A REVISION OF BANDLER – KOHOUT COMPOSITIONS OF RELATIONS

B. De Baets*

Department of Applied Mathematics and Computer Science, University of Gent, Krijgslaan 281 (S9), 9000 Gent, Belgium

E. Kerre

Department of Applied Mathematics and Computer Science, University of Gent, Krijgslaan 281 (S9), 9000 Gent, Belgium

Abstract: This paper contains a thorough investigation of the compositions of relations defined by Bandler and Kohout. It is shown that these compositions bear some shortcomings and improved definitions are suggested. Similar ideas are used to define new images of a set under a relation. Possible relationships among these images and among the compositions are investigated. An extensive overview of the properties, such as monotonicity and interaction with union and intersection, of the images and the compositions is given. Finally, the associativity properties of the compositions are examined.

1. Introduction

In 1980 W. Bandler and L. Kohout [1] introduced several new compositions of relations, called products in their terminology, based on the notions of aftersets and foresets of relations. They immediately

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extended these compositions to fuzzy relations using fuzzy implication operators [2]. Bandler and Kohout claim that fuzzy relational compositions constitute a tool for the analysis and synthesis of complex natural and artificial systems. The list of application areas is appealing, including medical diagnosis [3] and information retrieval systems [4,5]. Their definitions have been generally accepted and have even become common property in fuzzy set theory.

An attempt to apply Bandler and Kohout's ideas on the concept of *direct image of a set under a relation* brought to light that the definitions of their compositions bear some shortcomings [6]. These become even more important when one considers the fuzzy counterparts of these definitions. The purpose of this paper is to provide more accurate versions of these compositions of relations and to give an extensive overview of their relevant properties.

2. Preliminary definitions

A relation R from a universe X to a universe Y is a subset of $X \times Y$, i.e. $R \subseteq X \times Y$. The formula $(x, y) \in R$ is abbreviated as xRy, and one says that x is in relation R with y.

Definition 2.1. The afterset xR of $x \in X$ and the foreset Ry of $y \in Y$ are defined as

$$xR = \{y \mid xRy\}$$
$$Ry = \{x \mid xRy\}.$$

Definition 2.2. The domain $\mathrm{dom}(R)$ and the range $\mathrm{rng}(R)$ of R are defined as

$$\mathrm{dom}(R) = \{x \mid xR \neq \emptyset\}$$
$$\mathrm{rng}(R) = \{y \mid Ry \neq \emptyset\}.$$

Definition 2.3. The converse relation R^T of R is the relation from Y to X defined by

$$yR^T x \Leftrightarrow xRy.$$

The complement $\mathrm{co}\,R$ of R is the relation from X to Y defined by

$$x(\mathrm{co}\,R)y \Leftrightarrow \neg(xRy).$$

Notice that $\mathrm{dom}(R^T) = \mathrm{rng}(R)$, $\mathrm{rng}(R^T) = \mathrm{dom}(R)$ and $\mathrm{co}(R^T) = (\mathrm{co}\,R)^T$.

Consider an arbitrary family $(R_i)_{i \in I}$ of relations from X to Y indexed by I.

Definition 2.4. The union $\bigcup_{i \in I} R_i$ of the family $(R_i)_{i \in I}$ is the relation from X to Y defined by

$$x\left(\bigcup_{i \in I} R_i\right)y \Leftrightarrow (\exists i \in I)(xR_iy).$$

The intersection $\bigcap_{i \in I} R_i$ of the family $(R_i)_{i \in I}$ is the relation from X to Y defined by

$$x\left(\bigcap_{i \in I} R_i\right)y \Leftrightarrow (\forall i \in I)(xR_iy).$$

All of these operations can be expressed in terms of after- and foresets in the following way [9].

Properties 2.1.

1. $$xR = R^T x$$
$$Ry = yR^T$$

2. $$x(\mathrm{co}\,R) = \mathrm{co}(xR)$$
$$(\mathrm{co}\,R)y = \mathrm{co}(Ry)$$

3. $$x\left(\bigcup_{i \in I} R_i\right) = \bigcup_{i \in I} xR_i$$
$$\left(\bigcup_{i \in I} R_i\right)y = \bigcup_{i \in I} R_iy$$

4. $$x\left(\bigcap_{i \in I} R_i\right) = \bigcap_{i \in I} xR_i$$
$$\left(\bigcap_{i \in I} R_i\right)y = \bigcap_{i \in I} R_iy.$$

3. Images of a set under a relation

3.1. Definition

Consider a relation R from X to Y and a subset A of X. The classical definition of the *direct* image of the set A under the relation R is given as follows

$$R(A) = \{y \mid (\exists x \in A)(xRy)\}.$$

The direct image $R(A)$ is the set of those elements of Y that are in relation R^T with *at least one* element of A. The intention of this subsection is to refine the direct image $R(A)$ in order to distinguish those

elements of Y that are in relation R^T with *all* elements of A and those elements of Y that are in relation R^T with elements of A *only*. This refinement is achieved in the following definition.

Definition 3.1.   (Images of a set under a relation)

$$R(A) = \{y \mid A \cap Ry \neq \emptyset\}$$
$$R^{\triangleleft}(A) = \{y \mid \emptyset \subset A \subseteq Ry\}$$
$$R^{\triangleright}(A) = \{y \mid \emptyset \subset Ry \subseteq A\}$$
$$R^{\circ}(A) = \{y \mid \emptyset \subset A = Ry\}.$$

This definition provides four different images of a set under a relation. It is obvious that the first definition coincides with the classical definition of the direct image of a set under a relation. The second image $R^{\triangleleft}(A)$ is called the *subdirect* image of A under R, while the third image $R^{\triangleright}(A)$ is called the *superdirect* image of A under R. The fourth image $R^{\circ}(A)$ is called the *square* image of A under R.

