

GEOMETRICAL CONVEXITY AND GENERALIZATIONS OF THE BOHR–MOLLERUP THEOREM ON THE GAMMA FUNCTION

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Abstract: The main result is the following. If $g : (0, \infty) \rightarrow (0, \infty)$ is geometrically convex on an interval (a, ∞) , for some $a \geq 0$, and satisfies the functional equation

$$g(x+1) = xg(x), \quad x \in (0, \infty); \quad g(1) = 1,$$

then g is the Γ function. This result improves the classical Bohr-Mollerup theorem. We also proved that the geometrical convexity of g on (a, ∞) can be replaced by geometrical Jensen convexity on (a, ∞) – i.e. $g(\sqrt{xy}) \leq \sqrt{g(x)g(y)}$ for $x, y > a$ – and some weak regularity conditions.

Introduction

The Euler Γ function is characterized as the only *logarithmically convex* function $g : (0, \infty) \rightarrow (0, \infty)$, satisfying the functional equation

$$(1) \quad g(x+1) = x \cdot g(x), \quad x \in (0, \infty), \quad \text{with } g(1) = 1.$$

This is the well-known theorem of H. Bohr and J. Møllerup [3], pp. 149-164, published in 1922. Nine years later E. Artin [1] gave a very elegant and easy proof of it. An elementary and nice exposition of this proof can be found in Chapter 8 of W. Rudin's book [11]. W. Krull showed in his paper [5], which he called a marginal note to Artin's "*Einführung in die Theorie der Γ Funktion*", that this result can be obtained by characterizing the convex solutions of a class of linear finite difference equations (see also M. Kuczma, [6], p. 128.)

A. E. Mayer [8] showed that in the Bohr-Møllerup theorem the condition of the logarithmical convexity cannot be replaced by that of convexity. In particular, cf. H.-H. Kairies [4], for every sufficiently small $c > 0$, the function $g : (0, \infty) \rightarrow (0, \infty)$ given by

$$g(x) = \Gamma(x) \exp(c \sin 2\pi x), \quad x \in (0, \infty),$$

satisfies (1) and is convex on $(0, \infty)$.

The convexity of a function g is meant in the classical sense. Thus g is *convex* on an interval (a, b) if and only if for each triplet of numbers $x, y, z \in (a, b)$ with $x < y$ and $x \neq z \neq y$ the following inequality holds:

$$\frac{g(x) - g(z)}{x - z} \leq \frac{g(y) - g(z)}{y - z}.$$

Moreover g is *logarithmically convex* means that $\log \circ g$ is convex. We say that a function $g : (0, \infty) \rightarrow (0, \infty)$ is *geometrically convex* if

$$g(x^\lambda \cdot y^{1-\lambda}) \leq g(x)^\lambda \cdot g(y)^{1-\lambda} \quad \text{for all } \lambda \in (0, 1); \quad x, y \in (0, \infty).$$

Obviously g is geometrically convex on $(0, \infty)$ if and only if its exponential conjugate, i.e. the function $\log \circ g \circ \exp$ is convex on \mathbb{R} .

In Section 1 we will present the following theorem:

The only function $g : (0, \infty) \rightarrow (0, \infty)$ satisfying (1) and geometrically convex on a neighbourhood of the infinity is the Γ function.

We also prove that this result essentially improves the Bohr-Møllerup theorem. In this context the above mentioned examples of convex solutions g of equation (1) such that $g \neq \Gamma$ show that the geometrical convexity is a more appropriate characterization of the Γ function than convexity or even the logarithmical convexity.

In Section 2, using some well-known weak conditions which ensure the continuity of Jensen convex functions, we give some characterizations of the Γ function under the assumption of the geometrical Jensen convexity of the function g .

1. The main result

We start this section with the following obvious remarks.

