Mathematica Pannonica
4/2 (1993), 191 – 203

ON SOME GENERALISATION OF RECURRENT MANIFOLDS

Stanisław **Ewert-Krzemieniewski**

Instytut Matematyki, Politechnika Szczecińska, 70-310 Szczecin, Al. Piastów 17, Poland

Received September 1991

AMS Subject Classification: 53 B 20, 53 B 30

Keywords: Pseudo-Riemannian geometry, recurrent manifolds, conformally flat manifolds, subprojective spaces.

Abstract: The paper deals with some generalisation of recurrent manifolds. First, a few general statements are proved. Next, conformally flat manifolds are considered and the local structure theorem is proved. It appears that the manifolds under consideration belong to the class of subprojective spaces.

1. Introduction

Investigating conformally flat Riemannian manifolds of class one, i.e. manifolds characterized by the property that at least $n-1$ principal normal curvatures — the eigenvalues of the second fundamental form — are equal to one another, R. N. Sen and M. C. Chaki ([5]) found that if the remaining one is zero, then the curvature tensor satisfies

(1) $$R_{hijk,l} = 2a_l R_{hijk} + a_h R_{lijk} + a_i R_{hljk} + a_j R_{hilk} + a_k R_{hijl},$$

where the "comma" denotes covariant derivative with respect to the metric. Hereafter, Riemannian manifolds with condition (1) imposed on the curvature tensor were examined by M. C. Chaki ([1]) and M. C. Chaki and U. C. De ([2]). The first author called such manifolds *pseudo symmetric* since, as he claimed, the *locally symmetric* manifold satisfies (1) with $a_l = 0$.

In the present paper we shall consider a Riemannian manifold with not necessarily definite metric g whose curvature tensor satisfies (1). First of all we shall show that on a recurrent manifold, i.e. on a manifold whose curvature tensor satisfies

(2) $$R_{hijk/l}R_{pqrs} - R_{hijk}R_{pqrs/l} = 0,$$

the condition (1) holds at each point where the curvature tensor does not vanish. We shall also give further motivation for consideration of the condition (1). Moreover, we shall prove that on a neighbourhood of a generic point the vector a_l is a gradient. Then conformally flat manifolds satisfying (1) will be considered. We shall give necessary and sufficient conditions so that conformally flat manifold satisfies (1). Finally, the local structure theorem will be proved.

Throughout the paper all manifolds under consideration are assumed to be smooth connected Hausdorff manifolds and their metrics need not be definite. In the sequel we shall use the following lemmas.

Lemma 1 ([6]). *The curvature tensor of an arbitrary manifold (M, g) satisfies the equation*

$$R_{hijk/[lm]} + R_{jklm/[hi]} + R_{lmhi/[jk]} = 0.$$

Lemma 2 ([3], Lemma 1). *Let M be a Riemannian manifold of dimension $n \geq 3$. If B_{hijk} is a tensor field on M such that*

$$B_{hijk} = -B_{ihjk} = B_{jkhi}, \quad B_{hijk} + B_{hjki} + B_{hkij} = 0,$$

$$B_{hijk/[lm]} = 0,$$

and a_l, A_l are vector fields on M satisfying

$$a_r R^r{}_{ijk} = g_{ij}A_k - g_{ik}A_j,$$

then

$$A_l \left[B_{hijk} - \frac{S}{n(n-1)}(g_{ij}g_{hk} - g_{ik}g_{hj}) \right] = 0,$$

where $S = B_{pqrs}g^{ps}g^{qr}$.

A space of affine connection is said to be *subprojective* if both:

(a) under the mapping onto pseudo-euclidean space, the image of each geodesic is contained in two-dimensional plane,

(b) all such planes have either a common point or are parallel to each other ([4], p. 164).

Lemma 3 ([4], p. 184). *A Riemannian manifold (M, g) $(\dim M \geq 3)$ is subprojective iff it is conformally flat and*

$$R_{ij} - \frac{R}{2(n-1)} g_{ij} = P(v) g_{ij} + Q(v) v_{,i} v_{,j},$$

where v is some non-constant function.