For a non-empty set A, the subdirect image $R^{\triangleleft}(A)$ can be written as

$$R^{\triangleleft}(A) = \{y \mid (\forall x \in A)(xRy)\}.$$

This explains why the direct and subdirect image are called *existential* and *universal* compositions by Izumi, Tanaka and Asai [8]. They are also called *upper* and *lower* images by Dubois and Prade [7].

Remark 3.1.

- The non-emptiness condition $\emptyset \subset A$ in the definition of $R^{\triangleleft}(A)$ seems superfluous at first sight and could be evaded by restricting the definition to a non-empty set A. Without the condition $\emptyset \subset A$ it would follow that $R^{\triangleleft}(\emptyset) = Y$, which is unacceptable. Neither Izumi, Tanaka and Asai nor Dubois and Prade have observed the necessity of this non-emptiness condition.

- The non-emptiness condition $\emptyset \subset Ry$ in the definition of $R^{\triangleright}(A)$ has stronger consequences. Without the condition $\emptyset \subset Ry$ it would follow that $\mathrm{co}(\mathrm{rng}(R)) \subseteq R^{\triangleright}(A)$, which is unacceptable again. The condition $\emptyset \subset Ry$ ensures that $R^{\triangleright}(A)$ contains those elements of Y that are in relation R^T with elements of A only and that actually do so.

- The non-emptiness conditions imply that all images are contained in $\mathrm{rng}(R)$ and that the images of the empty set under a relation all yield the empty set.

An equivalent way of defining the new images is the following

$$R^{\triangleleft}(A) = R(A) \cap \{y \mid A \subseteq Ry\}$$
$$R^{\triangleright}(A) = R(A) \cap \{y \mid Ry \subseteq A\}$$
$$R^{\circ}(A) = R(A) \cap \{y \mid A = Ry\}.$$

These definitions interpret the non-emptiness conditions in a different way. Adopting these alternative definitions unnecessarily complicates the proofs of the properties of the new images. The condition $y \in R(A)$ not only implies the non-emptiness of A and Ry but also a certain overlap. This overlap is afterwards tested in a stricter way by the conditions of inclusion or equality in the second components of the above intersections.

Example 3.1. The images of a set under a relation can be illustrated on an example from medical diagnosis. Consider a set of patients X and a set of symptoms Y. Let R be the relation from X to Y defined by

$$xRy \Leftrightarrow \text{patient } x \text{ shows symptom } y.$$

Let A be the non-empty set of female patients in the population X, then the images of A under R are given by

- $R(A)$ is the set of symptoms shown by at least one female patient,
- $R^{\triangleleft}(A)$ is the set of symptoms shown by all female patients,
- $R^{\triangleright}(A)$ is the set of symptoms shown by at least one female patient and not by any male patient,
- $R^{\circ}(A)$ is the set of symptoms shown by all female patients and not by any male patient.

A close examination of the relationships between the images and the properties of the images is discussed in the following subsections. A few of these relationships and properties can also be found in [7,8].

3.2. Relationships between the images

3.2.1. Containment.
A first series of properties concerns the refining nature of the images.

Properties 3.1. (Containment)

$$R^{\circ}(A) = R^{\triangleleft}(A) \cap R^{\triangleright}(A)$$
$$R^{\circ}(A) \subseteq R^{\triangleleft}(A) \subseteq R(A)$$
$$R^{\circ}(A) \subseteq R^{\triangleright}(A) \subseteq R(A).$$

3.2.2. Relationships.
This paragraph investigates among other things how the determination of a subdirect or a superdirect image

can be converted into the determination of a classical direct image. These properties not merely have an aesthetical character but also a functional one. As will become clear, they assist in discovering and proving new properties. The new images can be expressed in terms of the direct image in the following way.

Properties 3.2.

$$R^{\triangleleft}(A) = \operatorname{co}((\operatorname{co} R)(A)) \qquad\qquad\text{if } A \neq \emptyset$$
$$R^{\triangleright}(A) = \operatorname{co}(R(\operatorname{co} A)) \cap \operatorname{rng}(R)$$
$$R^{\circ}(A) = \operatorname{co}((\operatorname{co} R)(A)) \cap \operatorname{co}(R(\operatorname{co} A)) \qquad\text{if } A \neq \emptyset.$$

Proof. As an example, the third equality is proven.

$$y \in \operatorname{co}((\operatorname{co} R)(A)) \cap \operatorname{co}(R(\operatorname{co} A))$$
$$\Leftrightarrow \neg(A \cap (\operatorname{co} R)y \neq \emptyset) \ \wedge \ \neg(\operatorname{co} A \cap Ry \neq \emptyset)$$
$$\Leftrightarrow A \cap \operatorname{co}(Ry) = \emptyset \ \wedge \ \operatorname{co} A \cap Ry = \emptyset$$
$$\Leftrightarrow \emptyset \subset A \subseteq Ry \ \wedge \ Ry \subseteq A$$
$$\Leftrightarrow \emptyset \subset A = Ry$$
$$\Leftrightarrow y \in R^{\circ}(A). \quad \Diamond$$

The following relationships express the direct image in terms of the subdirect or the superdirect image.

