Mathematica Pannonica
3/2 (1992), 115 – 123

THE CONGRUENCE LATTICE OF IMPLICATION ALGEBRAS*

Sándor **Radelecki**

Mathematical Institute, University of Miskolc, 3515 Miskolc-Egyetemváros, Hungary.

Received April 1991

AMS Subject Classification: 08 A 02, 08 B 10, 20 L 05, 06 D 99.

Keywords: 3-congruence permutable, 3-congruence modular, 3-congruence distributive, 3-filtral variety, minimal quasivariety, implication lattice, tolerance-trivial algebra.

Abstract: The variety of implication algebras is a minimal quasivariety. It is 3-filtral but not 2-filtral. An implication algebra A is tolerance-trivial iff $(A, \leq)$ is a lattice, where the partial ordering "$\leq$" is defined as follows: $a \leq b \Leftrightarrow \exists x \in A$ such that $b = x \cdot a$.

1. Introduction

Implication algebras are groupoids with a simple binary operation, which yields a partially order. This derived order structure can be considered as a generalization of Boolean lattices (see Prop.2).

Definition 1 ([1], [9]). A groupoid $(A, \cdot)$ is called an *implication algebra* if the operation " $\cdot$ " satisfies the following axioms:

$$(a \cdot b) \cdot a = a$$
$$(a \cdot b) \cdot b = (b \cdot a) \cdot a$$
$$a \cdot (b \cdot c) = b \cdot (a \cdot c).$$

* Research partially supported by Hungarian National Foundation for Scientific Research grant Nr. 1813.

Example. If $(B, \vee, \wedge, 0, 1, ^-)$ is a Boolean algebra then $(B, \to)$ and $(B, /)$, where $a \to b = a^- \vee b$ and $a / b = a^- \wedge b$ for all $a, b \in B$, are both implication algebras.

Remark. If the algebra above is the Boolean algebra of propositional calculus then "$\to$" represents ordinary implication.

Implication algebras are examples of algebraic varieties which are *3-permutable, 3-congruence distributive* and *3-congruence modular* but are not either *congruence permutable* or *2-distributive* or *2-modular*: [9], [4].

In this paper we shall prove a new property of implication algebras, namely that they are *3-filtral* but not *2-filtral* (§2) and we shall characterize those implication algebras on which every compatible tolerance is a congruence (§3)

Let us first review a few concepts:

A variety V is *congruence permutable (congruence 3-permutable)* if $\Theta_1 \circ \circ \Theta_2 = \Theta_2 \circ \Theta_1$ $(\Theta_1 \circ \Theta_2 \circ \Theta_1 = \Theta_2 \circ \Theta_1 \circ \Theta_2)$ for any two congruences $\Theta_1, \Theta_2 \in \text{Con } A$ and for any $A \in V$ (where " $\circ$ " is the relational product of congruences); *3-congruence modularity* and *3-congruence distributivity* mean that the systems of equations of H.P. Gumm and B. Johnson respectively for congruence modularity and congruence distributivity consist of at least 3+1 terms.

For example *3-distributivity* means that the following system of equations (where $n, i \in \mathrm{N}$; $q_o, q_1, \ldots, q_n$ are 3-variable terms):

$$(1) \qquad \begin{aligned} &q_o(x,y,z) = x, \quad q_n(x,y,z) = z \\ &q_i(x,y,x) = x, \quad 0 \le i \le n \\ &q_i(x,x,y) = q_{i+1}(x,x,y), \;\; i \text{ even} \\ &q_i(x,y,y) = q_{i+1}(x,y,y), \;\; i \text{ odd} \end{aligned}$$

must contain at least 3+1 terms, i.e.: $n = 3$.

For implication algebras these terms are:

$$(2) \qquad \begin{aligned} &q_0(x,y,z) = x, && q_3(x,y,z) = z \\ &q_1(x,y,z) = [y \cdot (z \cdot x)] \cdot x, && q_2(x,y,z) = (x \cdot y) \cdot z \end{aligned}$$

for all $x, y, z \in A$.

