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ON QUASI–CONTINUOUS FUNCTIONS HAVING DARBOUX PROPERTY

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Abstract: Some classes of quasi - continuous, Darboux like functions are studied. The maximal additive and multiplicative families for these classes are characterized. A necessary and sufficient condition for f to be the uniform limit of a sequence of quasi - continuous functions having the Darboux property is given.

1. Introduction. We shall consider the following families of real functions defined on some interval I:

$\mathcal{C}$onst – the class of all constant functions;

$\mathcal{C}$ – the class of all continuous functions;

$\mathcal{A}$ – the class of all almost continuous functions (in the sense of Stallings ([20]); $f : X \to Y$ is said to be almost continuous if for every open set $G \subset X \times Y$ containing f, there exists a continuous function $g : X \to Y$ lying entirely in G;

$\mathcal{C}$onn – the class of all connectivity functions; $f : X \to Y$ is a connectivity function if for every connected subset C of X, $f|C$ is a connected subset of $X \times Y$;

$\mathcal{D}$ – the class of all Darboux functions;

$\mathcal{B}_1$ – the family of all functions of the first class of Baire;

lsc(usc) – the class of all lower (upper) semicontinuous functions;

$\mathcal{M}$ – the class of Darboux functions f for which if x_0 is a right (left) hand sided point of discontinuity of f, then $f(x_0) = 0$ and there exists a sequence (x_n) such that $f(x_n) = 0$ and $x_n \searrow x_0$ $(x_n \nearrow x_0)$ ([8] and [14]);

$\mathcal{Q}$ – the class of all quasi-continuous functions; a function $f : X \to Y$ is quasi-continuous at a point x_0 iff $x_0 \in \overline{\operatorname{int} f^{-1}(V)}$ for every neighbourhood V of $f(x_0)$ ([15]);

$\mathcal{U}_0(\mathcal{U})$ – the class of all functions defined on I such that for every subinterval $J \subset I$ (and for every set A of the cardinality less than the continuum) the set $f(J)$ (respectively $f(J \setminus A)$) is dense in the interval $[\inf f|J, \sup f|J]$ ([4]); it is remarked in [4] that in these definitions the interval $[\inf f|J, \sup f|J]$ can be replaced by the interval $[f(a), f(b)]$, where $J = (a, b)$;

$\mathcal{Y}$ – the family of all functions with the Young property, i.e. functions which are bilaterally dense in themselves ([21]); some authors call functions having this property peripherally continuous ([2], [9]). (We make no distinction between a function and its graph.)

The inclusions $\mathcal{A} \subsetneq \mathcal{C}\mathrm{onn} \subsetneq \mathcal{D}$ are noticed in [1], the inclusions $\mathcal{D} \subsetneq \mathcal{U} \subsetneq \mathcal{U}_0 \subsetneq \mathcal{Y}$ follow from [4]. The inclusion $\mathcal{M} \subset \mathcal{B}_1$ is remarked in [14]. Now we shall prove the inclusion $\mathcal{M} \subset \mathcal{Q}$.

Lemma 1. *If $f \in \mathcal{M}$ and x_0 is a point of right-hand (left-hand) sided discontinuity of f then there exists a sequence (x_n) of points at which f is right-hand sided or left-hand sided continuous with $f(x_n) = 0$ and $x_n \searrow x_0$ $(x_n \nearrow x_0)$.*

Proof. Let us assume that f is right-hand sided discontinuous at some point x_0, $U = (x_0, x_0 + \varepsilon)$ for some $\varepsilon > 0$ and U contains no point of continuity of f at which f has the value zero. Observe that the set $B = \{x \in U : f(x) = 0\}$ is nowhere-dense and non-empty. Let (I_n) be a sequence of all components of the set $U \setminus \overline{B}$. Notice that $f(x) = 0$ for every $x \in \overline{B}$. Thus if $I_n = (a, b)$, then $f(a) = f(b) = 0$ and f is right-hand (left-hand) sided continuous at the point a (respectively, b). Hence there are points in $U \cap B$ at which f is right-hand or left-hand sided continuous. ◊

It follows easily from this lemma that for every point x_0 at which a function $f \in \mathcal{M}$ is discontinuous there exists a sequence (x_n) of continuity points of f such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} f(x_n) = f(x_0)$, and this condition implies quasi-continuity of f at x_0 (see e.g. [10]).

