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ON AN INTEGRAL EQUATION

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Abstract: A class G of functions f is introduced such that the integral equation $x(t) = \int_0^t f(s, x(s))ds$ has solutions. This class is more general than the class of functions f satisfying the classical Carathéodory's conditions.

Let $\mathbb{R}$ be the set of all reals, $I = [t_0, t_0 + a]$, $J = \{x \in \mathbb{R}^n : |x - x_0| \leq b\}$. A function $f : I \times J \to \mathbb{R}^n$ satisfies the *Carathéodory's conditions* (C) if

(i) almost all sections $f_t(x) = f(t, x)$ $(t \in I,\ x \in J)$ are continuous,

(ii) all sections $f^x(t) = f(t, x)$ $(t \in I,\ x \in J)$ are measurable (in the sense of Lebesgue), and

(iii) there is an integrable function (in the sense of Lebesgue) $m : I \to \mathbb{R}$ such that $|f(t, x)| \leq m(t)$ for every $(t, x) \in I \times J$.

It is well known the following theorem:

Theorem 0. ([2], p.7–8, Th. 1). *Suppose that $f : I \times J \to \mathbb{R}^n$ is a function satisfying the conditions* (C) *and d is a number such that $0 < d \leq a$, $\overline{g}(t_0 + d) = \int_{t_0}^{t_0+d} m(s)ds \leq b$. Then there is an absolutely*

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continuous function $h : [t_0, t_0 + d] \to J$ *such that* $h(t_0) = x_0$, *and* $h(t) = x_0 + \int_{t_0}^{t} f(s, h(s))ds$.

In this paper we prove that Th. 0 remains true if the conditions (C) will be replaced by more general conditions (G).

We say that a function $f : I \times J \to \mathbb{R}^n$ satisfies the *conditions* (G) if

(j) for every continuous function $h : I \to J$ the superposition $t \to \to f(t, h(t))$ is measurable,

(jj) there exists an integrable function $m : I \to \mathbb{R}$ such that $|f(t, x)| \leq \leq m(t)$ for every $(t, x) \in I \times J$, and

(jjj) there is a sequence of functions $f_k : I \times J \to \mathbb{R}^n$, satisfying the conditions (C) with $|f_k(t, x)| \leq m(t)$ for $(t, x) \in I \times J$, $k = 1, 2, \ldots$, and such that for every subsequence (f_{k_n}), for every sequence of continuous functions $g_n : I \to J$ which converges uniformly on I to a function g and for every $t \in I$ there is a strictly increasing sequence (n_i) of positive integers such that

$$\lim_{i \to \infty} \int_{t_0}^{t} f_{k_{n_i}}(s, g_{n_i}(s))ds = \int_{t_0}^{t} f(s, g(s))ds .$$

Theorem 1. *Suppose that* $f : I \times J \to \mathbb{R}^n$ *satisfies the conditions* (G) *and* d *is a number such that* $0 < d \leq a$, *and* $\overline{g}(t_0 + d) = \int_{t_0}^{t_0+d} m(s)ds \leq b$. *Then there is an absolutely continuous function* $h : [t_0, t_0 + d] \to J$ *such that*

$$h(t) = x_0 + \int_{t_0}^{t} f(s, h(s))ds , \quad t \in [t_0, t_0 + d] .$$

Proof. Since f satisfies the conditions (G), there is a sequence of functions f_k satisfying the condition (jjj). Without loss of generality we may assume that $|f_k(t, x)| \leq m(t)$ for $(t, x) \in I \times J$, $k = 1, 2, \ldots$. Since each f_k $(k = 1, 2, \ldots)$ satisfies the conditions (C), by Th. 0 there are absolutely continuous functions $h_k : [t_0, t_0 + d] \to J$ which satisfy the integral equations

$$h_k(t) = x_0 + \int_{t_0}^{t} f_k(s, h_k(s))ds \quad \text{for} \quad t \in [t_0, t_0 + d].$$

Remark that the functions h_k $(k = 1, 2, \dots)$ are uniformly bounded and equicontinuous on $[t_0, t_0 + d]$. By the Ascoli-Arzela Theorem, there is a subsequence $(h_{k_i})_i$ which converges uniformly on $[t_0, t_0 + d]$ to a continuous function $h : [t_0, t_0 + d] \to \mathbb{R}^n$. We shall prove that

$$h(t) = x_0 + \int_{t_0}^{t} f(s, h(s))ds \quad \text{for} \quad t \in [t_0, t_0 + d].$$