Filtral varieties can be defined using the notion of *product congruence:*

Let A be the subdirect product of algebras A_i ($i \in I$) and let a_i denote the i-th component of $a \in A$ belonging to A_i. A congruence $\varphi \in \operatorname{Con} A$, is called the *product of the congruences* $\varphi_i \in \operatorname{Con} A_i$, $i \in I$ if $a \,\varphi\, b$ exactly when $a_i \,\varphi_i\, b_i$ for all $i \in I$. We write $\varphi = \prod_{i \in I} \varphi_i$.

Definition 2 ([7],[8]). A variety $\mathcal{V}$ is called an *ideal variety* iff for all $A \in \mathcal{V}$ every compact congruence on A is a product congruence.

Definition 3 ([7],[8]). A variety $\mathcal{V}$ is called *filtral* if it is an ideal variety and it is *semi-simple* i.e. all its subdirect irreducible algebras are congruence-simple.

We shall denote the class of subdirect irreducible algebras of a variety $\mathcal{V}$ by SI$\mathcal{V}$, and the variety of implication algebras by $\mathcal{V}$(I). E.Fried and E. Kiss [5] gave the following characterization of filtral varieties by term functions (see also [8]):

Theorem ([5],[8]).: *A variety $\mathcal{V}$ is filtral iff there is an $n \in$ N and there are 3-variable terms $f_0, f_1, \dots, f_n$ $(n > 1)$ such that for any x, y, z in any algebra of $\mathcal{V}$ we have:*

$$\text{(3)}\qquad \begin{aligned}
&\text{(a)} \quad f_0(x,y,z) = x, \quad f_n(x,y,z) = z,\\
&\text{(b)} \quad f_i(x,y,x) = x, \quad (\textit{for all } i : 0 \le i \le n),\\
&\text{(c)} \quad f_i(x,x,z) = f_{i+1}(x,x,z), \quad \textit{for } i \textit{ even},\\
&\text{(d)} \quad \textit{for all } A \in \mathrm{SI}\mathcal{V} \quad \textit{and} \quad x,y,z \in A,\ x \ne y:\\
&\qquad\ f_i(x,y,z) = f_{i+1}(x,y,z), \quad \textit{for } i \textit{ odd}.
\end{aligned}$$

Proceding in the same way as in characterization of congruence modular and congruence distributive varieties by a system of term equation, we can use the following concept:

Definition 4. According to the theorem above, if the system (3) of equations for $\mathcal{V}$ needs at least $n+1$ terms, then $\mathcal{V}$ is called *n-filtral.* Eg. $\mathcal{V}$ is *3-filtral* if $n = 3$ and f_0, f_1, f_2, f_3 satisfy conditions (3).

Let us now list some properties of implication algebras:

Property 1 ([1]). Let be A an implication algebra. We can define an partially ordering relation "$\le$" on A as follows:

$$a \le b \Leftrightarrow \exists x \in A : b = x \cdot a.$$

J.C.Abbott has shown [1] that this relation is isotone on the left and antitone on the right with respect to "$\cdot$" (i.e. $\forall c \in A$, if $a \le b : c \cdot a \le c \cdot b$ and $a \cdot c \ge b \cdot c$); furthermore $(A, \le)$ is a semilattice with identity, i.e. $\sup\{a, b\} = (a \cdot b) \cdot b$ exists for all $a, b \in A$ and there is an element

$1 \in A$ such that $x \leq 1$ for all $x \in A$. "$\leq$" can be defined using 1, since $a \leq b \Leftrightarrow a \cdot b = 1$.

Property 2 ([1]). If $(A, \leq)$ is the semilattice corresponding to the implication algebra $(A, \cdot)$, then every principal filter $(\{x|x \geq a\}, \leq)$ is a Boolean lattice. Vice versa in every semilattice with the above mentioned property one can define a binary operation "$\cdot$" for which $(A, \cdot)$ is an implication algebra in the following way:

$$a \cdot b = (a \vee b)_b^-$$

where $(a \vee b)_b^-$ denotes the complement of $a \vee b$ in the Boolean lattice $(\{x|x \geq b\}, \leq)$.

Property 3 ([1]). For a pair $a, b \in A$, $\inf\{a, b\}$ exists exactly when $\{a, b\}$ has a common lower bound $c \in A$. In that case $\inf\{a, b\} = [a \cdot (b \cdot c)] \cdot c$.

Remark ([1]). $(A, \leq)$ is a Boolean lattice iff it has a least element, denoted by 0 $(0 \leq x$, for all $x \in A)$.