Remark 1. If the function $g : (0, \infty) \rightarrow (0, \infty)$ is a solution of (1) then $\varphi : (0, \infty) \rightarrow \mathbb{R}$, given by $\varphi = \log \circ g$ is a solution of the functional equation

$$(2) \quad \varphi(x + 1) = \log x + \varphi(x), \quad x \in (0, \infty), \quad \text{with } \varphi(1) = 0.$$

From (2) we get, by induction, for all $n \in \mathbb{N}$

$$(3) \quad \varphi(n + 1 + x) = \varphi(x) + \log [x(x + 1) \cdots (x + n)], \quad x \in (0, \infty).$$

Remark 2. A function $g : (0, \infty) \rightarrow (0, \infty)$ is geometrically convex on (a, ∞) , $a \geq 0$, if and only if the function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ defined by $\phi = \log \circ g \circ \exp$ is convex on $(\log a, \infty)$.

Now we can prove

Theorem 1. Suppose that $g : (0, \infty) \rightarrow (0, \infty)$ is a solution of (1) and g is geometrically convex on an interval (a, ∞) for some $a \geq 0$. Then $g \equiv \Gamma$.

Proof. Let $g : (0, \infty) \rightarrow (0, \infty)$ be a solution of (1), geometrically convex on the interval (a, ∞) . Put $\varphi = \log \circ g$ and $\phi = \log \circ g \circ \exp$ as in Remarks 1 and 2, respectively. By Remark 2 the function ϕ is convex on $(\log a, \infty)$. Take arbitrary $n \in \mathbb{N}$ with $n > a$ and $x \in (0, 1)$, and put

$$\begin{aligned} x_1 &= \log n, & x_2 &= \log(n + 1), \\ x_3 &= \log(n + 1 + x), & x_4 &= \log(n + 2). \end{aligned}$$

So we have

$$\log a < x_1 < x_2 < x_3 < x_4.$$

From the convexity of ϕ on the interval $(\log a, \infty)$ follows:

$$\frac{\phi(x_2) - \phi(x_1)}{x_2 - x_1} \leq \frac{\phi(x_3) - \phi(x_2)}{x_3 - x_2} \leq \frac{\phi(x_4) - \phi(x_2)}{x_4 - x_2}.$$

Since $\varphi(n) = \log [(n - 1)!]$, this inequality yields

$$\frac{\log n}{x_2 - x_1} \leq \frac{\varphi(n + 1 + x) - \log n!}{x_3 - x_2} \leq \frac{\log(n + 1)}{x_4 - x_2}.$$

Subtracting from this inequality $\frac{\log n}{x_2 - x_1}$ and multiplying by $(x_3 - x_2) > 0$ yields

$$\begin{aligned} 0 &\leq \varphi(n + 1 + x) - \log n! - \frac{x_3 - x_2}{x_2 - x_1} \log n \leq \\ &\leq \frac{x_3 - x_2}{x_4 - x_2} \log(n + 1) - \frac{x_3 - x_2}{x_2 - x_1} \log n. \end{aligned}$$

Put

$$\Theta_n := \frac{x_3 - x_2}{x_4 - x_2} \log(n+1) - \frac{x_3 - x_2}{x_2 - x_1} \log n.$$

Using (3) and the explicit expressions for the x_i 's, we get

$$0 \leq \varphi(x) - \log \left[\frac{n!}{x(x+1) \cdots (x+n)} \cdot n^{\frac{\log(n+1+x) - \log(n+1)}{\log(n+1) - \log n}} \right] \leq \Theta_n.$$

We will show that $\lim_{n \rightarrow \infty} \Theta_n = 0$. The inequality $x_4 - x_2 > x_3 - x_2$ implies

$$\begin{aligned} 0 \leq \Theta_n &< \frac{x_4 - x_2}{x_4 - x_2} \cdot \log(n+1) - \frac{x_4 - x_2}{x_2 - x_1} \cdot \log n = \\ &= \log(n+1) - \delta(n) \cdot \log n = \log \left[\frac{n}{n^{\delta(n)}} + \frac{1}{n^{\delta(n)}} \right], \end{aligned}$$

where $\delta(n) = \frac{x_4 - x_2}{x_2 - x_1}$. Hence we have

$$\lim_{n \rightarrow \infty} \delta(n) = \lim_{n \rightarrow \infty} \frac{\log(n+2) - \log(n+1)}{\log(n+1) - \log n} = 1.$$