Lemma 2. (a) *A function f is quasi-continuous and satisfies the Young condition iff for every $x_0 \in I$ there exist two sequences (x_n) and (z_n) of continuity points of f such that $x_n \nearrow x_0$, $z_n \searrow x_0$ and $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(z_n) = f(x_0)$ (this condition must be interpreted unilaterally for end-points of I).*

(b) *Let f be quasi-continuous. Then $f \in \mathcal{U}$ iff for each $x \in I$ the unilateral cluster sets of f at x are intervals and contain $f(x)$.*

Proof. (a) follows immediately from the fact that $f : I \to \mathbb{R}$ is quasi-continuous at some point x_0 iff there exists a sequence (x_n) of continuity points such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} f(x_n) = f(x_0)$, i.e. $f|C(f)$ is $\mathfrak{c}$-dense in f, where we denote by $C(f)$ the set of all continuity points of f (see e.g. [10], Lemma 2). We can also write the following condition: $f \in \mathcal{QY}$ iff $f(x_0) \in C^-(f|C(f), x_0) \cap C^+(f|C(f), x_0)$ for each $x_0 \in I$. (By $C^-(f, x)$ and $C^+(f, x)$ we denote the left-hand and right-hand sided cluster sets of f at a point x.)

(b) follows from the fact that $f|C(f)$ is $\mathfrak{c}$-dense in f and the following characterization of the classes $\mathcal{U}_0$ and $\mathcal{U}$, which is proved in [4], theorems 3.1 and 3.2:

(i) $f \in \mathcal{U}_0$ iff for each $x \in I$ the unilateral cluster sets of f at x are intervals and contain $f(x)$;

(ii) $f \in \mathcal{U}$ iff $f \in \mathcal{U}_0$ and f is $\mathfrak{c}$-dense in itself. ◊

For the classes of real functions defined on an interval I we can state

$$\begin{array}{c} \mathcal{Q} \\ \cup\!\!\!| \\ \mathcal{C}\mathrm{onst} \subsetneq \mathcal{C} \subsetneq \mathcal{M} \subsetneq \mathcal{A} \subsetneq \mathcal{C}\mathrm{onn} \subsetneq \mathcal{D} \subsetneq \mathcal{U} \subsetneq \mathcal{U}_0 \subsetneq \mathcal{Y}. \\ |\!\!\!\cap \\ \mathcal{B}_1 \end{array}$$

In the class $\mathcal{B}_1$ we have the following equalities:

$$\mathcal{AB}_1 = \mathcal{C}\mathrm{onn}\,\mathcal{B}_1 = \mathcal{DB}_1 = \mathcal{U}_0\mathcal{B}_1 = \mathcal{UB}_1 = \mathcal{YB}_1 \quad \text{see [1] and [3].}$$

In the first part of the present paper we remark that in the class $\mathcal{Q}$ the following inclusions hold:

$$\mathcal{AQ} \subsetneq \mathcal{C}\mathrm{onn}\,\mathcal{Q} \subsetneq \mathcal{DQ} \subsetneq \mathcal{U}_0\mathcal{Q} = \mathcal{UQ} \subsetneq \mathcal{YQ}.$$