Definition 5 ([1]). If $(A, \cdot)$ is an implication algebra and if the derived partially ordered set $(A, \leq)$ is a lattice (i.e. for all $a, b \in A$ $\inf\{a, b\} = a \wedge b$ exists), then $(A, \leq)$ (and $(A, \cdot, \leq)$ as well) is called an *implication lattice*.

2. The variety and congruences of implication algebras

One of the most notable properties of implication algebras is that is a one-to-one correspondence between their congruences and their filters.

A subset $F \subseteq A$ of a partially ordered set $(A, \leq)$ is called a *filter* if for all $a \in F$ and $x \in A$, $x \geq a \Rightarrow x \in F$ and if $\inf\{x_1, x_2\} = x_1 \wedge x_2$ exists for $x_1, x_2 \in F$, then $x_1 \wedge x_2 \in F$. E.g. $[a] = \{x \in A | x \leq a\}$ is a filter, called the *principal filter* belonging to a. By Property 1 if $a \neq b$ then $[a] \neq [b]$.

One can easily show that the intersection of a given family $\{F_i\}_{i \in I}$, $I \neq \emptyset$ of filters of $(A, \leq)$ is also a filter; $\coprod_{i \in I} F_i$ can be defined as the intersection of all filters containing the set $\bigcup_{i \in I} F_i$. If $\mathcal{F}_A$ denotes the set of all filters of an implication algebra $(A, \cdot)$, then $(\mathcal{F}_A, \coprod, \bigcap, A, \{1\})$ is a distributive complete lattice with 1 and 0.

From now on let $\Theta[a]$ denote the congruence class of Θ belonging to $a \in A$, i.e.: $\Theta[a] = \{x \in A | x \,\Theta\, a\}$.

Property 4 ([1]). *The mapping* $i : \operatorname{Con} A \to \mathcal{F}_A$, $i(\Theta) = \Theta[1]$ *is an isomorphism between* $(\operatorname{Con} A, \wedge, \vee, 1_A, 0_A)$ *and* $(\mathcal{F}_A, \bigcap, \coprod, A, \{1\})$. *For any* $F \in \mathcal{F}_A$, $i^{-1}(F) = \Theta_F$, *where* $a \,\Theta_F\, b \Leftrightarrow a \cdot b,\ b \cdot a \in F$ (i^{-1} denotes the inverse of the mapping i).

Proposition 1. *The variety of implication algebras is a minimal quasivariety.*

Proof. We begin by showing that $\mathcal{V}(\mathrm{I})$ has only one subdirect irreducible algebra, namely the 2-element one.

Let $A \in \mathrm{SI}\mathcal{V}(\mathrm{I})$, γ its monolit, and F_γ the filter belonging to γ. Since $\gamma \leq \Theta$ for all $\Theta \in \operatorname{Con} A (\Theta \neq 0_A)$, therefore $F_\gamma \subseteq \bigcap_{x \in A} [x]$ and so there exists an $a \in F_\gamma$ such that $F_\gamma = [a] = \{1, a\}$ and

$$(4) \qquad a \geq x \text{ for all } x \in A \setminus \{1\}.$$

Suppose now that there exists an $x \in A \setminus \{1\}$ such that $x \neq a$. Since $([x], \leq)$ is a Boolean lattice (see Prop.2) and $a \in [x]$, there exists an $a^- \in [x]$ such that $a^- \wedge a = x$, and $a^- \vee a = 1$.

Now (4) gives $a^- \leq a \neq 1$ - which is a contradiction. Thus $A = \{1, a\}$, i.e. A has two elements.

Two element implication algebras are isomorphic to each other and so $\mathrm{SI}\mathcal{V}(\mathrm{I})$ contains only one non-trivial algebra (and this one is congruence and subalgebra simple at the same time).

A locally finite variety $\mathcal{V}$ is a minimal quasivariety exactly when it has only one SI algebra and this can be embedded into every non-trivial $B \in \mathcal{V}$ (see [2], Cor.2).

By [1] the number of elements in any free implication algebra generated by n elements is at most 2^{2^n}. Therefore any finitely generated implication algebra is finite and so $\mathcal{V}(\mathrm{I})$ is locally finite.