Lemma 4 ([4], p. 176). *If (M, g) is a subprojective Riemannian manifold, then in a neighbourhood of each point there exists a coordinate system $x^1, \dots, x^n$ such that the metric takes the form either*

$$ds^2 = (dx^1)^2 + p^2(x^1) ds_1^2, \tag{3}$$

where $ds_1^2 = f_{ab} dx^a dx^b$ is a metric of an $(n-1)$-dimensional space of constant curvature $\overline{R}$, or

$$ds^2 = 2dx^1 dx^2 + p^2(x^1) ds_2^2, \tag{4}$$

where $ds_2^2 = h_{ab} dx^a dx^b$ is a metric of an $(n-2)$-dimensional pseudo-euclidean space.

2. General results

Proposition 1. *If the curvature tensor of the manifold M satisfies*

$$R_{pqrs,t} = b_t R_{pqrs}, \tag{5}$$

then the relation (1) holds on M, where $a_l = \frac{1}{4} b_l$.

Proof. From (5), in virtue of the Bianchi identity, we obtain

$$b_t R_{pqrs} = b_s R_{pqrt} + b_r R_{pqts}. \tag{6}$$

On the other hand, we have

$$R_{hijk,l} = \frac{2}{4} b_l R_{hijk} + \frac{1}{4} b_l R_{hijk} + \frac{1}{4} b_l R_{hijk}.$$

Applying (6) to the second and to the third component on the left hand side of the above equation, respectively to the indices (h, i, l) and (j, k, l), we easily obtain (1). ◊

Proposition 2. *If*

$$R_{hijk,l} = \sum_p \overset{p}{v}_{i_1} R_{i_2 i_3 i_4 i_5}, \tag{7}$$

where the sum includes all permutation p of the indices (h, i, j, k, l) and $\left\{ \overset{p}{v} = \left(\overset{p}{v}_1, \dots, \overset{p}{v}_n \right) \right\}$ is a set of some vectors, then there exists a vector a_l such that relation (1) holds.

Proof. It is easy to verify that there exist vectors a_l, b_l, such that

$$\sum_q \overset{q}{u}_l R_{i_1 i_2 i_3 i_4} = a_l R_{hijk} + b_l R_{hkij},$$

where the sum includes all permutations q of the indices (h, i, j, k). Hence the equation (7) takes the form

$$\begin{aligned} R_{hijk,l} = a_l R_{hijk} + b_l R_{hkij} + c_h R_{lijk} + d_h R_{lkij} + e_i R_{hljk} + \\ + f_i R_{hklj} + g_j R_{hilk} + h_j R_{hkil} + i_k R_{hijl} + j_k R_{hlij} \end{aligned} \tag{8}$$

for some vectors $a_l, \ldots, j_l$. Changing in (8) indices (h, i, j, k) into (k, j, i, h) respectively and adding the obtained equation to (8) we find

$$\begin{aligned} 2R_{hijk,l} = 2(a_l R_{hijk} + b_l R_{hkij}) + \\ + A_h R_{lijk} + B_h R_{lkij} + C_i R_{hljk} + D_i R_{hklj} + \\ + A_k R_{hijl} + B_k R_{hlij} + C_j R_{hilk} + D_j R_{hkil} \end{aligned} \tag{9}$$

for vectors a_l, b_l and some vectors A_l, B_l, C_l, D_l. Alternating (9) in (h, i) and the resulting equation in (j, k) we get

$$8R_{hijk,l} = w_l R_{hijk} + t_h R_{lijk} + t_i R_{hljk} + t_j R_{hilk} + t_k R_{hijl}, \tag{10}$$

where $t_h = 2A_h + 2C_h - B_h - D_h$, $w_l = 8a_l - 4b_l$. Permuting in (10) indices (h, i, l) and adding the obtained equations to (10), we find

$$t_h R_{lijk} + t_i R_{hljk} = -\frac{1}{2}(w_l R_{hijk} + w_h R_{iljk} + w_i R_{lhjk}) + t_l R_{hijk}.$$

Analogically, permuting in (10) indices (j, k, l), we obtain

$$t_j R_{hilk} + t_k R_{hijl} = -\frac{1}{2}(w_l R_{hijk} + w_j R_{hikl} + w_k R_{hilj}) + t_l R_{hijk}.$$

