}^{2l} \dim M(2l-p,1^p) = 2^{2l-1}$$

and this completes the proof of the theorem. ◊

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