On the other hand for every nontrivial $B \in \mathcal{V}(\mathrm{I})$ and $x \in B$, $x \neq 1$, $\{1, x\}$ is a two-element subalgebra of B and thus $\mathcal{V}(\mathrm{I})$ satisfies all previous conditions. $\diamondsuit$

Corollary 1. *Every implication algebra* $(A, \cdot)$ *is a subdirect power of two element implication algebra* $(\{1, a\}, \cdot)$.

Theorem 1. *The variety of implication algebras is 3-filtral but not 2-filtral.*

Proof. Assuming that $\mathcal{V}(\mathrm{I})$ is 2-filtral means there are three 3-variable terms f_0, f_1, f_2 sufficient for $\mathcal{V}(\mathrm{I})$ in the system (3) of equations. But in this case from (3) we get that $\mathcal{V}(\mathrm{I})$ is 2-distributive, contradicting [8].

To prove that $\mathcal{V}(\mathrm{I})$ is 3-filtral we shall use the terms q_0, q_1, q_2, q_3 from (2)-which were used first for distributivity. Let us check the identities of (3):

(a) is clear;

(b) $q_i(x,y,z) = x, \quad 0 \le i \le 2$ (by distributivity - (1));

(c) From (1) we have $q_0(x,x,z,) = q_1(x,x,z)$ and $q_2(x,x,z) = = q_3(x,x,z)$;

(d) Let x,y,z be elements of the subdirect irreducible algebra $(\{0,1\},\cdot)$ and let $x \neq y$:
If $x = 0$ and $y = 1$ then $q_1(0,1,z) = [1\cdot(z\cdot 0)]\cdot 0 = (z\cdot 0)\cdot 0 = = \sup\{z,0\} = z$, $q_2(0,1,z) = (0\cdot 1)\cdot z = z$;
If $x = 1$ and $y = 0$ then $q_1(1,0,z) = [0\cdot(z\cdot 1)]\cdot 1 = 1$, $q_2(1,0,z) = = (1\cdot 0)\cdot z = 0\cdot z$. Since $0\cdot 0 = 1$ and $0\cdot 1 = 1$, we have $0\cdot z = 1$.
To sum up: if $x \neq y$ then $q_1(x,y,z) = q_2(x,y,z)$ and so all the identities of (3) are satisfied. $\Diamond$

Corollary 2. *Every compact* $\Theta \in \mathrm{Con}\,A\,(A \in \mathcal{V}(\mathrm{I}))$ *has a complement.*

Proof. By [7] (and [8]) if $\mathcal{V}(\mathrm{I})$ is filtral then every compact congruence on $\mathcal{V}$ has a complement. $\Diamond$

Let $\mathrm{Con}^c A$ denote the lattice of compact congruences of A ; $\mathrm{Con}^{*c} A$ is the same lattice together with the element "1_A" and let $\mathcal{B}(\mathrm{Con}^{*c} A)$ be the Boolean lattice generated by $\mathrm{Con}^{*c} A$. (This one always exists, see [6]). Denoting the complement of $\Theta \in \mathrm{Con}\,A$ by Θ^-, let us define the operation " $*$ " on $\mathrm{Con}\,A$ as follows: $\Theta * \varphi = \Theta^- \vee \varphi$. (This way we obtain from $\mathcal{B}(\mathrm{Con}^{*c} A)$ an implication algebra in which, by [1], $(A,\cdot)$ can be dually embedded). Let Θ_a denote the congruence belonging to the principal filter [a] ($a \in A$), (and at the same time to the element $a \in A$ as well).

Proposition 2. *Let* $(A,\cdot)$ *be an implication algebra and* $(A,\le)$ *the derived partially ordered set. The following statements are equivalent:*

(i) $(A,\le)$ *is a Boolean lattice*;

(ii) $(A,\le)$ *and* $(\mathrm{Con}^{*c} A,\le)$ *are dually order-isomorphic*;

(iii) $(A,\cdot)$ *and* $(\mathcal{B}(\mathrm{Con}^{*c} A),*)$ *are dually isomorphic implication algebras.*

Proof. (i) $\Rightarrow$ (ii) by [11]. (For a more general construction see [6]).

(ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (ii): Since $\mathrm{Con}^{*c} A$ and $\mathcal{B}(\mathrm{Con}^{*c} A)$ both have a greatest element, $(A,\le)$ has a least element and therefore by [1] it is a Boolean algebra.

(i) $\Rightarrow$ (iii): If $\Theta \in \mathrm{Con}^{*c} A$, then Θ can be written as a finite union of principal filters $[a_1], \ldots, [a_n]$ $(a_1, \ldots, a_n \in A,\ n \in \mathbb{N})$. Since $(A, \leq)$ is a lattice, $[a_1] \amalg \ldots \amalg [a_n] = [a_1 \wedge \ldots \wedge a_n]$ and therefore $\Theta[1]$ is a principal filter, i.e. there is an $a_\Theta \in A$ such that $[a_\Theta] = \Theta[1]$.

If $\overline{a}$ denotes the complement of a and $\Theta_{\overline{a}}$ the corresponding congruence then $[a] \cap [\overline{a}] = \{x | x \geq a \text{ and } x \geq \overline{a}\} = \{x | x \geq 1\} = \{1\}$, so $\Theta_a \wedge \Theta_{\overline{a}} = 0_A$ and $[a] \cup [\overline{a}] = [a \wedge \overline{a}] = [0] = A$, i.e.: $\Theta_a \vee \Theta_{\overline{a}} = 1_A$. Hence Θ_a and $\Theta_{\overline{a}}$ are complements of each other; furthermore since for all $\Theta \in \mathrm{Con}^c A$ there is an $a \in A$ such that $\Theta_a = \Theta$, $\overline{\Theta} \in \mathrm{Con}^c A$ holds as well (for all $\Theta \in \mathrm{Con}^c A$). However, this means that $\mathrm{Con}^{*c} A = \mathcal{B}(\mathrm{Con}^{*c} A)$ and by (i)$\Leftrightarrow$ (ii) $(A, \leq)$ and $(\mathcal{B}(\mathrm{Con}^{*c} A), \leq)$ are dually order isomorphic Boolean algebras. But in that case, by [1] again, they are dually isomorphic as implication algebras. $\Diamond$

3. Reflexive, compatible relations on implication algebras

A compatible relation $\rho \leq A \times A$ on $(A, \cdot)$ is called a *compatible tolerance* if ρ is reflexive and symmetric ([3]).

Definition 6 ([3]). An algebra $A \in \mathcal{V}$ is called *tolerance-trivial* (*T-trivial*) if every compatible tolerance on A is a congruence (i.e. transitive as well).

Theorem 2. *Let $(A, \cdot)$ be an implication algebra. Then the following statements are equivalent:*

(i) *Every reflexive compatible relation on $(A, \cdot)$ is a congruence;*

(ii) *$(A, \cdot)$ is tolerance-trivial;*

(iii) *$(A, \leq)$ is an implication lattice.*

Proof. (i) $\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (iii): Let us define a relation ρ as follows: $a \,\rho\, b \Leftrightarrow$ there is a $k \in A$ such that $a \geq k$ and $b \geq k$. By definition ρ is reflexive and symmetric. Let us show that ρ is compatible as well. Consider $c \,\rho\, d$ $(c, d \in A)$. This means that there is an $l \in A$ such that $c \geq l$ and $d \geq l$. Then $ca \geq a \geq k$ and $db \geq b \geq k$, while $ac \geq c \geq l$ and $bd \geq d \geq l$, thus $ca \,\rho\, db$ and $ac \,\rho\, bd$, ie.: ρ is compatible. By (ii) ρ is a congruence and $1 \,\rho\, a$ for any $a \in A$. Therefore $\rho = 1_A$. However, this means that for any $a, b \in A$, $\{a, b\}$ has a lower bound $m \in A$. By Prop.3 of [1] inf $\{a, b\}$ exists for all $a, b \in A$ and hence $(A, \leq)$ is an implication lattice.

(iii) $\Rightarrow$ (i): Let us now assume that $(A, \cdot)$ is an implication lattice. Using the idea of [4] (Th.8) first we show that if $(A, \leq)$ is a Boolean lattice then it satisfies (i). Indeed in that case there is a $0 \in A$ such that $0 \leq x$ for all $x \in A$ and by [1] again the complement of a, denoted by $\overline{a}$, can be obtained as $\overline{a} = a \cdot 0$. Since $a \vee b = (a \cdot b) \cdot b, a \wedge b = [a \cdot (b \cdot 0)] \cdot 0$, every compatible relation on $(A, \cdot)$ is also a compatible relation on $(A, \wedge, \vee, 1, 0, ^-)$. But since this algebra belongs to a Mal'cev variety all its reflexive compatible relations are congruences [3].