Properties 3.3.

$$R(A) = \operatorname{co}((\operatorname{co} R)^{\triangleleft}(A)) \qquad\text{if } A \neq \emptyset$$
$$R(A) = \operatorname{co}(R^{\triangleright}(\operatorname{co} A)) \cap \operatorname{rng}(R).$$

Proof. The second equality can be proven as follows.

$$y \in \operatorname{co}(R^{\triangleright}(\operatorname{co} A)) \cap \operatorname{rng}(R)$$
$$\Leftrightarrow \neg(\emptyset \subset Ry \subseteq \operatorname{co} A) \ \wedge \ Ry \neq \emptyset$$
$$\Leftrightarrow (\neg(\emptyset \subset Ry) \ \vee \ \neg(Ry \subseteq \operatorname{co} A)) \ \wedge \ Ry \neq \emptyset$$
$$\Leftrightarrow (Ry = \emptyset \ \vee \ \neg(Ry \subseteq \operatorname{co} A)) \ \wedge \ Ry \neq \emptyset$$
$$\Leftrightarrow \neg(Ry \subseteq \operatorname{co} A)$$
$$\Leftrightarrow (\exists x \in Ry)(x \in A)$$
$$\Leftrightarrow A \cap Ry \neq \emptyset$$
$$\Leftrightarrow y \in R(A). \quad \Diamond$$

Other interesting relationships are these existing between the subdirect and the superdirect image.

Properties 3.4.

$$R^{\triangleleft}(A) = (\operatorname{co} R)^{\triangleright}(\operatorname{co} A) \cup \operatorname{co}(\operatorname{rng}(\operatorname{co} R)) \qquad \textit{if } A \neq \emptyset$$
$$R^{\triangleright}(A) = (\operatorname{co} R)^{\triangleleft}(\operatorname{co} A) \cap \operatorname{rng}(R) \qquad\qquad \textit{if } \operatorname{co} A \neq \emptyset.$$

Proof. As an example, the first equality is proven.

$$y \in (\operatorname{co} R)^{\triangleright}(\operatorname{co} A) \cup \operatorname{co}(\operatorname{rng}(\operatorname{co} R))$$

$$\Leftrightarrow \emptyset \subset (\operatorname{co} R)y \subseteq \operatorname{co} A \ \lor \ \neg((\operatorname{co} R)y \neq \emptyset)$$
$$\Leftrightarrow (\emptyset \subset \operatorname{co}(Ry) \ \land \ \operatorname{co}(Ry) \subseteq \operatorname{co} A) \ \lor \ \operatorname{co}(Ry) = \emptyset$$
$$\Leftrightarrow (Ry \subset X \ \land \ \emptyset \subset A \subseteq Ry) \ \lor \ Ry = X$$
$$\Leftrightarrow \emptyset \subset A \subseteq Ry$$
$$\Leftrightarrow y \in R^{\triangleleft}(A). \quad \Diamond$$

To conclude this paragraph, the following property for the square image is mentioned.

Property 3.5.

$$(\operatorname{co} R)^{\circ}(\operatorname{co} A) = R^{\circ}(A) \qquad \textit{if } A \neq \emptyset \textit{ and } \operatorname{co} A \neq \emptyset.$$

Proof.

$$y \in (\operatorname{co} R)^{\circ}(\operatorname{co} A)$$

$$\Leftrightarrow \emptyset \subset \operatorname{co} A = (\operatorname{co} R)y \Leftrightarrow \emptyset \subset \operatorname{co} A \ \land \ \operatorname{co} A = \operatorname{co}(Ry)$$
$$\Leftrightarrow \operatorname{co} A \neq \emptyset \ \land \ A = Ry \Leftrightarrow \emptyset \subset A \subseteq Ry \Leftrightarrow y \in R^{\circ}(A). \ \Diamond$$

3.2.3. Expressions in terms of aftersets. Using properties 3.2 expressing the new images in terms of the direct image, definitions 3.1 can be written in the following elegant way.

Properties 3.6. (Expressions in terms of aftersets)

$$R(A) = \bigcup_{x \in A} xR$$

$$R^{\triangleleft}(A) = \bigcap_{x \in A} xR \qquad\qquad\qquad \textit{if } A \neq \emptyset$$

$$R^{\triangleright}(A) = \bigcap_{x \in \operatorname{co} A} \operatorname{co}(xR) \cap \operatorname{rng}(R)$$

$$R^{\circ}(A) = \left(\bigcap_{x \in A} xR \right) \cap \left(\bigcap_{x \in \operatorname{co} A} \operatorname{co}(xR) \right) \qquad \textit{if } A \neq \emptyset.$$

Proof. As an example, the second equality is proven.

$$R^{\triangleleft}(A) = \text{co}((\text{co } R)(A)) = \text{co}\left(\bigcup_{x \in A} x(\text{co } R)\right) = \text{co}\left(\bigcup_{x \in A} \text{co}(xR)\right)$$
$$= \bigcap_{x \in A} \text{co}(\text{co}(xR)) = \bigcap_{x \in A} xR. \quad \Diamond$$

3.3. Monotonicity

In this subsection the relationship is investigated between the images of a subset of a set and the images of this latter set under a given relation. The same investigation is carried out for the images of a given set under a subrelation of a relation and under this latter relation. This investigation leads to the following results.

Properties 3.7. (Monotonicity)

$$A_1 \subseteq A_2 \Rightarrow R(A_1) \subseteq R(A_2)$$
$$\emptyset \subset A_1 \subseteq A_2 \Rightarrow R^{\triangleleft}(A_2) \subseteq R^{\triangleleft}(A_1)$$
$$A_1 \subseteq A_2 \Rightarrow R^{\triangleright}(A_1) \subseteq R^{\triangleright}(A_2)$$
$$R_1 \subseteq R_2 \Rightarrow R_1(A) \subseteq R_2(A)$$
$$R_1 \subseteq R_2 \Rightarrow R_1^{\triangleleft}(A) \subseteq R_2^{\triangleleft}(A)$$
$$(\text{rng}(R_1) = \text{rng}(R_2) \land R_1 \subseteq R_2) \Rightarrow R_2^{\triangleright}(A) \subseteq R_1^{\triangleright}(A).$$

Proof. Only the proof of the last property is mentioned.

$$y \in R_2^{\triangleright}(A) \Leftrightarrow \emptyset \subset R_2 y \subseteq A \Rightarrow \emptyset \subset R_1 y \subseteq A \Leftrightarrow y \in R_1^{\triangleright}(A). \quad \Diamond$$

Remark 3.2. From the proof of the last property it actually follows that the condition $\text{rng}(R_2) \subseteq \text{rng}(R_1)$ has to be added. Since the condition $R_1 \subseteq R_2$ already implies that $\text{rng}(R_1) \subseteq \text{rng}(R_2)$ it is obvious that immediately the condition $\text{rng}(R_1) = \text{rng}(R_2)$ can be used.