Now $\delta(n) > \frac{n}{n+1}$ by Cauchy's mean value theorem and $x_4 - x_2 = \log \frac{n+2}{n+1} < \log \frac{n+1}{n} = x_2 - x_1$, that is $1 > \delta(n)$. Hence

$$1 < \frac{n}{n^{\delta(n)}} < \frac{n}{n^{n/(n+1)}} = n^{\frac{1}{n+1}} \rightarrow 1 \quad \text{for } n \rightarrow \infty,$$

therefore

$$\begin{aligned} \lim_{n \rightarrow \infty} \log \left[\frac{n}{n^{\delta(n)}} + \frac{1}{n^{\delta(n)}} \right] &= \log \left[\lim_{n \rightarrow \infty} \frac{n}{n^{\delta(n)}} + \lim_{n \rightarrow \infty} \frac{1}{n^{\delta(n)}} \right] = \\ &= \log(1+0) = 0. \end{aligned}$$

This means $\lim_{n \rightarrow \infty} \Theta_n = 0$.

So $\varphi(x)$ and henceforth also $g(x)$ is uniquely defined for each x of the interval $(0, 1)$ and, while $\varphi(1) = 0$ by definition, also at $x = 1$. By the functional equation (1) the function g is uniquely defined on all of $(0, \infty)$. Since we know that the Γ function is geometrically convex on a neighborhood of ∞ (see Remark 4, below) the proof is complete. \diamond

To show the relation between the Bohr-Mollerup theorem and our Th. 1 we need some auxiliary results.

Lemma 1. *Suppose that $g : (a, \infty) \rightarrow (0, \infty)$, $a \geq 0$, is increasing and logarithmically convex on (a, ∞) . Then g is geometrically convex on (a, ∞) .*

Proof. Take arbitrary $x, y \in (a, \infty)$. Since $a < x^\lambda y^{1-\lambda} \leq \lambda x + (1-\lambda)y$ for $\lambda \in (0, 1)$, making use of the monotonicity and convexity of $\log g$, we have

$$\begin{aligned} \log g(x^\lambda y^{1-\lambda}) &\leq \log g(\lambda x + (1-\lambda)y) \leq \\ &\leq \lambda \log g(x) + (1-\lambda) \log g(y) = \log (g(x)^\lambda g(y)^{1-\lambda}). \end{aligned}$$

Hence $g(x^\lambda y^{1-\lambda}) \leq g(x)^\lambda g(y)^{1-\lambda}$, i.e. the function g is geometrically convex on (a, ∞) . \diamond

Remark 3. The function $g : (0, \infty) \rightarrow (0, \infty)$, given by

$$g(x) = \exp(-\sqrt{x+1})$$

is decreasing and logarithmically convex on $(0, \infty)$. Moreover, it is easy to verify that the function $\log \circ g \circ \exp$ is strictly concave on \mathbb{R} , which means that g is strictly geometrically concave on $(0, \infty)$. Thus, in Lemma 1, the supposition that g is increasing turns out to be indispensable.

Lemma 2. *If a function $g : (0, \infty) \rightarrow (0, \infty)$ satisfies (1) and is logarithmically convex on a neighbourhood of ∞ , then there exists an $a \geq 0$ such that g is increasing on (a, ∞) .*

Proof. By assumption $\log \circ g$ is convex on (b, ∞) for some $b \geq 0$. Thus the right derivative $(\log \circ g)'_+(x) = g'_+(x)(g(x))^{-1}$ exists for all $x \in (b, \infty)$ and is an increasing function on (b, ∞) . Suppose now that $g'_+(x) < 0$ for all $x \in (b, \infty)$. Then, of course, g would be decreasing on (b, ∞) . But this is a contradiction because $g(n) = (n-1)!$. Thus there exists an $a \geq b$ such that $g'_+(a)$ is nonnegative. In view of the monotonicity of $(\log \circ g)'_+ = \frac{g'_+}{g}$ we have $g'_+(x) \geq 0$ for all $x \geq a$. Consequently, the function g is increasing on (a, ∞) . \diamond

Remark 4. The Γ function is logarithmically convex on $(0, \infty)$, see e.g. Rudin [11], p. 192. Hence, by Lemma 2 and Lemma 1, Γ is also geometrically convex at least on the interval $(2, \infty)$ (cf. the proof of Lemma 2).