Let $\mathcal{Z}$ be a class of real functions. We define the maximal additive (multiplicative, latticelike, respectively) class for $\mathcal{Z}$ as the class of all such functions $f \in \mathcal{Z}$, for which $f+g \in \mathcal{Z}$ ($fg \in \mathcal{Z}$ or $\max(f,g) \in \mathcal{Z}$ and $\min(f,g) \in \mathcal{Z}$, respectively) whenever $g \in \mathcal{Z}$. The adequate classes we denote by $\mathcal{M}_a(\mathcal{Z})$, $\mathcal{M}_m(\mathcal{Z})$ and $\mathcal{M}_\ell(\mathcal{Z})$. Moreover let $\mathcal{M}_{\min}(\mathcal{Z}) = \{f \in \mathcal{Z}\colon \text{ if } g \in \mathcal{Z} \text{ then } \min(f,g) \in \mathcal{Z}\}$ and $\mathcal{M}_{\max}(\mathcal{Z}) = \{f \in \mathcal{Z}\colon \text{ if } g \in \mathcal{Z} \text{ then } \max(f,g) \in \mathcal{Z}\}$. Note that $\mathcal{M}_\ell(\mathcal{Z}) = \mathcal{M}_{\min}(\mathcal{Z}) \cap \mathcal{M}_{\max}(\mathcal{Z})$.

The following equalities are known:

$\mathcal{K}$	$\mathcal{M}_a(\mathcal{K})$	$\mathcal{M}_m(\mathcal{K})$	$\mathcal{M}_{\max}(\mathcal{K})$	$\mathcal{M}_{\min}(\mathcal{K})$	$\mathcal{M}_\ell(\mathcal{K})$
$\mathcal{D}$	Const ([19])	Const ([19])	$\mathcal{D}$usc ([7])	$\mathcal{D}$lsc ([7])	$\mathcal{C}$
$\mathcal{DB}_1$	$\mathcal{C}$ ([3])	$\mathcal{M}$ ([8])	$\mathcal{D}$usc ([7])	$\mathcal{D}$lsc ([7])	$\mathcal{C}$
$\mathcal{A}$	$\mathcal{C}$ ([14])	$\mathcal{M}$ ([14])	?	?	$\mathcal{C}$ ([14])
$\mathcal{C}$onn	$\mathcal{C}$ ([14])	$\mathcal{M}$ ([14])	?	?	$\mathcal{C}$ ([14])

Recently D. Banaszewski and K. Banaszewski proved the following results:

$\mathcal{K}$	$\mathcal{M}_a(\mathcal{K})$	$\mathcal{M}_m(\mathcal{K})$	$\mathcal{M}_{\max}(\mathcal{K})$	$\mathcal{M}_{\min}(\mathcal{K})$	$\mathcal{M}_\ell(\mathcal{K})$
$\mathcal{Y}$	$\mathcal{C}$ ([23])	$\mathcal{M}$ ([23])	$\mathcal{C}$ ([23])	$\mathcal{C}$ ([23])	$\mathcal{C}$
$\mathcal{QDB}_1$	$\mathcal{C}$ ([22])	$\mathcal{M}$ ([22])	$\mathcal{QD}$usc ([22])	$\mathcal{QD}$lsc ([22])	$\mathcal{C}$

In the second part of the present paper we shall add next lines to this table, namely,

$\mathcal{QD}$	Const	Const	$\mathcal{QD}$usc	$\mathcal{QD}$lsc	$\mathcal{C}$
$\mathcal{QA}$	$\mathcal{C}$	$\mathcal{M}$	?	?	$\mathcal{C}$
$\mathcal{QC}$onn	$\mathcal{C}$	$\mathcal{M}$	?	?	$\mathcal{C}$

It is well-known that a uniform limit of Darboux functions can be a function without the Darboux property. It was proved in [4] that a function f is a uniform limit of Darboux functions iff $f \in \mathcal{U}$. Since the classes $\mathcal{B}_1$ and $\mathcal{U}$ are closed with respect to uniform limits and $\mathcal{DB}_1 = \mathcal{UB}_1$, the class $\mathcal{DB}_1$ is closed with respect to uniform limits too (see e.g, [3]). The class $\mathcal{Q}$ is closed with respect to this operation too, but the class $\mathcal{DQ}$ is not.