3.4. Interaction with union and intersection

In this subsection the relationship is investigated between the images of the union and the intersection of an arbitrary family of sets under a given relation and the union and the intersection of the images of each of these sets under that given relation.

Consider a relation R from X to Y and an arbitrary non-empty family $(A_i)_{i \in I}$ of sets in X.

Properties 3.8. (Interaction with union)

$$R\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} R(A_i)$$

$$R^{\triangleleft}\left(\bigcup_{i\in I} A_i\right) = \bigcap_{i\in I} R^{\triangleleft}(A_i) \qquad if \ (\forall i\in I)(A_i \neq \emptyset)$$

$$R^{\triangleright}\left(\bigcup_{i\in I} A_i\right) \supseteq \bigcup_{i\in I} R^{\triangleright}(A_i).$$

Proofs.

1. $y \in R\left(\bigcup_{i\in I} A_i\right) \Leftrightarrow \left(\bigcup_{i\in I} A_i\right) \cap Ry \neq \emptyset \Leftrightarrow \bigcup_{i\in I}(A_i \cap Ry) \neq \emptyset$

    $\Leftrightarrow (\exists i \in I)(A_i \cap Ry \neq \emptyset) \Leftrightarrow y \in \bigcup_{i\in I} R(A_i)$

2. $y \in R^{\triangleleft}\left(\bigcup_{i\in I} A_i\right) \Leftrightarrow \emptyset \subset \left(\bigcup_{i\in I} A_i\right) \subseteq Ry \Leftrightarrow (\forall i \in I)(\emptyset \subset A_i \subseteq Ry)$

    $\Leftrightarrow y \in \bigcap_{i\in I} R^{\triangleleft}(A_i)$

3. $y \in \bigcup_{i\in I} R^{\triangleright}(A_i) \Leftrightarrow (\exists i \in I)(\emptyset \subset Ry \subseteq A_i) \Rightarrow \emptyset \subset Ry \subseteq \bigcup_{i\in I} A_i$

    $\Leftrightarrow y \in R^{\triangleright}\left(\bigcup_{i\in I} A_i\right).$ ◊

Without the condition $(\forall i \in I)(A_i \neq \emptyset)$ one can deduce the following interaction of the subdirect image with union.

Property 3.9.

$$\bigcap_{i\in I} R^{\triangleleft}(A_i) \subseteq R^{\triangleleft}\left(\bigcup_{i\in I} A_i\right) \subseteq \bigcup_{i\in I} R^{\triangleleft}(A_i).$$

Proof.

$$y \in \bigcap_{i\in I} R^{\triangleleft}(A_i) \Leftrightarrow (\forall i \in I)(\emptyset \subset A_i \subseteq Ry) \Rightarrow \emptyset \subset \bigcup_{i\in I} A_i \subseteq Ry$$

$$\Leftrightarrow y \in R^{\triangleleft}\left(\bigcup_{i\in I} A_i\right) \Rightarrow (\exists i \in I)(\emptyset \subset A_i \subseteq Ry)$$

$$\Leftrightarrow y \in \bigcup_{i\in I} R^{\triangleleft}(A_i). \quad ◊$$

Properties 3.10. (Interaction with intersection)

$$R\Big(\bigcap_{i\in I} A_i\Big) \subseteq \bigcap_{i\in I} R(A_i)$$

$$R^\triangleleft\Big(\bigcap_{i\in I} A_i\Big) \supseteq \bigcup_{i\in I} R^\triangleleft(A_i) \qquad \text{if } \bigcap_{i\in I} A_i \neq \emptyset$$

$$R^\triangleright\Big(\bigcap_{i\in I} A_i\Big) = \bigcap_{i\in I} R^\triangleright(A_i).$$

The same investigation is carried out for the relationship between the images of a given set under the union and the intersection of an arbitrary family of relations and the union and the intersection of the images of that given set under each of these relations.

Consider an arbitrary non-empty family $(R_i)_{i\in I}$ of relations from X to Y and a set A in X.

Properties 3.11. (Interaction with union)

$$\Big(\bigcup_{i\in I} R_i\Big)(A) = \bigcup_{i\in I} R_i(A)$$

$$\Big(\bigcup_{i\in I} R_i\Big)^\triangleleft(A) \supseteq \bigcup_{i\in I} R_i{}^\triangleleft(A)$$

$$\bigcap_{i\in I} R_i{}^\triangleright(A) \subseteq \Big(\bigcup_{i\in I} R_i\Big)^\triangleright(A) \subseteq \bigcup_{i\in I} R_i{}^\triangleright(A).$$

Properties 3.12. (Interaction with intersection)

$$\Big(\bigcap_{i\in I} R_i\Big)(A) \subseteq \bigcap_{i\in I} R_i(A)$$

$$\Big(\bigcap_{i\in I} R_i\Big)^\triangleleft(A) = \bigcap_{i\in I} R_i{}^\triangleleft(A)$$

$$\Big(\bigcap_{i\in I} R_i\Big)^\triangleright(A) \supseteq \bigcup_{i\in I} R_i{}^\triangleright(A) \qquad \text{if } (\forall y \in Y)\Big(\bigcap_{i\in I} R_i y \neq \emptyset\Big).$$

4. Compositions of relations

4.1. Introduction

As already mentioned in the introduction, this paper has been inspired by the results of Bandler and Kohout [1] concerning compositions of relations.