Substituting the above equalities into (10) we have

$$8R_{hijk,l} = 2t_l R_{hijk} + \frac{1}{2}(w_h R_{lijk} + w_i R_{hljk} + w_j R_{hilk} + w_k R_{hijl}),$$

whence, in virtue of (10), follows relation (1). ◊

Proposition 3. *If on a manifold M relation* (1) *holds, then*

$$(a_{l,m} - a_{m,l}) R_{hijk} = 0.$$

Moreover,

$$\begin{aligned} R_{hijk,[lm]} = \\ = A_{hm} R_{lijk} - A_{hl} R_{mijk} + A_{im} R_{hljk} - A_{il} R_{hmjk} + \\ + A_{jm} R_{hilk} - A_{jl} R_{himk} + A_{km} R_{hijl} - A_{kl} R_{hijm}, \end{aligned} \tag{11}$$

where $A_{hm} = a_{h,m} - a_h a_m$.

Proof. Differentiating covariantly (1) we get

$$(12)\quad \begin{aligned} R_{hijkl[lm]} =& 2(A_{lm} - A_{ml})R_{hijk} + \\ & + A_{hm}R_{lijk} - A_{hl}R_{mijk} + A_{im}R_{hljk} - A_{il}R_{hmjk} + \\ & + A_{jm}R_{hilk} - A_{jl}R_{himk} + A_{km}R_{hijl} - A_{kl}R_{hijm}. \end{aligned}$$

Permuting cyclically (12) in pairs (h,i), (j,k), (l,m), adding the resulting equations to (12) and applying Lemma 1 we obtain

$$(13)\quad \begin{aligned} & 2(B_{lm}R_{hijk} + B_{hi}R_{jklm} + B_{jk}R_{lmhi}) + \\ & + B_{hm}R_{lijk} - B_{hl}R_{mijk} + B_{im}R_{hljk} - B_{il}R_{hmjk} + \\ & + B_{jm}R_{hilk} - B_{jl}R_{himk} + B_{km}R_{hijl} - B_{kl}R_{hijm} + \\ & + B_{ji}R_{hklm} - B_{jh}R_{iklm} + B_{ki}R_{jhlm} - B_{kh}R_{jilm} = 0, \end{aligned}$$

where $B_{lm} = a_{l/m} - a_{m/l} = -B_{ml}$.

Suppose that for some (p,q) inequality $B_{pq} \neq 0$ holds. Putting $h = j = l = p$, $i = k = m = q$ into (13) we obtain $R_{pqpq} = 0$. Hence, putting $h = j = l = p$, $i = k = q$ we find $R_{pqpm} = 0$ for an arbitrary m. Therefore, putting $h = j = p$, $i = k = q$ we get $R_{pqlm} = 0$ for arbitrary l, m. Moreover, for $h = j = p$ and $i = q$ we have

$$(14)\quad 3B_{pq}R_{pklm} + B_{pm}R_{lqpk} - B_{pl}R_{mqpk} + B_{qm}R_{plpk} - B_{ql}R_{pmpk} = 0.$$

Substituting in (14) $l = p$ we find

$$(15)\qquad R_{pkpm} = 0$$

for arbitrary k, m. Then, putting in (14) $m = q$, we get $R_{pklq} = 0$ for arbitrary k, l, which, together with (15) and (14) yields $R_{pklm} = 0$ for arbitrary k, l, m. Finally, putting in (13) $h = p$ and $i = q$ we obtain $R_{jklm} = 0$ for all j, k, l, m. Last of all, (11) results from (12) and the first part of our proposition. ◊

From Prop. 3 we obtain

Theorem 1. *Let (M,g) be a manifold whose curvature tensor does not vanish on a dense subset of M. If on (M,g) relation* (1) *holds, then the vector a_l is locally a gradient.*

3. Conformally flat manifolds

Let M ($\dim M \geq 3$) be a conformally flat manifold. Then on M the following well-known relations hold:

$$R_{hijk} = \frac{1}{n-2}\left[g_{ij}R_{hk} - g_{ik}R_{hj} + g_{hk}R_{ij} - g_{hj}R_{ik}\right] - \tag{16}$$
$$- \frac{R}{(n-1)(n-2)}(g_{ij}g_{hk} - g_{ik}g_{hj}),$$

$$R_{ij,k} - R_{ik,j} = \frac{1}{2(n-1)}(g_{ij}R_{,k} - g_{ik}R_{,j}). \tag{17}$$

