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DUAL EXTREMUM PRINCIPLES FOR A HOMOGENEOUS DIRICHLET PROBLEM FOR A PARABOLIC EQUATION

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Abstract: In most variational principles for time depending problems there is the great disadvantage that the constructed functionals do not have any extremality properties. In this paper we construct two functionals: concave F_a and convex F_b such that if they attain the same critical value at v and w, respectively, then $u = u(v) = u(w)$ is a solution of the parabolic equation with the homogeneous Dirichlet condition. The clue of this construction is an idea of multiplying the equation by partial derivative of function v with respect to independent variable t. We give also a convenient estimation for the solution u by the difference of the values of these corresponding functionals. Our consideration we based on the variational principles of Herrera and Sewell [1] and our result [3] on the variational formulation for the initial-boundary value problem.

1. Preliminaries

The parabolic equation is considered in a bounded domain $Q := = \Omega \times (0;T)$ where $T > 0$ and Ω is an open bounded domain of $\mathbf{R}^m$

with a Lipschitz boundary $\partial\Omega$ (in the sense of Nečas [2]). We denote the parts of boundary ∂Q by $\Omega_0 := \Omega\times\{0\}$, $\Omega_T := \Omega\times\{T\}$, $\Gamma := \partial\Omega\times\times(0;T)$. We define a Sobolev-type space

$$H := \{v\in L^2(Q) : D_t v, D_i v, D_i D_t v \in L^2(Q) \text{ for } i=1,\dots,m\}$$

where D_i, D_t are distributional derivatives. This space is a Hilbert space with the scalar product given by

$$(u,v)_H := \int_Q [uv + D_t u D_t v + D_i u D_i v + D_i D_t u D_i D_t v]\,dx\,dt$$

and the corresponding norm $\|\cdot\|_H$. There and below we shall use the Einstein convention for the sum:

$$D_i u D_i v := \sum_{i=1}^{m} D_i u D_i v \quad \text{and} \quad (D_i u)^2 := D_i u D_i u.$$

For a formal description of the boundary condition on Γ we introduce a closed linear subspace V of H.

Definition. Let $C^\infty_{0x}(\overline{Q})$ be the set of all infinitely differentiable in $\overline{Q}$ functions which vanish in some neighbourhood of $\overline{\Gamma}$. Then V is the closure of $C^\infty_{0x}(\overline{Q})$ in H.

Under the assumption on the boundary $\partial\Omega$ the measure γ on Γ and the space $L^2(\Gamma)$ are well defined and there are the linear and continuous operators of traces (cf. [2]):

$$\mathrm{Tr}_0 \colon H \to H^1(\Omega) \quad \mathrm{Tr}_T \colon H \to H^1(\Omega) \quad \mathrm{Tr}_\Gamma \colon H \to L^2(\Gamma)$$

where $H^1(\Omega) := \{v\in L^2(\Omega) : D_i v \in L^2(\Omega) \text{ for } i=1,\dots,m\}$ is the Sobolev space with the norm $\|v\|_{H^1(\Omega)} := \left(\int_\Omega [(D_i v)^2 + v^2]dx\right)^{\frac{1}{2}}$, and $H^1_0(\Omega) := \{v\in H^1(\Omega) : v = 0 \text{ on } \partial\Omega\}$. We remark that there exists $\mathrm{Tr}_\Gamma(D_t v)\in L^2(\Gamma)$ the trace of the derivative $D_t v$ on Γ. From the definition of the space V it follows (cf. [3]):

Lemma 1. $\mathrm{Tr}_0 v, \mathrm{Tr}_T v \in H^1_0(\Omega)$ *and* $\mathrm{Tr}_\Gamma v = \mathrm{Tr}_\Gamma(D_t v) = 0$ *for all* $v\in V$.

Let $k\geq 0$ be a constant. Let $a, a^*\colon V\times V\to \mathbb{R}$ be bilinear forms given by

(1.1)

$$a(u,v) := \int_Q [D_t u D_t v + D_i u D_i D_t v + k u D_t v]\,dx\,dt + \int_{\Omega_0} [D_i u D_i v + k uv]\,dx;$$

(1.2)

$$a^*(u,v) := \int_Q [D_t u D_t v - D_i u D_i D_t v - k u D_t v] dx dt + \int_{\Omega_T} [D_i u D_i v + k u v] dx.$$

These forms are adjoint i.e.

(1.3) $\qquad a(u,v) = a^*(v,u) \quad \text{for all} \quad u,v \in V.$

The bilinear form $e\colon V \times V \to \mathbf{R}$ given by

(1.4) $$e(u,v) := \frac{1}{2}[a(u,v) + a^*(u,v)]$$

is symmetric and strictly positive. Its integral shape is the following:

$$e(u,v) = \int_Q D_t u D_t v \, dx dt + \frac{1}{2}\int_{\Omega_0} [D_i u D_i v + k u v] dx + \frac{1}{2}\int_{\Omega_T} [D_i u D_i v + k u v] dx.$$

This permits us to introduce a new norm in the space V.