In the last part of this paper we shall prove that a function f is a uniform limit of quasi-continuous functions having Darboux property iff $f \in \mathcal{QU}$. Notice also that a real function defined on $\mathbb{R}$ is a pointwise limit of some sequence of functions from the class $\mathcal{QD}$ iff it is pointwise discontinuous ([12]).

2. We start with some universal construction of quasi-continuous functions having Darboux property. Let $A \subset \mathbb{R}$ be a set $\mathfrak{c}$-dense in itself (where $\mathfrak{c}$ denotes the cardinality of the continuum) and let B be a subset of $\mathbb{R}$. Let $\mathcal{D}^*(A, B)$ denote the class of all functions $f : A \to B$ which take on every $y \in B$ in every non-empty interval of A (i.e. a set of the form $A \cap (a, b)$ for some $a, b \in \mathbb{R}$). It is well-known that the family $\mathcal{D}^*(A, B)$ is non-empty (see e.g. [5]).

Let $I = [0,1]$, $C \subset I$ be the Cantor set and for each $n \in \mathbb{N}$ let $\mathcal{J}_n$ be the family of all components of the set $I \setminus C$ of the n-th order (i.e. such components of $I \setminus C$ which length is equal to 3^{-n}). Let $A = I \setminus \bigcup\{\overline{J} : J \in \bigcup_{n=1}^{\infty} \mathcal{J}_n\}$. Notice that this set is $\mathfrak{c}$-dense in itself. Let (q_n) be a sequence of all rationals such that for every rational q the set $\{n : q_n = q\}$ is infinite. Then for a given function $\varphi \in \mathcal{D}^*(A, \mathbb{R})$ the function $f : I \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \varphi(x) & \text{for } x \in A \\ q_n & \text{for } x \in \bigcup\{\overline{J}, J \in \mathcal{J}_n\},\ n \in \mathbb{N} \end{cases}$$

is quasi-continuous and has the Darboux property.

Now we shall employ this method to construct some example of a quasi-continuous function with the Darboux property but not connected. It is easy to find (by transfinite induction) a function $\varphi \in \mathcal{D}^*(A, \mathbb{R})$ such that $\varphi(x) \neq -x$ for each $x \in A$. We define a function $f : I \to \mathbb{R}$ in the following way:

$$f(x) = \begin{cases} \varphi(x) & \text{for } x \in A \\ q_n & \text{for } x \in \bigcup\{\overline{J} : J \in \mathcal{J}_n \text{ and } q_n \notin J\},\ n \in \mathbb{N} \\ x + 1 & \text{otherwise.} \end{cases}$$

Then $f \in \mathcal{QD}$ and $\operatorname{rng} f = \mathbb{R}$ but $f \cap \{(x, x) : x \in I\} = \emptyset$ and therefore f is not connected.

Notice also that the function f which was constructed by J. Jastrzębski in [13] is quasi-continuous and connected but not almost continuous. Moreover, the function $g : I \to \mathbb{R}$ defined by $g(x) = q_n$ for $x \in \bigcup\{\overline{J} : J \in \mathcal{J}_n\}$ and $g(x) = 0$ otherwise, belongs to the class $\mathcal{QU}$ but g does not have the Darboux property. Finally, the function $h : I \to \mathbb{R}$, $h(x) = \sin(1/x)$ for $x \in (0, 1]$ and $h(0) = 0$ is quasi-continuous and almost continuous but h is not continuous. Thus all inclusions $\mathcal{C} \subsetneq \mathcal{AQ} \subsetneq \mathcal{C}\mathrm{onn}\,\mathcal{Q} \subsetneq \mathcal{DQ} \subsetneq \mathcal{UQ}$ are proper. The equality

$\mathcal{QU}_0 = \mathcal{UQ}$ follows from Lemma 2 (b). Now let $(I_n)_n$ be a sequence of all components of the complement of the Cantor set such that the unions $\bigcup_{n=1}^{\infty} I_{2n+1}$ and $\bigcup_{n=1}^{\infty} I_{2n}$ are dense in C and let f be the characteristic function of the set $C \cup \bigcup_{n=1}^{\infty} I_{2n}$. Then $f \in \mathcal{YQ} \setminus \mathcal{U}_0\mathcal{Q}$.

