Mathematica Pannonica
3/2 (1992), 107 – 113

REMARKS ON LOCALLY CLOSED SETS

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Received December 1991

AMS Subject Classification: 54 A 05, 54 A 10; 54 D 15

Keywords: Locally closed set, α-topology, semi-open set.

Abstract: This paper provides a useful characterization of $LC(X, \tau^\alpha)$, i.e. the family of locally closed subset of (X, τ^α), where τ^α denotes the α-topology of a topological space (X, τ). In addition, we consider various statements about the family of locally closed subsets of an arbitrary space and examine the relationships between these statements.

1. Introduction and preliminaries

Recently there has been some interest in the notion of a locally closed subset of a topological space. According to Bourbaki [5] a subset S of a space (X, τ) is called locally closed if it is the intersection of an open set and a closed set. Ganster and Reilly used locally closed sets in [7] and [8] to define the concept of LC-continuity, i.e. a function $f : X \to Y$ is LC-continuous if the inverse with respect to f of any open set in Y is locally closed in X. This enabled them to produce a decomposition of continuity for functions between arbitrary topological spaces. Later on, Jelić [9] extended their results to the bitopological setting by providing a decomposition of pairwise continuity and, quite recently, Balachandran and Sundaram studied several variations of LC-continuity in [4] and [13]. Finally, locally closed sets have been used by Aho and Nieminen [1] in their study of α-spaces and irresolvability.

In this paper we begin by characterizing LC(X, τ^α), i.e. the family of locally closed subsets of (X, τ^α) where τ^α denotes the associated α-topology of a space (X, τ). Our first result points out that the family LC(X, τ^α) has already been investigated by Kuratowski [10] in a different context and, moreover, that LC(X, τ^α) coincides with the collection of δ-sets in (X, τ) [6]. We then move on to consider various statements about the family of locally closed subsets of an arbitrary space (X, τ) and examine the relationships between these statements.

Let (X, τ) be a topological space. For a subset S of X, the closure and the interior of S with respect to (X, τ) will be denoted by cl S and int S respectively.

Definition 1.1. A subset S of a space (X, τ) is called

(i) *semi open* if $S \subseteq \text{cl}\,(\text{int}\, S)$;

(ii) *semi-closed* if $X - S$ is semi-open, or, equivalently, if $\text{int}\,(\text{cl}\, S) \subseteq S$;

(iii) an *α-set* if $S \subseteq \text{int}\,(\text{cl}\,(\text{int}\, S))$;

(iv) nwd (=*nowhere dense*) if $\text{int}\,(\text{cl}\, S) = \emptyset$.

The collections of semi-open sets, semi-closed sets and α-sets in (X, τ) will be denoted by SO(X, τ), $SC(X, \tau)$ and τ^α respectively. Njåstad [11] has shown that τ^α is a topology on X with the following properties: $\tau \subseteq \tau^\alpha$, $(\tau^\alpha)^\alpha = \tau^\alpha$ and $S \in \tau^\alpha$ if and only if $S = U - N$ where $U \in \tau$ and N is nwd in (X, τ). Hence $\tau = \tau^\alpha$ if and only if every nwd set in (X, τ) is closed. Clearly every α-set is semi-open and every nwd set in (X, τ) is semi-closed. Andrijević [2] has observed that SO(X, τ^α) = SO(X, τ), and that $N \subseteq X$ is nwd in (X, τ^α) if and only if N is nwd in (X, τ).

Definition 1.2. A subset S of (X, τ) is called

(i) *locally closed* if $S = U \cap F$ where U is open and F is closed, or, equivalently, if $S = U \cap \mathrm{cl}\, S$ for some open set U;

(ii) *co-locally closed* if $X - S$ is locally closed, or, equivalently, if $S = U \cup F$ where U is open and F is closed and nwd.