Consider a relation R from X to Y and a relation S from Y to Z. The classical definition of the composition of the relations R and S is given as follows

$$R \circ S = \{(x, z) \mid (\exists y \in Y)(xRy \wedge ySz)\}.$$

The composition $R \circ S$ is a relation from X to Z, consisting of those couples (x, z) for which there exists *at least one* element of Y that is in relation R^T with x and that is in relation S with z. The relation $R \circ S$ is read as R *before* S or R *followed by* S. This definition can be written in terms of after- and foresets in the following way

$$R \circ S = \{(x, z) \mid xR \cap Sz \neq \emptyset\}.$$

Bandler and Kohout have introduced the following new compositions.

Definition 4.1. (Bandler-Kohout compositions)

$$R \triangleleft_{bk} S = \{(x, z) \mid xR \subseteq Sz\}$$
$$R \triangleright_{bk} S = \{(x, z) \mid Sz \subseteq xR\}$$
$$R \diamond_{bk} S = \{(x, z) \mid xR = Sz\}.$$

These compositions are called *products* by Bandler and Kohout, more specifically *round product* $\circ$, *subproduct* $\triangleleft_{bk}$, *superproduct* $\triangleright_{bk}$ and *square product* $\diamond_{bk}$. The subproduct and superproduct are also called *triangular products*.

It is clear that definitions 3.1 of the new images have been inspired by definitions 4.1. It is surprising that the definitions of Bandler and Kohout do not mention any non-emptiness condition. This is a regrettable shortcoming. One easily verifies that

$$\text{co}(\text{dom}(R)) \times Z \subseteq R \triangleleft_{bk} S$$
$$X \times \text{co}(\text{rng}(S)) \subseteq R \triangleright_{bk} S.$$

The first expression means that x is in relation $R \triangleleft_{bk} S$ with all elements of Z even if there is no element of Y that is in relation R^T with x. A similar remark holds for the second expression. In this way, the compositions $R \triangleleft_{bk} S$, $R \triangleright_{bk} S$ and $R \diamond_{bk} S$ can contain a lot of unwanted couples. It is clear that only those couples can be accepted for which both components are involved in the relations. An apparent solution would be to consider only those relations R for which $\text{dom}(R) = X$ and $\text{rng}(R) = Y$. This becomes too big a restriction when one wants to consider several relations between the same universes. It is unrealistic that all of these relations would have the same domain and range.

The intention of this section is to improve the definitions of Bandler and Kohout and to carry out a close examination of the properties of the compositions and the relationships between these compositions.

4.2. Definition

Consider a relation R from X to Y and a relation S from Y to Z.
Definition 4.2. (De Baets-Kerre)

$$R \lhd S = \{(x,z) \mid \emptyset \subset xR \subseteq Sz\}$$
$$R \rhd S = \{(x,z) \mid \emptyset \subset Sz \subseteq xR\}$$
$$R \diamond S = \{(x,z) \mid \emptyset \subset xR = Sz\}.$$

These compositions are called the sub-, super- and square composition.
Remark 4.1. The non-emptiness conditions imply that all compositions are contained in $\mathrm{dom}(R) \times \mathrm{rng}(R)$. As for the images, an equivalent way of defining the new compositions is the following

$$R \lhd S = (R \circ S) \cap \{(x,z) \mid xR \subseteq Sz\}$$
$$R \rhd S = (R \circ S) \cap \{(x,z) \mid Sz \subseteq xR\}$$
$$R \diamond S = (R \circ S) \cap \{(x,z) \mid xR = Sz\}.$$

Example 4.1. The compositions of two relations can also be illustrated on an example from medical diagnosis. Consider a set of patients X, a set of symptoms Y and a set of illnesses Z. Let R be the relation from X to Y defined by

$$xRy \Leftrightarrow \text{patient } x \text{ shows symptom } y$$

and S the relation from Y to Z defined by

$$ySz \Leftrightarrow y \text{ is a symptom of illness } z.$$

The compositions of R and S are given by
- $x(R \circ S)z \Leftrightarrow$ patient x shows at least one symptom of illness z,
- $x(R \lhd S)z \Leftrightarrow$ all symptoms shown by patient x are symptoms of illness z (and patient x shows at least one symptom),
- $x(R \rhd S)z \Leftrightarrow$ patient x shows all symptoms of illness z (and patient x shows at least one symptom),
- $x(R \diamond S)z \Leftrightarrow$ the symptoms shown by patient x are exactly those of illness z (and patient x shows at least one symptom).

4.3. Relationships between the compositions

4.3.1. Containment. A first series of properties concerns the refining nature of the compositions.

Properties 4.1. (Containment)
$$R \diamond S = (R \triangleleft S) \cap (R \triangleright S)$$
$$R \diamond S \subseteq R \triangleleft S \subseteq R \circ S$$
$$R \diamond S \subseteq R \triangleright S \subseteq R \circ S.$$

4.3.2. Relationships. A similar investigation is carried out as for the images. The new compositions can be expressed in terms of the classical composition in the following way.