Theorem 2. *Let M ($\dim M \geq 3$) be a conformally flat manifold whose curvature tensor satisfies (1). If $a_l(x) \neq 0$ ($x \in M$), then there exists a neighbourhood of x such that*

$$R_{ij} = Fg_{ij} + Ha_ia_j, \tag{18}$$

F, H being functions, and

$$R_{hk,l} = 2F(a_hg_{lk} + a_kg_{hl}) + 4Ha_ha_ka_l + \tag{19}$$
$$+ \frac{R}{n-1}(2g_{hk}a_l - g_{lk}a_h - g_{hl}a_k).$$

Proof. From (1), by contraction with g^{ij}, we obtain

$$R_{hk,l} = 2a_lR_{hk} + a_hR_{lk} + a_kR_{hl} + a_rR^r{}_{khl} + a_rR^r{}_{hkl}, \tag{20}$$

whence

$$R_{,l} = 2a_lR + 4A_l, \tag{21}$$

where $A_l = a_rR^r{}_l$. Moreover, we have

$$R_{ij,k} - R_{ik,j} = 3a_rR^r{}_{ijk} + R_{ij}a_k - R_{ik}a_j. \tag{22}$$

Substituting (22) and (21) into (17) we find

$$3a_rR^r{}_{ijk} + R_{ij}a_k - R_{ik}a_j - \tag{23}$$
$$- \frac{1}{n-1}\left[g_{ij}(Ra_k + 2A_k) - g_{ij}(Ra_j + 2A_j)\right] = 0.$$

On the other hand, transvecting (16) with a^h, we have

$$a_rR^r{}_{ijk} = \frac{1}{n-2}[g_{ij}A_k - g_{ik}A_j + R_{ij}a_k - R_{ik}a_j] - \tag{24}$$
$$- \frac{R}{(n-1)(n-2)}(g_{ij}a_k - g_{ik}a_j).$$

Substituting (24) into (23) we obtain

$$R_{ij}a_k - R_{ik}a_j = \frac{R}{n-1}(g_{ij}a_k - g_{ik}a_j) - \frac{1}{n-1}(g_{ij}A_k - g_{ik}A_j). \tag{25}$$

Moreover, substituting (25) into (23), we get

(26) $$a_r R^r{}_{ijk} = \frac{1}{n-1}(g_{ij}A_k - g_{ik}A_j),$$

which, by transvection with a^i, results in

(27) $$a_j A_k = a_k A_j.$$

Now, at each point where a_l does not vanish, we can choose a vector z^j satisfying $a_r z^r = 1$. Transvecting (27) with z^j we find

(28) $$A_k = a_k A_r z^r.$$

Thus, transvecting (25) with z^k and applying (28), we get

(29) $$R_{ij} = F(g_{ij} - z_i a_j) + R_{ir} z^r a_j,$$

where $A_r z^r + (n-1)F = R$. Hence, in virtue of the symmetry of R_{ij}, relation (18) results from (29).

Finally, substituting (18), (26) and (28) into (20) and eliminating $A_r z^r$, we obtain (19). ◊

Theorem 3. *Let M ($\dim M \geq 3$) be a conformally flat and semi-symmetric (i.e. $R_{hijk/[lm]} = 0$) manifold and its curvature tensor satisfies (1). Then relation $a_r R^r{}_m = 0$ holds on M.*

Proof. We can assume that $a_l(x) \neq 0$. Then, in view of the equality (26), from Lemma 2 results

$$A_l\Big[R_{hijk} - \frac{R}{n(n-1)}(g_{ij}g_{hk} - g_{ik}g_{hj}\Big] = 0.$$

Suppose that $A_l(y) \neq 0$ for some $y \in M$. Then, in some neighbourhood $U \ni y$, the scalar curvature is constant, whence, in virtue of (21), we have $a_l R + 2A_l = 0$. On the other hand, the relation $R_{ij} = \frac{R}{n}g_{ij}$ holds on U, therefore $A_j = \frac{R}{n}a_j$. Hence A_j must vanish on U, which is a contradiction. This completes the proof. ◊