The function g given in Remark 3 shows also that in Lemma 2 the assumption of g to be a solution of (1) is essential. However, the following more general result is true too: *If a function $g : (b, \infty) \rightarrow (0, \infty)$ is logarithmically convex and is not a decreasing function, then there exists an $a \geq b$ such that g is increasing on (a, ∞) .*

Now we can see, using the lemmas above, that the following result which is a generalization of the Bohr-Mollerup theorem (cf. H.-H. Kairies [4], p. 50) follows from Th. 1.

Theorem 2. *If $g : (0, \infty) \rightarrow (0, \infty)$ is a solution of (1) and g is logarithmically convex on an interval (a, ∞) for some $a \geq 0$, then $g \equiv \Gamma$.*

Remark 5. Lemmas 1 and 2 prove that Th. 1 (as well as its consequence, Th. 2) generalizes the Bohr-Mollerup theorem. To see that Th. 1 is an essential improvement of this classical result notice that all power functions $g(x) = x^p$, $x \in (0, \infty)$, with $p > 0$, are geometrically convex but not logarithmically convex. Note also that $g(x) = x^p$ is not convex for $p \in (0, 1)$.

2. Generalizations of Theorem 1 for Jensen convex functions

We introduce the following analogue of a Jensen convex function. Let $I \subseteq (0, \infty)$ be an open interval. A function $g : I \rightarrow (0, \infty)$ is said to be *geometrically Jensen convex on I* if

$$g(\sqrt{xy}) \leq \sqrt{g(x)g(y)} \quad \text{for all } x, y \in I.$$

Remark 6. It is easy to see that $g : I \rightarrow (0, \infty)$ is geometrically Jensen convex iff the function $\log \circ g \circ \exp$ is Jensen convex on the interval $\log(I)$. Furthermore, it is well-known that every continuous (geometrically) Jensen convex function is (geometrically) convex.

Using this remark and the well-known theorems of F. Bernstein-G. Doetsch [2], W. Sierpiński [12], A. Ostrowski [10] and M. R. Mehdi [9] (cf. also M. Kuczma [7]), which give some very weak sufficient conditions for a Jensen convex function to be continuous, we can formulate the theorems of the previous section in a more general form.

Corollary 1. *Suppose that $g : (0, \infty) \rightarrow (0, \infty)$ is bounded above on a neighbourhood of a point and geometrically Jensen convex on an interval (a, ∞) for some $a \geq 0$. If g satisfies (1), then $g \equiv \Gamma$.*

Proof. By assumption there are $x_0 \in (0, \infty)$, $r > 0$ and $M > 0$ such that $g(x) \leq M$ for all $x \in (x_0 - r, x_0 + r)$. Choose $n \in \mathbb{N}$ such that $(n + x_0 - r, n + x_0 + r) \subset (a, \infty)$. Hence by equation (1) we have:

$$\begin{aligned} g(x+n) &= x(x+1) \cdots (x+n-1)g(x) \leq \\ &\leq (x_0+r+n)^n M, \quad x \in (x_0-r, x_0+r). \end{aligned}$$

Thus g is bounded above on $U := ((x_0 + n) - r, (x_0 + n) + r) \subset (a, \infty)$. It follows that the function $\log \circ g \circ \exp$ is bounded above on the interval $\log(U) \subset (\log a, \infty)$. The Bernstein-Doetsch theorem (cf. Kuczma [7], p. 145, Th. 2) implies that $\log \circ g \circ \exp$ is continuous on $(\log a, \infty)$. Remark 6 yields that $\log \circ g \circ \exp$ is convex on the interval $(\log a, \infty)$,

consequently, g is geometrically convex on (a, ∞) . Now Cor. 1 results from Th. 1. \diamond