Evidently, $h_k(t_0) = x_0$ $(k = 1, 2, \dots)$. So $h(t_0) = \lim_{i \to \infty} h_{k_i}(t_0) = x_0$. Fix $t \in [t_0, t_0 + d]$. There exists a subsequence $(m_j)_j$ of the sequence $(k_i)_i$ such that

$$\lim_{j \to \infty} \int_{t_0}^{t} f_{m_j}(s, h_{m_j}(s))ds = \int_{t_0}^{t} f(s, h(s))ds.$$

Since

$$h_{m_j}(t) = x_0 + \int_{t_0}^{t} f_{m_j}(s, h_{m_j}(s))ds,$$

and

$$\lim_{j \to \infty} h_{m_j}(t) = h(t),$$

we obtain by (jjj) the relation

$$h(t) = \lim_{j \to \infty} h_{m_j}(t) = \lim_{j \to \infty} \left(x_0 + \int_{t_0}^{t} f_{m_j}(s, h_{m_j}(s))ds \right) =$$

$$= x_0 + \lim_{j \to \infty} \int_{t_0}^{t} f_{m_j}(s. h_{m_j}(s))ds = x_0 + \int_{t_0}^{t} f(s, h(s))ds. \quad \Diamond$$

From Th. 1 it follows immediately

Corollary 1. *If a function* $f : I \times J \to \mathbb{R}^n$ *satisfying the conditions* (G) *is such that for every continuous function* $h : [t_0, t_0 + d] \to J$ *the superposition* $t \to f(t, h(t))$ *is a derivative then there exists a solution of the Cauchy's problem* $y'(t) = f(t, y(t))$, $y(t_0) = x_0$, *defined on* $[t_0, t_0 + d]$.

Recollect that $g : [t_0, t_0 + d] \to \mathbb{R}^n$ is a derivative at a point t if

$$\lim_{r \to t} \int_t^r g(s)ds/(r-t) = g(t) \quad ([1] \text{ or } [3]).$$

Theorem 2. *If $f, g : I \times J \to \mathbb{R}^n$ are functions satisfying the conditions* (G) *then the sum $f + g$ satisfies the conditions* (G).

Proof. Evidently, the sum $f + g$ satisfies the conditions (j), (jj). Let (f_k), (g_k) be sequences of functions satisfying the condition (jjj) for f and g, respectively. Obviously, the sums $f_k + g_k$ $(k = 1, 2, \dots)$ satisfy the conditions (C). Suppose that a sequence of continuous functions $h_k : I \to J$ converges uniformly on I to a function h. Fix $t \in I$. Let (k_n) be a strictly increasing sequence of positive integers. By (jjj) there are a subsequence (n_i) of the sequence $(1, 2, \dots)$ and a subsequence (i_j) of (n_i) such that

$$\lim_{i \to \infty} \int_{t_0}^t f_{k_{n_i}}(s, h_{n_i}(s)) = \int_{t_0}^t f(s, h(s))ds, \text{ and}$$

$$\lim_{j \to \infty} \int_{t_0}^t g_{k_{i_j}}(s, h_{i_j}(s)) = \int_{t_0}^t g(s, h(s))ds.$$

Consequently,

$$\lim_{j \to \infty} \int_{t_0}^t \Big(f_{k_{i_j}}(s, h_{i_j}(s)) + g_{k_{i_j}}(s, h_{i_j}(s)) \Big) ds =$$

$$= \int_{t_0}^t (f(s, h(s)))ds + \int_{t_0}^t g(s, h(s))ds. \quad \Diamond$$

Remark 1. Analogously as above we may prove that *the product kf satisfies the conditions* (G) *whenever $k \in \mathbb{R}$ is a constant and the function $f : I \times J \to \mathbb{R}^n$ satisfies the conditions* (G).

Remark 2. From Remark 1 and Th. 2 it follows that *the space $G = \{f : I \times J \to \mathbb{R}^n : f$ satisfies (G)$\}$ with the metric $\rho(f, g) = \min(1, \sup\{|f(t, x) - g(t, x)| : (t, x) \in I \times J\})$ is a linear metric space.*

Theorem 3. *Assume the Continuum Hypothesis. Then the set $C = \{f : I \times J \to \mathbb{R}^n : f$ satisfies (C) $\}$ is closed and nowhere dense in G.*