Corollary. *Under assumptions of* Th. 3, *if $a_l \neq 0$, then*

$$R_{ij} = \frac{R}{n-1}g_{ij} + Ha_i a_j,$$

$$R_{hk/l} = \frac{R}{n-1}(2g_{hk}a_l + g_{lk}a_h + g_{hl}a_k) + 4Ha_h a_k a_l.$$

Theorem 4. *Let M ($\dim M \geq 3$) be a manifold whose Ricci tensor is of the form* (18) *and satisfies* (19)*. Then we have on M*

(30) $$R_{hk/l} - R_{hl/k} - \frac{1}{2(n-1)}(g_{hk}R_{/l} - g_{hl}R_{/k}) = 0.$$

Proof. To prove (30) it is enough to express $R_{/l}$ in terms of R and F.

By contraction of (18) with g^{ij} we get $R = Fn + Ha_r a^r$ and contraction of (19) with g^{hk} yields

$$R_{/l} = [4F + 4Ha_r a^r + 2R]a_l.$$

Hence

$$R_{/l} = [6R - 4(n-1)F]a_l. \tag{31}$$

Applying (19) and (31), by direct calculations, we check (30). ◊

Theorem 5. *Let M be a manifold whose Weyl conformal curvature tensor vanishes and the Ricci tensor and its covariant derivative are of the forms* (18) *and* (19). *Then the relation* (1) *holds on M.*

Proof. From (16) and (18) it follows

$$R_{hijk} = \frac{1}{n-2}\Big[\Big(2F - \frac{R}{n-1}\Big)(g_{ij}g_{hk} - g_{ik}g_{hj}) +$$
$$+ H(g_{ij}a_h a_k - g_{ik}a_h a_j + g_{hk}a_i a_j - g_{hj}a_i a_k)\Big].$$

On the other hand, differentiating covariantly (16) and substituting (19) and (31), by a straightforward calculation we check that condition (1) is satisfied on M. ◊

Proposition 4. *Let (M,g) be a conformally flat manifold whose curvature tensor satisfies* (1). *If $a_l(x) \neq 0$, $R_{hijk}(x) \neq 0$ $(x \in M)$, then there exists a neighbourhood $U \ni x$ and a function a defined on U, satisfying $a_{/l} = a_l$, such that $F = F(a)$, $H = H(a)$, $B = a_r a^r = B(a)$, where F, H are given by* (18).

Proof. According to Prop. 3, if $R_{hijk}(x) \neq 0$, then there exists a neighbourhood $U \ni x$ and a function a defined on U such that $a_{/l} = a_l$. We shall prove that both the functions F, H defined by (18) as well as $B = a_r a^r$ depend on a.

Differentiating covariantly (18) and substituting to (17), in virtue of Prop. 3, we get

$$g_{ij}F_k + H_k a_i a_j + Ha_{i/k}a_j - g_{ik}F_j - H_j a_i a_k - Ha_{i/j}a_k = \tag{32}$$

$$= \frac{1}{2(n-1)}[g_{ij}(nF_k + H_k B + 2Ha_{r/k}a^r) - g_{ik}(nF_j + H_j B + 2Ha_{r/j}a^r)],$$

where $F_k = F_{/k}$, $H_k = H_{/k}$ and $B_{/k} = 2a_{r/k}a^r$. Contracting (32) with g^{ij} we obtain

$$\frac{n-2}{2}F_k = -\frac{1}{2}BH_k + H_r a^r a_k + Ha_{r/s}g^{rs}a_k, \tag{33}$$

whence, multiplying by a_i and alternating in (i,k), we find

(34) $$(n-2)(a_iF_k - a_kF_i) = -B(a_iH_k - a_kH_i).$$

Moreover, transvecting (32) with a^i and applying (33) we have

(35) $$B(a_jH_k - a_kH_j) = -H(a_ja_{r/k}a^r - a_ka_{r/j}a^r).$$

On the other hand, substituting (18) into the left hand side of (19) we get

(36) $$\begin{aligned} F_lg_{hk} + H_la_ha_k + Ha_{h/l}a_k + Ha_ha_{k/l} &= 2F(a_hg_{lk} + a_kg_{hl}) + \\ &+ 4Ha_ha_ka_l + \frac{nF+HB}{n-1}(2g_{hk}a_l - g_{lk}a_h - g_{hl}a_k), \end{aligned}$$