We will denote the collections of all locally closed sets and co-locally closed sets of (X, τ) by $\mathrm{LC}(X, \tau)$ and co-$\mathrm{LC}(X, \tau)$ respectively. Note that Stone [12] has used the term *FG* for a locally closed subset. A dense subset of (X, τ) is locally closed if and only if it is open. More generally, Ganster and Reilly [7] have pointed out that, if $S \subseteq X$ is nearly open, i.e. if $S \subseteq \mathrm{int}\,(\mathrm{cl}\, S)$, then S is locally closed if and only if S is open. It is easy to check that (X, τ) is submaximal, i.e. every dense set is open, if and only if every subset of X is locally closed. Finally, spaces in which singletons are locally closed are called $\mathrm{T_D}$-spaces [3].

No separation axioms are assumed unless explicitly stated.

2. Locally closed sets in α-spaces

Let (X, τ) be a topological space and let us denote by J the ideal of nwd subsets of (X, τ). On page 69 in [10] Kuratowski defined a subset $A \subseteq X$ to be open mod J if there exists an open set G such that $A - G \in J$ and $G - A \in J$.

Proposition 2.1 (see page 69 in [10]). *Let J denote the ideal of nwd sets in a space (X, τ). Then*

1) *open sets are open* $\mathrm{mod} J$;
2) *closed sets are open* $\mathrm{mod} J$;
3) *if A, B are open* $\mathrm{mod} J$ *then $A \cap B$, $A \cup B$ and $X - A$ are open* $\mathrm{mod} J$;
4) *$A \subseteq X$ is open* $\mathrm{mod} J$ *if and only if $A = U \cup N$ where U is open and N is* nwd *in (X, τ).*

In order to state our main result in this section we need some more definitions. A subset S of a space (X, τ) is called semi-locally closed [13] if it is the intersection of a semi-open set and a semi-closed set. A subset S of (X, τ) is said to be a δ-set in (X, τ) [6] if $\mathrm{int}(\mathrm{cl} S) \subseteq \mathrm{cl}(\mathrm{int} S)$.

Theorem 2.2. *Let A be a subset of a space (X, τ) and let J denote the ideal of nwd subsets of (X, τ). Then the following are equivalent:*

1) $A \in \mathrm{LC}(X, \tau^\alpha)$;
2) *A is semi-locally closed* ;
3) *A is a δ-set* ;
4) *$A = U \cup N$ where U is open and N is* nwd *in (X, τ)* ;

5) *A is open* mod J.

Proof. 1) $\Rightarrow$ 2): This is obvious since every α-set is semi-open.
2) $\Rightarrow$ 3): Let $A = S \cap T$ where $S \in \mathrm{SO}(X,\tau)$ and $T \in SC(X,\tau)$, i.e. $S \subseteq \mathrm{cl}\,(\mathrm{int}\, S)$ and $\mathrm{int}\,(\mathrm{cl}\, T) \subseteq T$. Since $\mathrm{int}\,(\mathrm{cl}\, A) \subseteq \mathrm{int}\,(\mathrm{cl}\, T) \subseteq T$, we have $\mathrm{int}\,(\mathrm{cl}\, A) \subseteq \mathrm{int}\, T$. Since $A \subseteq S \subseteq \mathrm{cl}\,(\mathrm{int}\, S)$ we have $\mathrm{int}\,(\mathrm{cl}\, A) \subseteq \mathrm{cl}\,(\mathrm{int}\, S)$. Consequently, $\mathrm{int}\,(\mathrm{cl}\, A) \subseteq \mathrm{cl}\,(\mathrm{int}\, S) \cap \mathrm{int}\, T \subseteq \mathrm{cl}\,(\mathrm{int}\, S \cap \mathrm{int}\, T) = \mathrm{cl}\,(\mathrm{int}\, A)$. Hence A is a δ-set.
3) $\Rightarrow$ 4): Suppose that $\mathrm{int}\,(\mathrm{cl}\, A) \subseteq \mathrm{cl}\,(\mathrm{int}\, A)$ and let $U = \mathrm{int}\, A$ and $N = A - \mathrm{int}\, A$. We will show that N is nwd. Clearly $\mathrm{int}\,(\mathrm{cl}\, N) \subseteq \mathrm{int}\,(\mathrm{cl}\, A)$, and since $N \cap \mathrm{int}\, A = \emptyset$, we have $\mathrm{int}\,(\mathrm{cl}\, N) \cap \mathrm{cl}\,(\mathrm{int}\, A) = \emptyset$. So $\mathrm{int}\,(\mathrm{cl}\, N) = \emptyset$, i.e. N is nwd.
4) $\Rightarrow$ 5): See Prop. 2.1.
5) $\Rightarrow$ 1): Let A be open mod J. By Prop. 2.1., $X - A$ is open mod J, so $X - A = U \cup N$ where $U \in \tau$ and N is nwd in (X,τ). Hence $A = (X - N) \cap (X - U) \in LC(X,\tau^\alpha)$ since $X - N \in \tau^\alpha$ and $X - U$ is closed in (X,τ) and thus closed in (X,τ^α). $\Diamond$