3. Theorem 1. *Assume that $I = [0,1]$, $X, Y \subset \mathbb{R}$ are intervals, a, b, c are reals such that $a < b < c$ and $F : X \times Y \to \mathbb{R}$ satisfies the following conditions:*

(1) *$F_x : Y \to \mathbb{R}$, $F_x(y) = F(x,y)$ is continuous and $(F_x)^{-1}(b)$ is countable for each $x \in X$;*

(2) *$F^y : X \to \mathbb{R}$, $F^y(x) = F(x,y)$ is continuous and $(F^y)^{-1}(b)$ is countable for each $y \in Y$;*

(3) card $\{x \in X\colon \forall y \in Y\ F(x,y) \neq a\} < 2^\omega$;

(4) card $\{x \in X : \forall y \in Y\ F(x,y) \neq c\} < 2^\omega$.

Then for every non-constant, continuous function $f : I \to X$ there exists a Lebesgue measurable, quasi-continuous function $g : I \to Y$ with the Darboux property such that $F(f,g)$ does not have the Darboux property (compare with [24]).

Proof. Notice that the following condition follows from (1):

(1') $$\forall x \in X \quad \exists y(x) \in Y \quad F(x, y(x)) \neq b.$$

Let $f : I \to X$ be a non-constant, continuous function. Let D be the set of all points $x \in X$ for which the set $f^{-1}(x)$ has a positive measure. Then the set D is countable and it follows from (1) that the set $\{y \in Y\colon \exists x \in D\ F(x,y) = b\}$ is countable too. Thus there exists a countable, dense set $P \subset Y$ such that

(5) $$\forall x \in D \quad \forall p \in P \quad F(x,p) \neq b.$$

Moreover, we have also the following property

(6) $$\forall p \in P \quad m(\{z : F(f(z),p) = b\}) = 0,$$

where the symbol $m(A)$ denotes the Lebesgue measure of A. In fact, $\{z : F(f(z),p) = b\} = \bigcup\{f^{-1}(x) : F(x,p) = b\}$ and it follows from (2) and (5) that this union has a measure zero.

Let (p_n) be a sequence of all points of P such that for any $p \in P$ the set $\{n : p_n = p\}$ is infinite.

Now we shall modify the construction of quasi-continuous function having Darboux property from the second part of this paper. We choose (inductively) a sequence of finite families of open intervals $(\mathcal{J}_n)_{n=0}^{\infty}$ such that:

(7) $$\mathcal{J}_0 = \{\emptyset\};$$

(8) if L is a component of the set $I \setminus \bigcup\{J : J \in \mathcal{J}_k,\ k \leq n\}$ then there exists some $K \in \mathcal{J}_{n+1}$ such that $K \subset L$, and
$$m(L) > \sum_{K \in \mathcal{J}_{n+1}, K \subset L} m(K) \geq m(L)/3;$$

(9) $F(f(x), p_n) \neq b$ for each $x \in \bigcup\{\overline{J} : J \in \mathcal{J}_n\}$;

(10) if $J \in \mathcal{J}_n$ and K is an interval on which f is constant and $K \cap \overline{J} \neq \emptyset$, then $K \subset \overline{J}$;

(11) if d, e are the end-points of some interval $J \in \mathcal{J}_n$ then $f(e) \neq f(d)$.