whence, by contraction with g^{hk},

$$nF_l + BH_l + 2Ha_{r/l}a^r = [(2n+4)F + 6HB]a_l.$$

Multiplying by a_i and alternating in (i,l), in virtue of (35), we obtain

$$n(a_iF_l - a_lF_i) = B(a_iH_l - a_lH_i).$$

Comparing the last result with (34) we find

(37) $$a_iF_l - a_lF_i = 0.$$

Hence $F = F(a)$.

The last result enables us to prove $H = H(a)$. Multiplying (36) by a_m and alternating in (l,m), by the use of (37), we get

(38) $$\begin{aligned} &a_ha_k(H_la_m - H_ma_l) + \\ &+ Ha_k(a_{h/l}a_m - a_{h/m}a_l) + Ha_h(a_{k/l}a_m - a_{k/m}a_l) = \\ &= \left(2F - \frac{R}{n-1}\right)(a_ha_mg_{lk} - a_ha_lg_{mk} + a_ka_mg_{lh} - a_ka_lg_{mh}). \end{aligned}$$

Moreover, multiplying (36) by a_m and alternating in (h,m), in virtue of $F_l = F'a_l$, results in

$$\begin{aligned} \left(F' - \frac{2R}{n-1}\right)a_l(a_mg_{hk} - a_hg_{mk}) + Ha_k(a_{h/l}a_m - a_{m/l}a_h) = \\ = \left(2F - \frac{R}{n-1}\right)a_k(a_mg_{hl} - a_hg_{ml}) \end{aligned}$$

which, by symmetrisation in (k,l), gives

(39) $$H(a_{h/l}a_m - a_{m/l}a_h) = \left(2F - F' + \frac{R}{n-1}\right)(g_{hl}a_m - g_{ml}a_h).$$

Applying (39) to (38) we obtain

$$a_h a_k(H_l a_m - H_m a_l) = \tag{40}$$
$$= \left(F' - \frac{2R}{n-1}\right)(a_h a_m g_{lk} - a_h a_l g_{mk} + a_k a_m g_{lh} - a_k a_l g_{mk}),$$

whence, multiplying by a_p and alternating in (k,p), we get

$$\left(F' - \frac{2R}{n-1}\right)[a_m(g_{lk}a_p - g_{lp}a_k) - a_l(g_{mk}a_p - g_{mp}a_k)] = 0. \tag{41}$$

Suppose that $B \neq 0$. Transvecting (41) with a^p we find

$$F' - \frac{2R}{n-1} = 0. \tag{42}$$

On the other hand, if $B = 0$, then contraction of (41) with g^{lk} yields (42) again. Consequently, in virtue of (40), $H_l a_m - H_m a_l = 0$. Hence $H = H(a)$.

Finally, since $R = nF + HB$, $F = F(a)$, $H = H(a)$, in view of (42), we have $B = B(a)$. ◊

4. The local structure theorem

We are now in a position to prove our main result.

Theorem 6. (i) *Let (M,g) ($\dim M \geq 3$) be a conformally flat manifold whose curvature tensor satisfies (1). If $a_l(x) \neq 0$, $R_{hijk}(x) \neq 0$ ($x \in$ $\in M$), then in some neighbourhood of x there exists a coordinate system $x^1, \ldots, x^n$ such that the metric of M takes the form either (4) where p is a function of x^1 only such that*

$$p''(x) \neq 0, \quad p(x)p'''(x) - p'(x)p''(x) \neq 0, \tag{43}$$

or (3) where p is a function of x^1 only and

$$p'(x)^2 \neq -E \quad or \quad p''(x) \neq 0, \tag{44}$$

$$pp'' - (p')^2 \neq E \quad or \quad pp''' - p'p'' \neq 0 \quad at \quad x, \tag{45}$$

$$p'[pp'p''' + 3(p')^2p'' - 4p(p'')^2] = -E(3p'p'' + pp'''), \tag{46}$$

with that $E = [(n-1)(n-2)]^{-1}\overline{R}$, $\overline{R}$ being the scalar curvature in the metric ds_1^2.