Properties 4.2.
$$R \triangleleft S = \text{co}(R \circ (\text{co } S)) \cap (\text{dom}(R) \times Z)$$
$$R \triangleright S = \text{co}((\text{co } R) \circ S) \cap (X \times \text{rng}(S))$$
$$R \diamond S = \text{co}(R \circ (\text{co } S)) \cap \text{co}((\text{co } R) \circ S) \cap (\text{dom}(R) \times Z)$$
$$= \text{co}(R \circ (\text{co } S)) \cap \text{co}((\text{co } R) \circ S) \cap (X \times \text{rng}(S)).$$

Proof. As an example, the first equality is proven.
$$(x, z) \in \text{co}(R \circ (\text{co } S)) \cap (\text{dom}(R) \times Z)$$
$$\Leftrightarrow \neg((x, z) \in R \circ (\text{co } S)) \wedge (x \in \text{dom}(R))$$
$$\Leftrightarrow \neg(xR \cap (\text{co } S)z \neq \emptyset) \wedge xR \neq \emptyset$$
$$\Leftrightarrow xR \cap \text{co}(Sz) = \emptyset \wedge xR \neq \emptyset$$
$$\Leftrightarrow \emptyset \subset xR \subseteq Sz$$
$$\Leftrightarrow (x, z) \in R \triangleleft S. \quad \Diamond$$

The inverse relationships are given next.

Properties 4.3.
$$R \circ S = \text{co}(R \triangleleft (\text{co } S)) \cap (\text{dom}(R) \times Z)$$
$$R \circ S = \text{co}((\text{co } R) \triangleright S) \cap (X \times \text{rng}(S)).$$

Bandler and Kohout established the following relationships for their definitions
$$R \triangleright_{bk} S = (\text{co } R) \triangleleft_{bk} (\text{co } S)$$
$$R \diamond_{bk} S = (\text{co } R) \diamond_{bk} (\text{co } S).$$

Finding the correct relationships for the improved definitions is a delicate matter. These relationships between the subcomposition and the supercomposition are given next.

Properties 4.4.
$$R \triangleleft S = ((\text{co } R) \triangleright (\text{co } S) \cap (\text{dom}(R) \times Z)) \cup (\text{dom}(R) \times \text{co}(\text{rng}(\text{co } S)))$$
$$R \triangleright S = ((\text{co } R) \triangleleft (\text{co } S) \cap (X \times \text{rng}(S)) \cup (\text{co}(\text{dom}(\text{co } R)) \times \text{rng}(S)).$$

Proof. As an example, the second equality is proven. First note that
$$(x, z) \in \text{co}(\text{dom}(\text{co}\, R)) \times \text{rng}(S) \Leftrightarrow xR = Y \;\wedge\; Sz \neq \emptyset.$$
Then it follows
$$(x, z) \in ((\text{co}\, R) \triangleleft (\text{co}\, S) \cap (X \times \text{rng}(S))) \cup (\text{co}(\text{dom}(\text{co}\, R)) \times \text{rng}(S))$$
$$\Leftrightarrow (\emptyset \subset \text{co}(xR) \subseteq \text{co}(Sz) \;\wedge\; Sz \neq \emptyset) \;\vee\; (xR = Y \;\wedge\; Sz \neq \emptyset)$$
$$\Leftrightarrow (\emptyset \subset Sz \subseteq xR \;\wedge\; xR \neq Y) \;\vee\; (xR = Y \;\wedge\; Sz \neq \emptyset)$$
$$\Leftrightarrow (x, z) \in R \triangleright S. \quad \Diamond$$

To conclude this paragraph, the following relationship is mentioned.
Property 4.5.
$$R \diamond S = ((\text{co}\, R) \diamond (\text{co}\, S) \cap (\text{dom}(R) \times Z)) \cup (\text{co}(\text{dom}(\text{co}\, R)) \times \text{co}(\text{rng}(\text{co}\, S)))$$

Proof.
$$(x, z) \in ((\text{co}\, R) \diamond (\text{co}\, S) \cap (\text{dom}(R) \times Z)) \cup$$
$$\cup (\text{co}(\text{dom}(\text{co}\, R)) \times \text{co}(\text{rng}(\text{co}\, S)))$$
$$\Leftrightarrow (\emptyset \subset \text{co}(xR) = \text{co}(Sz) \;\wedge\; xR \neq \emptyset) \;\vee\; (xR = Sz = Y)$$
$$\Leftrightarrow (\emptyset \subset xR = Sz \;\wedge\; xR \neq Y) \;\vee\; (\emptyset \subset xR = Sz = Y)$$
$$\Leftrightarrow (x, z) \in R \diamond S. \quad \Diamond$$

4.3.3. Expressions in terms of after- and foresets. The classical composition can be written in terms of after- and foresets in the following way.
Property 4.6.
$$R \circ S = \bigcup_{y \in Y} (Ry \times yS).$$

Proof.
$$(x, z) \in \bigcup_{y \in Y} (Ry \times yS) \Leftrightarrow (\exists y \in Y)(x \in Ry \;\wedge\; z \in yS)$$
$$\Leftrightarrow (\exists y \in Y)(y \in xR \;\wedge\; y \in Sz)$$
$$\Leftrightarrow xR \cap Sz \neq \emptyset \Leftrightarrow (x, z) \in R \circ S. \quad \Diamond$$

In contrast with the expressions of the new images in terms of aftersets, the expressions of the new compositions in terms of after- and foresets are not so elegant. For instance, the subcomposition can be expressed as follows

$$R \triangleleft S = \left(\bigcap_{y \in Y} ((\mathrm{co}(Ry) \times Z) \cup (Ry \times yS)) \right) \cap \left(\bigcup_{y \in Y} (Ry \times Z) \right).$$

4.3.4. Convertibility. Important relationships exist between the converses of the compositions of two relations and the compositions of the converses of these relations. These convertibility properties are a welcome help when proving other properties.

Properties 4.7. (Convertibility)

$$(R \circ S)^T = S^T \circ R^T$$
$$(R \triangleleft S)^T = S^T \triangleright R^T$$
$$(R \triangleright S)^T = S^T \triangleleft R^T$$
$$(R \diamond S)^T = S^T \diamond R^T.$$

4.4. Monotonicity

In this subsection the relationship is investigated between the compositions of a subrelation of a relation and the compositions of this latter relation and a given relation.