Such a choice is possible. Indeed, let us assume that are have chosen a family $\mathcal{J}_n$. Let $L \in I \setminus \bigcup_{k \leq n} \bigcup \mathcal{J}_k$. Then the set $Z = L \cap \{z \in I : F(f(z), p_{n+1}) = b\}$ is closed and nowhere-dense. Moreover, it follows from (6) that Z has a measure zero. Let (L_m) be a finite sequence of components of $L \setminus Z$ such that $\sum_m m(L_m) \geq 2m(L)/3$. By (10), $f|L_m$ is constant on no neighbourhood of ends of L_m (for each m). Thus for each m we can choose a subinterval K_m of L_m which satisfies (9), (10) and (11) and with $m(K_m) \geq m(L_m)/2$. Finally we put $\mathcal{J}_{n+1} = \{K_m \subset L : L \in \mathcal{J}_n\}$ and observe that this family satisfies all conditions (8), (9), (10) and (11).

Now let $A = I \setminus \bigcup\{\overline{K} : K \in \mathcal{J}_n,\ n \in \mathbb{N}\}$. Evidently this set is $\mathfrak{c}$-dense in itself, nowhere-dense and has a measure zero. Additionally, it follows from (11) that f is not constant on any interval of A. Let $C = \overline{A}$. Then $C \setminus A$ is countable and f is constant on no interval of C. Hence we have the following property:

(12) for each subinterval J of I, if $J \cap A \neq \emptyset$ then the set $f(J \cap A)$ has the cardinality of the continuum.

Indeed, let us suppose that J is a closed subinterval of I such that $J \cap A \neq \emptyset$ and the set $f(J \cap A)$ has the cardinality less than the continuum. Because the set $C \setminus A$ is countable, the set $f(J \cap C)$ has the cardinality less than the continuum too. Since f is continuous and

$J \cap C$ is a compact set, the set $f(J \cap C)$ is closed and consequently, it is countable. Let (y_n) be a sequence of all points of $f(J \cap C)$ and for each $n \in \mathbb{N}$ let $C_n = J \cap C \cap f^{-1}(y_n)$. By (11) the sets C_n are nowhere-dense in $J \cap C$ and $\bigcup_{n=1}^{\infty} C_n = J \cap C$, which contradicts the Baire theorem. Therefore (12) holds.

Lemma 3. *If a set A is $\mathfrak{c}$-dense in itself and $f : A \to X$ is a continuous function which satisfies the condition* (12), *then there exists a function $\varphi \in \mathcal{D}^*(A, Y)$ such that $F(f(x), \varphi(x)) \neq b$ for each $x \in A$, $F(f(x_1), \varphi(x_1)) = a$ and $F(f(x_2), \varphi(x_2)) = c$ for some $x_1, x_2 \in A \cap J$ and each interval J for which $A \cap J \neq \emptyset$* (compare e.g. with [16]).

Proof (of Lemma 3). Let (I_n) be a sequence of all basis sets in A. We list all elements of the family $(I_n) \times Y$ in the sequence $(I_\gamma \times \{y_\gamma\})_{\gamma < 2^\omega}$ and choose (by induction) sequences $s_\gamma, t_\gamma, w_\gamma \in I_\gamma$, $t'_\gamma, w'_\gamma \in Y$ such that:

(13) $s_\gamma \in I_\gamma \setminus \{s_\beta, t_\beta, w_\beta : \beta < \gamma\}$ and $F(f(s_\gamma), y_\gamma) \neq b$,

(14) $t_\gamma \in I_\gamma \setminus (\{s_\beta, t_\beta, w_\beta : \beta < \gamma\} \cup \{s_\gamma\})$ and $F(f(t_\gamma), t'_\gamma) = a$,

(15) $w_\gamma \in I_\gamma \setminus (\{s_\beta, t_\beta, w_\beta : \beta < \gamma\} \cup \{s_\gamma, t_\gamma\})$ and $F(f(w_\gamma), w'_\gamma) = c$.