(ii) *Let U be an open subset of $\mathbb{R}^n$ ($n \geq 3$), endowed with the metric g given by (4) where p is a function of x^1 only such that (43) is satisfied on U. Then (U,g) is recurrent but non-locally symmetric conformally flat manifold.*

(iii) *Let U be an open subset of $\mathbb{R}^n$ ($n \geq 3$), endowed with the* *metric g given by* (3) *where p is a function of x^1 only such that* (44)–(46) *hold on U. Then (U,g) is non-recurrent conformally flat manifold satisfying* (1).

Proof. In virtue of Th. 2, Prop. 4 and Lemmas 3 and 4, the local form of the metric must be either (3) or (4). Straightforward computations show us that in the metric (4) the only components of the Christoffel symbols, the curvature tensor and its covariant derivative which may not vanish are

$$\Gamma^c_{ab} = \overline{\Gamma^c}_{ab}, \quad \Gamma^2_{ab} = -pp'h_{ab}, \quad \Gamma^c_{1b} = \frac{p'}{p}\delta^c_b,$$
$$R_{1ab1} = pp''h_{ab}, \quad R_{1ab1/1} = (pp''' - p'p'')h_{ab},$$

where $a, b, c = 3, \ldots, n$ and the dash denotes objects in the metric ds_2^2. Since $a_l(x) \neq 0$ and $R_{hijk}(x) \neq 0$, it follows that (43) hold at x.

On the other hand, in the metric (3) the only components of the Christoffel symbols and the curvature tensor which may not vanish are

$$\Gamma^c_{ab} = \overline{\Gamma^c}_{ab}, \quad \Gamma^1_{ab} = -pp'f_{ab}, \quad \Gamma^c_{1b} = \frac{p'}{p}\delta^c_b,$$
$$R_{abcd} = p^2(E + (p')^2)f_{abcd}, \quad R_{1ab1} = pp''f_{ab}, \tag{47}$$

where $a, b, c \ldots = 2, \ldots, n$, $E = [(n-1)(n-2)]^{-1}\overline{R}$, the dash denotes objects in the metric ds_1^2 and $f_{abcd} = f_{bc}f_{ad} - f_{bd}f_{ac}$. From (47) the inequalities (44) result.

Computing the components of the covariant derivative of the curvature tensor in the metric (3) we find

$$R_{abcd/1} = 2pp'(-E + pp'' - (p')^2)f_{abcd}, \tag{48}$$

$$R_{abcd/e} = 0, \tag{49}$$

$$R_{1bcd/1} = 0, \tag{50}$$

$$R_{1bcd/e} = pp'(-E + pp'' - (p')^2)f_{ebcd}, \tag{51}$$

$$R_{1bc1/1} = (pp''' - p'p'')f_{bc}, \tag{52}$$

$$R_{1bc1/e} = 0. \tag{53}$$

Since the relation (1) holds on M, we also have

$$R_{abcd/1} = 2a_1R_{abcd}, \tag{54}$$

$$R_{abcd/e} = 2a_eR_{abcd} + a_aR_{ebcd} + a_bR_{aecd} + a_cR_{abed} + a_dR_{abce}, \tag{55}$$

(56) $$R_{1bcd/1} = a_c R_{1b1d} + a_d R_{1bc1},$$

(57) $$R_{1bcd/e} = a_1 R_{ebcd},$$

(58) $$R_{1bc1/1} = 4a_1 R_{1bc1},$$

(59) $$R_{1bc1/e} = 2a_e R_{1bc1} + a_b R_{1ec1} + a_c R_{1be1}.$$

Now, from (48) and (54) we have $p'(-E+pp''-(p')^2) = a_1 p(E+(p')^2)$ and from (52) and (58) we get $pp'''-p'p'' = 4a_1 pp''$, whence (46) results.