Properties 4.8. (Monotonicity)

$$R_1 \subseteq R_2 \Rightarrow R_1 \circ S \subseteq R_2 \circ S$$
$$(\mathrm{dom}(R_1) = \mathrm{dom}(R_2) \wedge R_1 \subseteq R_2) \Rightarrow R_2 \triangleleft S \subseteq R_1 \triangleleft S$$
$$R_1 \subseteq R_2 \Rightarrow R_1 \triangleright S \subseteq R_2 \triangleright S$$
$$S_1 \subseteq S_2 \Rightarrow R \circ S_1 \subseteq R \circ S_2$$
$$S_1 \subseteq S_2 \Rightarrow R \triangleleft S_1 \subseteq R \triangleleft S_2$$
$$(\mathrm{rng}(S_1) = \mathrm{rng}(S_2) \wedge S_1 \subseteq S_2) \Rightarrow R \triangleright S_2 \subseteq R \triangleright S_1.$$

Proof. As an example, the second property is proven:

$$(x, z) \in R_2 \triangleleft S \Leftrightarrow \emptyset \subset xR_2 \subseteq Sz \Rightarrow \emptyset \subset xR_1 \subseteq Sz \Leftrightarrow (x, z) \in R_1 \triangleleft S. \ \Diamond$$

4.5. Interaction with union and intersection

As for the images the relationship is investigated between the compositions of the union and the intersection of an arbitrary family of relations and a given relation and the union and the intersection of the compositions of each of these relations and that given relation.

Consider an arbitrary non-empty family $(R_i)_{i \in I}$ of relations from X to Y and a relation S from Y to Z.

Properties 4.9. (Interaction with union)

$$\left(\bigcup_{i\in I} R_i\right) \circ S = \bigcup_{i\in I}(R_i \circ S)$$

$$\bigcap_{i\in I}(R_i \triangleleft S) \subseteq \left(\bigcup_{i\in I} R_i\right) \triangleleft S \subseteq \bigcup_{i\in I}(R_i \triangleleft S)$$

$$\left(\bigcup_{i\in I} R_i\right) \triangleright S \supseteq \bigcup_{i\in I}(R_i \triangleright S).$$

Properties 4.10. (Interaction with intersection)

$$\left(\bigcap_{i\in I} R_i\right) \circ S \subseteq \bigcap_{i\in I}(R_i \circ S)$$

$$\left(\bigcap_{i\in I} R_i\right) \triangleright S = \bigcap_{i\in I}(R_i \triangleright S).$$

Similarly, consider a relation R from X to Y and an arbitrary non-empty family $(S_i)_{i\in I}$ of relations from Y to Z.

Properties 4.11. (Interaction with union)

$$R \circ \left(\bigcup_{i\in I} S_i\right) = \bigcup_{i\in I}(R \circ S_i)$$

$$R \triangleleft \left(\bigcup_{i\in I} S_i\right) \supseteq \bigcup_{i\in I}(R \triangleleft S_i)$$

$$\bigcap_{i\in I}(R \triangleright S_i) \subseteq R \triangleright \left(\bigcup_{i\in I} S_i\right) \subseteq \bigcup_{i\in I}(R \triangleright S_i).$$

Properties 4.12. (Interaction with intersection)

$$R \circ \left(\bigcap_{i\in I} S_i\right) \subseteq \bigcap_{i\in I}(R \circ S_i)$$

$$R \triangleleft \left(\bigcap_{i\in I} S_i\right) = \bigcap_{i\in I}(R \triangleleft S_i).$$

4.6. Associativity

Compared to the study of the images, a new issue comes up in the study of the compositions, namely that of associativity. It is well-known that the classical composition of relations is associative. In this subsection other possible associativity properties are investigated.

Consider a relation R from X to Y, a relation S from Y to Z and a relation T from Z to U.

Properties 4.13. (Associativity)

$$R \circ (S \circ T) = (R \circ S) \circ T$$
$$R \circ (S \triangleright T) \subseteq (R \circ S) \triangleright T$$
$$R \triangleleft (S \circ T) \supseteq (R \triangleleft S) \circ T$$
$$R \triangleleft (S \triangleleft T) \subseteq (R \circ S) \triangleleft T$$
$$R \triangleleft (S \triangleright T) = (R \triangleleft S) \triangleright T$$
$$R \triangleright (S \circ T) \supseteq (R \triangleright S) \triangleright T .$$

Proofs.

2. First note that

$$(x, u) \in R \circ (S \triangleright T) \Leftrightarrow xR \cap (S \triangleright T)u \neq \emptyset$$
$$\Leftrightarrow (\exists y \in xR)(y \in (S \triangleright T)u)$$
$$\Leftrightarrow (\exists y \in xR)(\emptyset \subset Tu \subseteq yS)$$
$$\Leftrightarrow (\exists y \in xR)(\forall z \in Tu)(z \in yS) \wedge Tu \neq \emptyset \quad \text{(A)}$$

and

$$(x, u) \in (R \circ S) \triangleright T \Leftrightarrow \emptyset \subset Tu \subseteq x(R \circ S)$$
$$\Leftrightarrow (\forall z \in Tu)(z \in x(R \circ S)) \wedge Tu \neq \emptyset$$
$$\Leftrightarrow (\forall z \in Tu)(xR \cap Sz \neq \emptyset) \wedge Tu \neq \emptyset$$
$$\Leftrightarrow (\forall z \in Tu)(\exists y \in xR)(y \in Sz) \wedge Tu \neq \emptyset$$
$$\Leftrightarrow (\forall z \in Tu)(\exists y \in xR)(z \in yS) \wedge Tu \neq \emptyset \quad \text{(B)}.$$

Comparing expressions (A) and (B) shows that expression (B) follows from expression (A) but not conversely. Hence $R \circ (S \triangleright T) \subseteq (R \circ S) \triangleright T$.