Corollary 2.3. $\mathrm{SO}(X,\tau) \subseteq \mathrm{LC}(X,\tau^\alpha)$ *and* $\mathrm{SC}(X,\tau) \subseteq \mathrm{LC}(X,\tau^\alpha)$ *for every space* (X,τ).

Corollary 2.4. *If $f : (X,\tau) \to (Y,\sigma)$ is quasi-continuous, i.e. the inverse image of every open set is semi-open, then $f : (X,\tau^\alpha) \to (Y,\sigma)$ is* LC-*continuous.*

Remark 2.5. In a recent paper [6], Chattopadhyay and Bandyopadhyay study the collection T^δ of all δ-sets of a space (X,τ). Using Th. 2.2. one obtains straightforward proofs of many results in [6], e.g.

1) *T^δ is the discrete topology if and only if (X,τ^α) is submaximal* (since 1) $\Leftrightarrow$ 3) in Th. 2.2.);
2) $(\tau^\alpha)^\delta = T^\delta$ (since $(\tau^\alpha)^\alpha = \tau^\alpha$);
3) *$\tau = T^\delta$ if and only if every open set is closed.*

Finally let us observe that *(X,τ^α) is a* $\mathrm{T_D}$ *space if and only if every singleton is a δ-set in (X,τ).*

3. On the structure of LC(X,τ)

The topic of this section is the relationship between the following properties of a space (X,τ):

(A) $\mathrm{SO}(X,\tau) \subseteq \mathrm{LC}(X,\tau)$;
(B) co-$\mathrm{LC}(X,\tau) \subseteq \mathrm{LC}(X,\tau)$;
(C) $\mathrm{SC}(X,\tau) \subseteq \mathrm{LC}(X,\tau)$;

(D) Every nwd set in (X,τ) is locally closed in (X,τ).

Theorem 3.1. *For a space (X,τ) the following are equivalent:*

1) *(X,τ) satisfies* (A);
2) $\tau=\tau^{\alpha}$;
3) $\mathrm{LC}(X,\tau)=\mathrm{LC}(X,\tau^{\alpha})$.

Proof. 1) $\Rightarrow$ 2): Let N be nwd in (X,τ). Then $X-N$ is dense and semi-open in (X,τ), hence, by assumption, locally closed. Thus $X-N\in\tau$ and so N is closed in (X,τ). Hence $\tau=\tau^{\alpha}$.
2) $\Rightarrow$ 3): This is obvious.
3) $\Rightarrow$ 1): Let $S\in\mathrm{SO}(X,\tau)$. By Cor.2.3. we have $S\in\mathrm{LC}(X,\tau^{\alpha})$. Thus $S\in\mathrm{LC}(X,\tau)$ and so (X,τ) satisfies (A). $\Diamond$

Corollary 3.2. *For every space (X,τ), (X,τ^{α}) satisfies* (A).

Our next result follows immediately from Prop. 2.1. and Th. 3.1.

Theorem 3.3. *For a space (X,τ) the following holds:*

1) (A) *implies* (B);
2) (A) *implies* (C) *implies* (D).

We now provide examples to show that none of the implications in Th.3.3. can be reversed.