Definition. The function $\|v\|_V := \sqrt{e(v,v)}$ is the norm and $e(\cdot,\cdot)$ is a scalar product in the space V.

We remark that from the continuity of the operators of traces $\mathrm{Tr}_0, \mathrm{Tr}_T$ in $(V, \|\cdot\|_H)$ we have the estimate $\|v\|_V \leq C\|v\|_H$.

2. Formulation of the variational problems

For $f_Q \in L^2(Q)$ and $f_0 \in H_0^1(\Omega)$ we define a linear functional $f\colon V \to \mathbf{R}$ by

(2.1) $$< f, v > := \int_Q f_Q D_t v \, dx dt + \int_\Omega [D_i f_0 D_i v + k f_0 v] dx.$$

We have proved in [3] (Th. 1) the equivalence of the variational problem VP:

(2.2) $\qquad$ find $u \in V$ such that $a(u,v) = < f, v > \ (\forall v \in V)$

and the initial-boundary value problem IBV:

(2.3a) $\qquad D_t u - D_i D_i u + k u = f_Q \quad \text{in} \quad L^2(Q);$

(2.3b) $\qquad \mathrm{Tr}_0 u = f_0 \quad \text{in} \quad H_0^1(\Omega);$

(2.3c) $\qquad \mathrm{Tr}_\Gamma u = 0 \quad \text{in} \quad L^2(\Gamma).$

Moreover we have proved (Th. 3) that the variation of the functional $X\colon V \times V \to \mathbf{R}$ defined by

(2.4) $\qquad X(u_1,u_2) := a(u_1,u_2) - < f, u_2 > - < g, u_1 >$

is vanishing on the solutions $(u_1,u_2) \in V \times V$ of the following system of two adjoint variational equations:

$$\text{(2.5a)} \qquad a(u_1, v) = < f, v > \qquad (\forall v \in V)$$

$$\text{(2.5b)} \qquad a^*(u_2, v) = < g, v >$$

where $g\colon V \to \mathbf{R}$ is an arbitrary continuous linear functional. If the functional g is given by

$$\text{(2.6)} \quad < g, v > := \int_Q g_Q D_t v \, dx dt + \int_{\Omega_T} [D_i g_T D_i v + k g_T v] dx \; (\forall v \in V)$$

for $g_Q \in L^2(Q)$ and $g_T \in H_0^1(\Omega)$ then the equation (2.5b) is equivalent to the following terminal-boundary value problem:

$$\begin{aligned} D_t u + D_i D_i u - ku &= g_Q \quad \text{in } L^2(Q), \\ \mathrm{Tr}_T u &= g_T \quad \text{in } H_0^1(\Omega), \\ \mathrm{Tr}_\Gamma u &= 0 \quad \text{in } L^2(\Gamma). \end{aligned}$$

The simplest such functional is $g \equiv 0$.

In [1] Herrera and Sewell have presented the algebraic construction of the affine subspaces $\mathbf{D}_a$, $\mathbf{D}_b$ of an arbitrary vector space on which the functional similar to X is concave and convex respectively. Th. 4.3 from [1] can be written in the following form:

Theorem 1. *Let $X\colon V \times V \to \mathbf{R}$ be given by $X(u_1, u_2) := a(u_1, u_2) - < f, u_2 > - < g, u_1 >$ where the form $a\colon V \times V \to \mathbf{R}$ is bilinear, positive and f, g are the linear functionals on V. Let $\alpha, \beta, \gamma, \delta$ be real numbers such that*

$$\text{(2.7)} \qquad \alpha\beta < 0, \quad \gamma\delta > 0, \quad \alpha\delta - \beta\gamma \neq 0,$$

(2.8)

$$\mathbf{D}_a := \{(u_1, u_2) \in V \times V : \alpha a(u_1, z) - \beta a^*(u_2, z) = < \alpha f - \beta g, z >, \forall z \in V\},$$

(2.9)

$$\mathbf{D}_b := \{(u_1, u_2) \in V \times V : \gamma a(u_1, z) - \delta a^*(u_2, z) = < \gamma f - \delta g, z >, \forall z \in V\},$$

and $\|u\| := \sqrt{a(u, u)}$. Then

(i) *$(u_1, u_2) \in V \times V$ is a solution of system (2.5) iff $(u_1, u_2) \in \mathbf{D}_a \cap \mathbf{D}_b$;*