Now we define a function $\varphi : A \to Y$ by

$$\varphi(x) = \begin{cases} y_\gamma & \text{for } x = s_\gamma, \\ t'_\gamma & \text{for } x = t_\gamma, \\ w'_\gamma & \text{for } x = w_\gamma, \\ y(x) & \text{otherwise,} \end{cases}$$

where $\gamma < 2^\omega$ and $y(x)$ is defined in (1′). It is easy to verify that the function φ has the required properties. The proof of Lemma 3 is completed.

Now we can finish the proof of Th. 1. We define a function $g : I \to Y$ by $g(x) = p_n$ for $x \in \bigcup\{\overline{J} : J \in \mathcal{J}_n\}$, $n = 1, 2, \ldots$, and $g(x) = \varphi(x)$ for $x \in A$. It is easy to see that the function g is quasi-continuous, measurable and has the Darboux property. Instead the function $F(f, g)$ takes the values a, c and does not take the value b, and consequently, $F(f, g)$ does not have the Darboux property. ◊

Corollary 1. (1) *If we put $X = Y = \mathbb{R}$, $F(x, y) = x + y$, $a = -1$, $b = 0$ and $c = 1$, then we obtain the following inclusion: $\mathcal{M}_a(\mathcal{QD}) \cap \mathcal{C} \subset \mathcal{C}$onst.* Since the opposite inclusion is clear, *we have the equality $\mathcal{M}_a(\mathcal{QD}) \cap \mathcal{C} = \mathcal{C}$onst.*

(2) *We have also the equality $\mathcal{M}_m(\mathcal{QD}) \cap \mathcal{C} = \mathcal{C}$onst. The inclu-*

sion "$\supset$" is trivial. The second inclusion follows from Th. 1, if we put $X = Y = \mathbb{R}$, $F(x,y) = x \cdot y$, $a = 0$, $b = 1$ and $c = 2$.

(3) Similarly we can conclude that

$$\{f : I \to \mathbb{R} : f \in \mathcal{C} \text{ and } f/g \in \mathcal{D} \text{ for each } g \in \mathcal{QD},\ g : I \to \mathbb{R}_+\} = \mathcal{C}\text{onst}$$

and

$$\{f : I \to \mathbb{R}_+ : f \in \mathcal{C} \text{ and } g/f \in \mathcal{D} \text{ for each } g \in \mathcal{QD}\} = \{f : I \to \mathbb{R}_+ : f \in \mathcal{C}\text{onst}\}.$$

Lemma 4. *Let us assume that* $f \in \mathcal{QY}\text{usc}$ ($f \in \mathcal{QY}\text{lsc}$) *and* $g \in \mathcal{QU}$. *Then* $\max(f,g) \in \mathcal{Q}$ ($\min(f,g) \in \mathcal{Q}$). (Notice that the assumption $f, g \in \mathcal{Y}$ is necessary; we have $\mathcal{M}_{\min}(\mathcal{Q}) = \mathcal{M}_{\max}(\mathcal{Q}) = \mathcal{C}$ ([17])).

Proof. Observe that for quasi-continuous functions f, g the set $C(f) \cap \cap C(g)$ is residual in I and $\max(f,g)$ is continuous at every point from this set. Thus it is enough to prove that for each $x \in I$ there exists a sequence (x_n) of points of the set $C(f) \cap C(g)$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} \max(f,g)(x_n) = \max(f,g)(x)$. Let $x_0 \in I$. We shall consider three cases.

(a) $f(x_0) \ge g(x_0)$ and there exists a sequence (x_n) of points of $C(f) \cap C(g)$ such that $\lim_{n\to\infty} x_n = x_0$, $\lim_{n\to\infty} f(x_n) = f(x_0)$ and $f(x_n) \ge \ge g(x_n)$ for each $n \in \mathbb{N}$. Then $\lim_{n\to\infty} \max(f,g)(x_n) = \lim_{n\to\infty} f(x_n) = = f(x_0) = \max(f,g)(x_0)$ and therefore $\max(f,g)$ is quasi-continuous at x_0.