Proof. Of course, if $f \in C$ then f satisfies the conditions (j), (jj) and the functions $f_k = f$ $(k = 1, 2, \dots)$ satisfy all requirements of the condition (jjj). So $C \subset G$. Moreover, if a sequence of functions $f_k : I \times J \to \mathbb{R}^n$ satisfying the conditions (C) converges uniformly (with respect to the metric ρ from Remark 2) to a function f then f satisfies also the conditions (C). So C is a closed set in G with respect to the metric ρ. Fix $f \in C$ and $\varepsilon > 0$ $(\varepsilon < 1)$. Denote by ω the first ordinal number of the continuum power. Let $(h_\alpha)_{\alpha<\omega_1}$ be a transfinite sequence of all continuous functions $(h_\alpha : I \to J$ and let $(F_\alpha)_{\alpha<\omega_1}$ be a transfinite sequence of all closed subsets of $I \times J$ which are of positive (Lebesgue) measure and all sets $E_t = \{(t, x) : x \in J\}$, $t \in I$. Denote by $G(h_\alpha)$ the graph of the function h_α $(\alpha < \omega_1)$. By transfinite induction, there is a set

$$B = \{(t_\alpha, x_\alpha) \in I \times J : \alpha < \omega_1\}$$

such that

$$(t_\alpha, x_\alpha) \in F_\alpha - \bigcup_{\beta<\alpha} G(h_\beta), \quad \text{and} \quad x_\alpha \neq x_0 \quad \text{for} \quad \alpha < \omega_1 ,$$

and for each $t \in I$ the intersection $B \cap E_t$ contains a sequence $((t, x_k))_k$ such that $\lim_{k\to\infty} x_k = x_0$. Let $u \in \mathbb{R}^n$ be a point such that $|u| = 1$. Let us put

$$g(t, x) = \begin{cases} \varepsilon u & \text{for } (t, x) \in B \\ 0 & \text{otherwise} \end{cases}$$

and

$$h = f + g .$$

Evidently, $\rho(f, h) = \varepsilon$. To prove that $h \in G - C$ it suffices to show that $g \in G - C$. Since for each $\alpha < \omega_1$ the set $\{t \in I : g(t, h_\alpha(t)) \neq 0\}$ is countable, g satisfies the conditions (j), (jj) and

$$\int_I g(s, h_\alpha(s))ds = 0 \quad \text{for} \quad \alpha < \omega_1 .$$

Consequently, g satisfies the condition (jjj), and $g \in .G$. Fix $t \in I$. Since $g(t, x_0) = 0$ and x_0 is an accumulation point of the set $B \cap E_t$, the section g_t is not continuous at x_0. So $g \notin C$, and the proof is completed. ◊

Remark 3. In Th. 4 the Continuum Hypothesis can be replaced by the Martin's Axiom.

Example 1. Let $I = [0, 1]$, $J = [-1, 1]$, and let

$$f(t,x) = \begin{cases} 1 & \text{if } x = 0 \text{ and } 1/(2n+1) < t < 1/2n,\ n = 1,2,\dots \\ 0 & \text{otherwise.} \end{cases}$$

The function f is of Baire class 1. If the function $x : [0,d] \to J$ $(d \le 1)$ satisfies the equation

$$(*) \qquad x(t) = \int_0^t f(s, x(s))ds$$

then $x(0) = 0$, x is nondecreasing, and $x(t) > 0$ for $t > 0$ $(t \le d)$. But, in this case $f(s, x(s)) = 0$ for $s > 0$ and $x(t) = 0$ for $t \in [0,d]$. This contradiction proves that the integral equation $(*)$ has not an absolutely continuous solution, and consequently $f \notin G$.

Example 2. Let $I = [0,1]$ and $J = [-1,1]$. Denote by T_e and T_d, respectively the euclidean and the density topologies in $\mathbb{R}$ (for the definition of the density topology see [1]). There is an approximately continuous (i.e. (T_d, T_e) continuous) function $g : J \to [0,1]$ is such that $g[1/k] = 1$ for $k = 1,2\dots$, and $g(0) = 0$ (see [1]). Consequently, the function $f(t,x) = g(x)$ is a $(T_e \times T_d, T_e)$ continuous mapping. Assume that $f \in G$, Let (f_k) be a sequence of functions from C corresponding to f by the condition (jjj). For $k = 1,2,\dots$ there are an index n_k and a number y_k such that $n_{k+1} > n_k$, $|y_k - 1/k| < 1/k(k+1)$, and

$$\text{(i)} \qquad \left| \int_0^1 f_{n_k}(s, y_k)ds - \int_0^1 f(s, 1/k)ds \right| < 1/2.$$

Since

$$\int_0^1 f(s, 1/k)ds = 1,$$

it follows from (i) that

$$\int_0^1 f_{n_k}(s, y_k)ds > 1/2.$$

Then the sequence (y_k) converges uniformly to 0 and there is not a strictly increasing sequence (k_i) of positive integers such that

$$\lim_{i\to\infty}\int_0^1 f_{n_{k_i}}(s,y_{k_i})ds = \int_0^1 f(s,0)ds = 0\,.$$

So, $f \notin G$. Observe that the integral equation $(*)$ has a solution $x(t) = 0$ for $t \in I$.

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