(ii) *for all $(u_{a1}, u_{a2}) \in \mathbf{D}_a$ and $(u_{b1}, u_{b2}) \in \mathbf{D}_b$*

$$2\,[X(u_{b1}, u_{b2}) - X(u_{a1}, u_{a2})] = m_1 \|u_{a1} - u_{b1}\|^2 + m_2 \|u_{a2} - u_{b2}\|^2$$

where $m_1 := \frac{2\alpha\gamma}{\alpha\delta - \beta\gamma} > 0$; $m_2 := \frac{-2\beta\delta}{\alpha\delta - \beta\gamma} > 0$;

(iii) *if $X(u_{a1}, u_{a2}) = \max_{(u_1, u_2) \in \mathbf{D}_a} X(u_1, u_2) = \min_{(u_1, u_2) \in \mathbf{D}_b} X(u_1, u_2) = X(u_{b1}, u_{b2})$ then $(u_{a1}, u_{a2}) = (u_{b1}, u_{b2})$;*

(iv) *if $(u_1, u_2) \in V \times V$ is a solution of system (2.5) then*

$$2\left[X(u_{b1},u_{b2})-X(u_{a1},u_{a2})\right]=m_1\left[\|u_{a1}-u_1\|^2+\|u_{b1}-u_1\|^2\right]+$$
$$+m_2\left[\|u_{a2}-u_2\|^2+\|u_{b2}-u_2\|^2\right]$$

for all $(u_{a1},u_{a2})\in \mathbf{D}_a$ *and* $(u_{b1},u_{b2})\in \mathbf{D}_b$. ◊

The subspaces $\mathbf{D}_a$ and $\mathbf{D}_b$ are very important in this theorem. Next we shall do some characterizations of them.

3. Dual extremum principles

For a^*, f, g given by (1.2), (2.1), (2.6) we define the functional La_v on V by

(3.1) $$< La_v, z >:= \frac{1}{2\alpha}[a^*(v,z)+ < \alpha f - \beta g, z >]\ (\forall z \in V)$$

and the following auxiliary problem AP($\alpha\beta v$):

(3.2) find $u \in V$ such that $e(u,z) =< La_v, z >\ (\forall z \in V)$.

Definition. A is the set of all elements $v \in V$ such that there exists a solution of the auxiliary problem AP($\alpha\beta v$).

The equality $e(v,v) = \|v\|_V$ implies that the solution of this problem is unique. So, we can define the operator $S_a\colon A \to \mathbf{D}_a$

(3.3) $$S_a(v) := (u_a(v), u_a^*(v))$$

where $u_a(v)$ is a solution of auxiliary problem AP($\alpha\beta v$) and

(3.4) $$u_a^*(v) := \frac{1}{\beta}[v - \alpha u_a(v)].$$

Lemma 2. *The operator* S_a *is a bijection* A *on* $\mathbf{D}_a$.

Proof. From the definition (1.4) of the symmetric form e we have

$$\alpha a(u_1,z) - \beta a^*(u_2,z) = 2\alpha e(u_1,z) - \alpha a^*(u_1,z) - \beta a^*(u_2,z) =$$
$$=2\alpha e(u_1,z)-a^*(\alpha u_1+\beta u_2,z)=2\alpha e(u_1,z)-a^*(v,z)=<\alpha f-\beta g,z>.$$

Hence $(u_1,u_2)\in \mathbf{D}_a \Leftrightarrow u_1$ is a solution AP($\alpha\beta v$) for $v = \alpha u_1+\beta u_2$. The operator S_a is one to one because the mapping $v \mapsto La_v$ is one to one. Indeed, for $La_v = La_w$ we have $< La_v - La_w, z >= \frac{1}{2\alpha}a^*(v-w,z) = 0$ $(\forall z \in V)$ and for $z = v-w$, $a^*(v-w,v-w) = \|v-w\|_V^2 = 0$, so $v = w$. ◊

Using this bijection we define a new functional:

(3.5) $$F_a\colon A \to \mathbf{R} \qquad F_a := X \circ S_a$$

Taking $z = u_1$ in (2.8) we obtain for all $(u_1,u_2)\in \mathbf{D}_a$:

$$(3.6) \qquad a^*(u_2,u_1) = \frac{\alpha}{\beta}a(u_1,u_1) - \frac{1}{\beta} < \alpha f - \beta g, u_1 >$$

and from (1.3-4)

$$(3.7) \qquad a(u_1,u_2) = \frac{\alpha}{\beta}\|u_1\|_V^2 - \frac{1}{\beta} < f, \alpha u_1 > + < g, u_1 > .$$

Therefore for $(u_1,u_2) \in \mathbf{D}_a$ we have

$$(3.8) \qquad X(u_1,u_2) = \frac{\alpha}{\beta}\|u_1\|_V^2 - \frac{1}{\beta} < f, \alpha u_1 + \beta u_2 >$$

and for all $v \in V$

$$(3.9) \qquad F_a(v) = \frac{\alpha}{\beta}\|u_a(v)\|_V^2 - \frac{1}{\beta} < f, v > .$$