3. Follows immediately from the foregoing property using the convertibility properties.

$$T^T \circ (S^T \triangleright R^T) \subseteq (T^T \circ S^T) \triangleright R^T$$
$$\Leftrightarrow (T^T \circ (S^T \triangleright R^T))^T \subseteq ((T^T \circ S^T) \triangleright R^T)^T$$
$$\Leftrightarrow (S^T \triangleright R^T)^T \circ T \subseteq R \triangleleft (T^T \circ S^T)^T$$
$$\Leftrightarrow (R \triangleleft S) \circ T \subseteq R \triangleleft (S \circ T).$$

4. First note that

$$(x, u) \in R \triangleleft (S \triangleleft T) \Leftrightarrow \emptyset \subset xR \subseteq (S \triangleleft T)u$$

$$\Leftrightarrow xR \neq \emptyset \ \land \ (\forall y \in xR)(y \in (S \triangleleft T)u)$$

$$\Leftrightarrow xR \neq \emptyset \ \land \ (\forall y \in xR)(\emptyset \subset yS \subseteq Tu)$$

$$\Leftrightarrow xR \neq \emptyset \ \land \ (\forall y \in xR)(yS \neq \emptyset) \ \land \ (\forall y \in xR)(\forall z \in yS)(z \in Tu) \text{ (A)}$$

and

$$(x, u) \in (R \circ S) \triangleleft T \Leftrightarrow \emptyset \subset x(R \circ S) \subseteq Tu$$

$$\Leftrightarrow (\exists z \in Z)(z \in x(R \circ S)) \ \land \ (\forall z \in Z)(z \in x(R \circ S) \Rightarrow z \in Tu)$$

$$\Leftrightarrow (\exists z \in Z)(xR \cap Sz \neq \emptyset) \ \land \ (\forall z \in Z)(xR \cap Sz \neq \emptyset \Rightarrow z \in Tu) \text{ (B)}.$$

It is easy to show that expression (B) can be deduced from expression (A).

Indeed, from $xR \neq \emptyset \ \land \ (\forall y \in xR)(yS \neq \emptyset)$ it follows that $(\exists y \in Y)(\exists z \in Z)(y \in xR \land z \in yS)$ and thus $(\exists z \in Z)(xR \cap Sz \neq \emptyset)$. Consider $z \in Z$ such that $xR \cap Sz \neq \emptyset$. This means that $(\exists y \in Y)(y \in xR \land y \in Sz)$ and thus $(\exists y \in Y)(y \in xR \land z \in yS)$. With (A) it follows that $z \in Tu$, and hence (B). Notice that from (B) it does not follow that $(\forall y \in xR)(yS \neq \emptyset)$!

5. $$(x, u) \in R \triangleleft (S \triangleright T) \Leftrightarrow \emptyset \subset xR \subseteq (S \triangleright T)u$$

$$\Leftrightarrow xR \neq \emptyset \ \land \ (\forall y \in xR)(y \in (S \triangleright T)u)$$

$$\Leftrightarrow xR \neq \emptyset \ \land \ (\forall y \in xR)(\emptyset \subset Tu \subseteq yS)$$

$$\Leftrightarrow xR \neq \emptyset \ \land \ Tu \neq \emptyset \ \land \ (\forall y \in xR)(\forall z \in Tu)(z \in yS)$$

$$\Leftrightarrow xR \neq \emptyset \ \land \ Tu \neq \emptyset \ \land \ (\forall z \in Tu)(\forall y \in xR)(y \in Sz)$$

$$\Leftrightarrow Tu \neq \emptyset \ \land \ (\forall z \in Tu)(\emptyset \subset xR \subseteq Sz)$$

$$\Leftrightarrow \emptyset \subset Tu \subseteq x(R \triangleleft S)$$

$$\Leftrightarrow (x, u) \in (R \triangleleft S) \triangleright T.$$

6. Follows immediately from the fourth property using the convertibility properties. $\lozenge$

Remark 4.2.

- Only the first property is an example of genuine associativity, while the fifth property can be seen as some kind of mixed associativity. Due to the equalities in these properties the following notations are justified: $R \circ S \circ T$ and $R \triangleleft S \triangleright T$.
- For their definitions, Bandler and Kohout found the following associativity properties

$$R \triangleleft_{bk} (S \triangleleft_{bk} T) = (R \circ S) \triangleleft_{bk} T$$
$$R \triangleleft_{bk} (S \triangleright_{bk} T) = (R \triangleleft_{bk} S) \triangleright_{bk} T$$
$$R \triangleright_{bk} (S \circ T) = (R \triangleright_{bk} S) \triangleright_{bk} T .$$

Comparing these results with properties 4.13 shows that for the improved definitions the first and the third of these equalities become inequalities. Moreover, some new properties have been discovered.

- Concerning the fourth property it is easy to show that the equality does not hold in general. Let $X = \{x\}$, $Y = \{y_1, y_2\}$, $Z = \{z\}$ and $U = \{u\}$. Let $R = \{(x, y_1), (x, y_2)\}$, $S = \{(y_1, z)\}$ and $T = \{(z, u)\}$. It is easy to see that $(x, u) \in (R \circ S) \triangleleft T$ while $(x, u) \notin R \triangleleft (S \triangleleft T)$.

5. Conclusion

It has been shown that the compositions of Bandler and Kohout are subject to some improvement. Modified definitions have been suggested and have been studied intensively. Throughout the overview of their properties, it has been indicated that most of the properties of the Bandler-Kohout compositions are no longer valid for the new definitions. Similar to the new compositions, new images of a set under a relation have been introduced and have been discussed in detail. This paper serves as a reference for researchers working with compositions of relations.

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