Example 3.4. Let $X=\{a,\ b,\ c\}$ and $\tau=\{\emptyset,\ \{a\},\ X\}$. Then $\mathrm{LC}(X,\tau)=\{\emptyset,\{a\},\{b,c\},X\}=$ co-$\mathrm{LC}(X,\tau)$. Hence (X,τ) satisfies (B) but fails to satisfy (A) since $\{a,c\}\in\mathrm{SO}(X,\tau)-\mathrm{LC}(X,\tau)$.

Example 3.5. Let $X=\{a,b,c\}$ and $\tau=\{\emptyset,\ \{a\},\ \{a,b\},\ X\}$. Then $\mathrm{LC}(X,\tau)=\{\emptyset,\ \{a\},\ \{b\},\ \{c\},\ \{a,b\},\ \{b,c\},\ X\}$ and $\mathrm{SC}(X,\tau)=$ $=\{\emptyset,\{b\},\{c\},\{b,c\},X\}$. Hence (X,τ) satisfies (C) but not (A) since $\{a,c\}\in\mathrm{SO}(X,\tau)-\mathrm{LC}(X,\tau)$.

Example 3.6. Let $X=\{a,b,x,y,z\}$ and $\tau=\{\emptyset,\{a\},\{b\},\{a,b\},$ $\{a,b,x\},\{a,b,x,y\},\{a,b,x,z\},X\}$. If $Z=\{x,y,z\}$ then Z is a closed subspace of (X,τ) and the subspace topology $\tau|Z=\{\emptyset,\{x\},\{x,y\},\{x,$ $z\},Z\}$ is submaximal. Now, if $N\subseteq X$ is nwd in (X,τ) then $N\subseteq Z$ and $N\in\mathrm{LC}(Z,\tau|Z)$ and, since Z is closed in (X,τ), $N\in\mathrm{LC}(X,\tau)$. Hence (X,τ) satisfies (D).
On the other hand, if $A=\{a,y\}$ then $A\in\mathrm{SC}(X,\tau)$. We have, however, $x\in U\cap\mathrm{cl}\,A$ for any open set U containing y, so $A\notin$ $\notin\mathrm{LC}(X,\tau)$. Hence (X,τ) does not satisfy (C). $\Diamond$

In order to state and prove our final result let us say that a space (X,τ) is a T_1^* space if every nwd subset is a union of closed sets. Clearly every T_1 space is a T_1^* space, and the indiscrete topology on any set X having at least two points yields a T_1^* space which is not T_1.

Theorem 3.7. *For a space* (X,τ) *the following are equivalent:*

1) (X,τ) *satisfies* (A) ;
2) (X,τ) *satisfies* (B) *and* (D) ;
3) (X,τ) *is* T_1^* *and satisfies* (B) .

Proof. 1) $\Rightarrow$ 2): See Th. 3.3.

2) $\Rightarrow$ 3): Let N be nwd in (X,τ) and let $x \in N$. Then $\{x\} \in$ $\in \mathrm{LC}(X,\tau)$ by (D). Since (B) holds, $X - \{x\}$ is locally closed and dense and so open. Thus $\{x\}$ is closed, and hence (X,τ) is T_1^*.

3) $\Rightarrow$ 1): By Th. 3.1. we have to show that every nwd subset N of (X,τ) is closed. Let $x \in \mathrm{cl}\, N$. Since (X,τ) is T_1^* and $\mathrm{cl}\, N$ is nwd, $\{x\}$ is closed and so $\mathrm{cl}\, N \cap (X - \{x\}) \in \mathrm{LC}(X,\tau)$. Since (B) holds, $\{x\} \cup (X - \mathrm{cl}\, N)$ is locally closed and dense, hence an open neighbourhood of x. Consequently $N \cap (\{x\} \cup (X - \mathrm{cl}\, N))$ is nonempty and so $x \in N$. Thus N is closed. ◇

Corollary 3.8. *In general, the statements* (B) *and* (C) *are independent of each other.*

Corollary 3.9 [1]. *Let* (X,τ) *be a* T_D *space satisfying* (B). *Then* (X,τ) *satisfies* (A).

Proof. It is easy to show that (X,τ) is T_1^*. Now apply Th. 3.6. ◇

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