Replacing in (3.1–3.9) the numbers α, β by γ, δ we define respectively the auxiliary problem with the functional Lb_v, the set B, the bijection $S_b\colon B \to \mathbf{D}_b$, $S_b(w) := (u_b(w), u_b^*(w))$ and the functional $F_b\colon B \to \mathbf{R}$, $F_b := X \circ S_b$ such that

$$(3.10) \qquad F_b(w) = \frac{\gamma}{\delta}\|u_b(w)\|_V^2 - \frac{1}{\delta} < f, w > .$$

Corollary of Th. 1. *If the numbers* $\alpha, \beta, \gamma, \delta$ *satisfy* (2.7) *the functional* F_a *is concave,* F_b *is convex and* $F_a(v) \le F_b(w)$ *for all* $v \in A$ *and* $w \in B$.

The inequality is obvious from the definitions of F_a, F_b and from Th. 1 (ii). ◊

Definition. We call *dual extremal principles* the following extremal problems

$$(3.11) \qquad \text{find } \overline{v} \in A \text{ such that } F_a(\overline{v}) = \operatorname{Sup} F_a$$

where $\operatorname{Sup} F_a$ is the number $\sup\limits_{v \in A} F_a(v)$;

$$(3.12) \qquad \text{find } \overline{w} \in B \text{ such that } F_b(\overline{w}) = \operatorname{Inf} F_b$$

where $\operatorname{Inf} F_b$ is the number $\inf\limits_{w \in B} F_b(w)$.

Because $\operatorname{Sup} F_a = \operatorname{Sup} X\big|_{\mathbf{D}_a}$ and $\operatorname{Inf} F_b = \operatorname{Inf} X\big|_{\mathbf{D}_b}$ we can formulate two theorems.

Theorem 2. *If the dual extremal principles* (3.11–3.12) *have the solutions* $\overline{v} \in A$ *and* $\overline{w} \in B$ *such that* $F_a(\overline{v}) = F_b(\overline{w})$ *then the element* $(u_1,u_2) = S_a(\overline{v}) = S_b(\overline{w})$ *is a solution of the system of variational equations* (2.5) *and conversely if* (u_1,u_2) *is a solution of the system then the dual extremum principles have the solution* $\overline{v}$ *and* $\overline{w}$ *respectively and* $(u_1,u_2) = S_a(\overline{v}) = S_b(\overline{w})$.

Proof. If the dual extremum principles have the solution $\overline{v}$ and $\overline{w}$ such that $F_a(\overline{v}) = F_b(\overline{w})$ then from (iii) of Th. 1 we obtain $(u_1, u_2) := = S_a(\overline{v}) = S_b(\overline{w}) \in \mathbf{D}_a \cap \mathbf{D}_b$ and from (i) this element (u_1, u_2) is a solution of the system (2.5).

Let (u_1, u_2) be a solution of system (2.5). Then from (i) it results that (u_1, u_2) belongs to $\mathbf{D}_a \cap \mathbf{D}_b$. Hence for $\overline{v} := \alpha u_1 + \beta u_2 \in A$ and $\overline{w} := \gamma u_1 + \delta u_2 \in B$ the equation $F_a(\overline{v}) = X(u_1, u_2) = F_b(\overline{w})$ is satisfied. This means that $\overline{v}$ and $\overline{w}$ are the solutions of the dual extremum principles (3.11–3.12). $\Diamond$

Theorem 3. *For the solution (u_1, u_2) of the system (2.5) and any elements $v \in A$ and $w \in B$ there is the estimation*

$$2\,[F_b(w) - F_a(v)] = m_1 \left[\|u_a(v) - u_1\|_V^2 + \|u_b(w) - u_1\|_V^2\right] + \\ + m_2 \left[\|u_a^*(v) - u_2\|_V^2 + \|u_b^*(w) - u_2\|_V^2\right]$$

where $(u_a(v), u_a^*(v)) = S_a(v)$ and $(u_b(w), u_b^*(w)) = S_a(w)$.

Proof. It is obvious from (iv) of Th. 1. $\Diamond$

Corollary . *The element $u_1 = u_a(\overline{v}) = u_b(\overline{w})$ for the solution of the dual extremum principles such that $F_a(\overline{v}) = F_b(\overline{w})$ is a solution of the initial-boundary problems (2.3) and the estimation*

$$m_1 \left[\|u_a(v) - u_1\|_V^2 + \|u_b(w) - u_1\|_V^2\right] \leq 2\,[F_b(w) - F_a(v)]$$

is satisfied for all $v \in V$ and $w \in V$.

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