Now let $(A, \cdot)$ be an implication lattice and ρ a compatible reflexive relation on A. Let $a \, \rho \, b, b \, \rho \, c$ (for $a, b, c \in A$). Then $(a \wedge b) \wedge c = d$ exists and it is the greatest lower bound of $\{a, b, c\}$. The restriction of " $\cdot$ " to the principal filter $[d]$ is a Boolean algebra (with "0" element d) and $a, b, c \in [d]$.

On the other hand the restriction of ρ to $[d]$ is also compatible and reflexive and thus it is also a congruence on $([d], \cdot)$. But this means that $a \, \rho \, b \Rightarrow b \, \rho \, a$ and $a \, \rho \, b, b \, \rho \, c \Rightarrow a \, \rho \, c$. In conclusion ρ is a congruence on $(A, \cdot)$ as well. $\diamondsuit$

Corollary 3. *Let $(A, \cdot)$ be an implication algebra. If the derived structure $(A, \leq)$ is an implication lattice, then the congruences of $(A, \cdot)$ permute.*

Proof. In this case $(A, \cdot)$ is tolerance-trivial by Th.2. According to [10] every tolerance-trivial algebra has permutable congruences. $\diamondsuit$

Corollary 4. *For a finite implication algebra $(A, \cdot)$ the following statements are equivalent:*

(i) The derived partially ordered set $(A, \leq)$ is a Boolean lattice;

(ii) $(A, \cdot)$ is tolerance-trivial;

(iii) $(A, \cdot)$ *and* $(\mathrm{Con}\, A, *)$ *are dually isomorphic;*

(iv) $(A, \leq)$ *and* $(\mathrm{Con}\, A, \leq)$ *are dually order isomorphic.*

Proof. The proof is based on the fact that if A is finite then all its congruences are compact and so $\mathrm{Con}\, A = \mathrm{Con}^c A = \mathrm{Con}^{*c} A = \mathcal{B}(\mathrm{Con}^{*c} A)$. Applying Prop.2 we get Cor.4.

Acknowledgments. The author wishes to thank Professor E. W. Kiss and E. T. Schmidt for their helpful advices they gave during preparation of the paper, Professor G. Nyerges and Professor F. Soós for their help in redaction of it.

References

[1] ABBOTT, J. C.: Semi-Boolean algebra. *Mat. Vesnik* **4** (1967), 177 – 198.

[2] BERGMAN, C.H. and Mc KENZIE, R.: Minimal varieties and quasivarieties (Manuscript).

[3] CHAJDA, I.: Tolerance trivial algebras and varieties, *Acta Sci. Math.* (Szeged) **46** (1983), 35 – 40.

[4] CHAJDA, I. and ZELINKA, B.: Tolerances and congruences on implication algebras, *Czech. Math. J.* (Praha) **38 (113)** (1988), 207 – 217.

[5] FRIED, E. and KISS, E. W.: Connection between congruence-lattice and polynomial properties, *Algebra Universalis* **17** (1983), 227 – 262.

[6] GRÄTZER, G. and SCHMIDT, E. T.: On the generalized Boolean algebra generated by a distributive lattice, *Indagationes Math.* **18** (1958), 547 – 553.

[7] KISS, E. W.: Complemented and skew congruences, *Ann. Univ. Ferrara*-Ser VII. Sc. Mat. **29** (1983) 111 – 127.

[8] KISS, E. W.: Moduláris-varietások, Kandidátusi értekezés, 1986.

[9] MITSCHKE, A.: Implication algebras are 3-permutable and 3-distributive, *Algebra Universalis* **1** (1971), 182 – 186.

[10] RAFTERY, J. G., ROSENBERG, I. G., STURM, T.: Tolerance relations and BCK-algebras, *Math. Japonica* 36/3 (1991), 399 – 410.

[11] SCHMIDT, E. T.: A survey on congruence lattice representation (Manuscript).