Remark 7. It follows from Ostrowski's theorem [10] (cf. M. Kuczma [7], p. 210, Th. 1), that the Cor. 1 remains true on replacing the assumption " g is bounded above on a neighbourhood of a point" by " g is bounded above on a set $T \subset (0, \infty)$ such that the inner measure of T is positive".

Remark 8. It follows from Mehdi's theorem [9] (cf. M. Kuczma [7], p. 210, Th. 2), that the Cor. 1 remains true on replacing the assumption " g is bounded above on a neighbourhood of a point" by " g is bounded above on a set $T \subseteq (0, \infty)$ containing a second category set with the Baire property such that g is bounded above on T ".

In a similar way as Cor. 1, using now the Sierpiński theorem [12] (cf. also M. Kuczma [7], p. 218, Th. 2), we can prove

Corollary 2. Suppose that $g : (0, \infty) \rightarrow (0, \infty)$ is geometrically Jensen convex on an interval (a, ∞) for some $a \geq 0$, and there is a nonempty open interval $I \subset (0, \infty)$ such that the restriction $g|_I$ is measurable. If g satisfies (1), then $g \equiv \Gamma$.

Remark 9. In M. Kuczma's book [7] one can find some other weak conditions which guarantee the continuity of Jensen convex functions. They allow to formulate somewhat more general results than the above Cors. 1 and 2.

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References

- [1] ARTIN, E.: Einführung in die Theorie der Gammafunktion, *Hamburger Math. Einzelschr.*, Heft 1, B. G. Teubner, Leipzig/Berlin, 1931.
- [2] BERNSTEIN, F. and DOETSCH, G.: Zur Theorie der konvexen Funktionen, *Math. Ann.* **76** (1915), 514–526.
- [3] BOHR, H. and MOLLERUP, J.: Lærebog i matematisk Analyse III, Grænseprocesser. Jul. Gjellerup, København, 1922.
- [4] KAIRIES, H.-H.: Convexity in the theory of the Gamma function, In: *General Inequalities 1, Proceedings of the First International Conference on General Inequalities*, held in the Math. Research Inst. at Oberwolfach, Black Forest, 1976, ed. by E. F. Beckenbach, ISNM Vol. 41, Birkhäuser Verlag Basel, Stuttgart, 1978, 49–62.
- [5] KRULL, W.: Bemerkungen zur Differenzgleichung $g(x+1) - g(x) = \varphi(x)$, I. II, *Math. Nachr.* **1** (1948), 365–376; **2** (1949), 251–262.

- [6] KUCZMA, M.: Functional Equations in a Single Variable, *Monografie Matematyczne* 46 PWN - Polish Scientific Publishers, Warsaw 1968.
- [7] KUCZMA, M.: An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality, *Prace Nauk. Uniw. Śląsk.* 489, Polish Scientific Publishers, Warsaw-Cracow-Katowice, 1985.
- [8] MAYER, A. E.: Konvexe Lösungen der Funktionalgleichung $1/f(x+1) = xf(x)$, *Acta Math.* 70 (1939), 57-62.
- [9] MEHDI, M. R.: On convex functions, *J. London Math. Soc.* 39 (1964), 321-326.
- [10] OSTROWSKI, A.: Mathematische Miszellen. XIV. Über die Funktionalgleichung der Exponentialfunktion und verwandte Funktionalgleichungen, *Jahresbericht d. D.M.V.* 38 (1929), 54-62.
- [11] RUDIN, W.: Principles of mathematical analysis, (Third Edition), Mc Graw-Hill, Inc., New York 1985.
- [12] SIERPIŃSKI, W.: Sur les fonctions convexes mesurables, *Fund. Math.* 1 (1920), 125-129.