Finally, comparing (49) with (55) and (53) with (59) and taking into account (44), we infer that $a_e(x) = 0$. Therefore $a_1(x) \neq 0$ must hold. Since p' could not vanish on any open set containing x, we obtain (45). This completes the proof of (i).

The proof of (ii) is obvious. To prove (iii) one has to check that under conditions (44)–(46) the components (47)–(53) satisfied (54)–(59) whereas (2) is not satisfied. ◊

It is well-known that any recurrent manifold is always semi-symmetric. We shall prove that in the case of the manifolds under consideration this property need not hold.

Theorem 7. (i) *Let (M,g) $(\dim M \geq 3)$ be a conformally flat non-recurrent and semi-symmetric manifold whose curvature tensor satisfies* (1). *If $a_l(x) \neq 0$, $R_{hijk}(x) \neq 0$ $(x \in M)$, then in some neighbourhood of x there exists a coordinate system $x^1, \dots, x^n$ such that the metric g takes the form* (3) *where $p = Cx^1 + D$, $C \neq 0$ and D being constants.*

(ii) *Let U be an open subset of $\mathbb{R}^n$ $(n \geq 3)$, endowed with the metric g of the form* (3) *where $p = Cx^1 + D$, $C \neq 0$ and D being constants, $p \neq 0$ everywhere on U. Then (U, g) is a conformally flat non-recurrent and semi-symmetric manifold whose curvature tensor satisfies* (1).

Proof. On the account of Th. 6, in some neighbourhood of x there exists a coordinate system $x^1, \dots, x^n$ such that the metric of the manifold takes the form (3). In the metric (3) the only components of $R_{hk/[lm]}$ which may not vanish are $R_{1a/[1b]} = (n-2)p^{-1}p''(E+(p')^2-pp'')f_{ab}$. Hence we find that the only components of $R_{hijk/[lm]}$ not identically equal to zero are $R_{1cda/[1b]} = pp''(E+(p')^2-pp'')f_{bcda}$. However, $E + (p')^2 - pp'' = 0$ contradicts to (45). This completes the proof of (i). The proof of (ii) is obvious. ◊

From Th. 7 we get

Proposition 5. *There exist semi-symmetric conformally flat manifolds satisfying* (1) *which are not recurrent.*

Proposition 6. *There exist conformally flat manifolds satisfying* (1) *which are neither semi-symmetric nor recurrent.*

Proof. Let U be an open subset of R^n $(n \geq 3)$ determined by the inequalities either $x^1 > B\frac{\pi}{2} - C$ or $x^1 < -B\frac{\pi}{2} - C$, B and C being constants, $B > 0$, endowed with the metric of the form (3), where ds_1^2 is a metric of a flat space. If p is a function of x^1 only such that $p = A \exp\left[\int y(x^1)dx^1\right]$, $A =$ constant, $a > 0$, where $y = y(x^1)$ is given by the equation $\frac{1}{y} + B \operatorname{arctg}(By) = x^1 + C$, then (U, g) is a conformally flat manifold satisfying (1) which is neither recurrent nor semi-symmetric. $\Diamond$

References

[1] CHAKI, M. C.: *Analele Şt. Univ. "Al. I. Cuza" Iaşi* (ser. I: Matematică) **33** (1987), 53–58.

[2] CHAKI, M. C. and DE, U. C.: On pseudo symmetric spaces, *Acta Math. Hung.* **53**/3–4 (1989), 185–190.

[3] GRYCAK, W.: On generalized curvature tensors and symmetric (0, 2)-tensors with a symmetry condition imposed on the second derivative, *Tensor, N. S.* **33** (1979), 150–152.

[4] KRUCHKOVICH, G. J.: O prostranstvah V. F. Kagana, in: V. F. Kagan: Subproektivnye prostranstva, Gosudarstvennoe Izdatelstvo Fiziko-Matematicheskoi Literatury, Moskva, 1961 (in Russian).

[5] SEN, R. N. and CHAKI, M. C.: On curvature restrictions of a certain kind of conformally flat Riemannian space of class one, *Proc. Nat. Inst. Sci. India* **33** (Part A) (1967), 100–102.

[6] WALKER, A. G.: On Ruse's spaces of recurrent curvature, *Proc. Lond. Math. Soc.* **52** (1950), 36–64.