(b) $f(x_0) \ge g(x_0)$ and $f(x_n) < g(x_n)$ (if n is sufficiently big) for every sequence (x_n) of points of $C(f) \cap C(g)$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} f(x_n) = f(x_0)$. Since $f \in \mathcal{QY}$, there exists a sequence (x_n) such that $x_n \in C(f) \cap C(g)$, $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} f(x_n) = f(x_0)$. We can assume that $\lim_{n\to\infty} g(x_n)$ exists (finite or infinite). Then $\lim_{n\to\infty} g(x_n) \ge \ge \lim_{n\to\infty} f(x_n) = f(x_0)$. Since $g \in \mathcal{U}$, $C(g, x_0)$ is an interval ([4]) and therefore there exists a sequence (x'_n) such that $x'_n \in C(f) \cap \cap C(g)$, $\lim_{n\to\infty} x'_n = x_0$ and $\lim_{n\to\infty} g(x'_n) = f(x_0)$. Since f is upper semi-continuous, $\overline{\lim_{n\to\infty}} f(x'_n) \le f(x_0)$. Hence $\overline{\lim_{n\to\infty}} \max(f,g)(x'_n) = f(x_0)$ and there exists a subsequence (x'_{n_k}) of (x'_n) such that $\lim_{k\to\infty} \max(f,g) (x'_{n_k}) = f(x_0)$ and consequently, $\max(f,g)$ is quasi-continuous at the point x_0.

(c) $f(x_0) < g(x_0)$. Then there exists a sequence (x_n) of points such that $x_n \in C(f) \cap C(g)$, $\lim_{n\to\infty} x_n = x_0$, $\lim_{n\to\infty} g(x_n) = g(x_0) > f(x_0)$ and $g(x_n) > f(x_0)$ for each $n \in \mathbb{N}$. Since f is upper semicontinuous, $\overline{\lim_{n\to\infty}} f(x_n) \le f(x_0)$ and consequently, $\lim_{n\to\infty} \max(f,g)(x_n) = \lim_{n\to\infty} g(x_n) = g(x_0) = \max(f,g)(x_0)$. Hence $\max(f,g)$ is quasi-continuous at the point x_0. ◊

Lemma 5. *If $f \in \mathcal{M}$ and $g \in \mathcal{QY}$ then the product fg is quasi-continuous.*

Proof. Of course it is sufficient to prove that fg is quasi-continuous at every point x_0 at which f is not continuous. Then $f(x_0) = 0$ and, by Lemma 1, if f is not continuous at x_0 from the left (from the right) then there exists a sequence (x_n) of points at which f is unilaterally continuous such that $f(x_n) = 0$ for each n and $x_n \nearrow x_0$ ($x_n \searrow x_0$). For every $n \in \mathbb{N}$ we choose a unilateral neighbourhood U_n of x_n such that $|f(x)| < 1/(n \cdot |g(x_n)|)$ if $g(x_n) \neq 0$ and $|f(x)| < 1/n$ whenever $g(x_n) = 0$, for each $x \in U_n$. Since $g \in \mathcal{QY}$, Lemma 2 (a) implies that for every $n \in \mathbb{N}$ there exists $z_n \in U_n \cap (x_n - 1/n, x_n + 1/n) \cap C(f) \cap C(g)$ for which $|g(z_n) - g(x_n)| < \varepsilon_n$, where $\varepsilon_n = 1$ if $g(x_n) = 0$ and $\varepsilon_n = |g(x_n)|$ otherwise. Then fg is continuous at each z_n, $\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} (fg)(z_n) = 0 = (fg)(x_0)$. This implies the quasi-continuity of fg at the point x_0. ◊

We shall apply also the following two lemmata, which were proved in [14].

Lemma 6. *Let Φ be some property of functions, let $\mathcal{X}_1$ be the class of all functions $f : X \to \mathbb{R}$ (where X is a topological space) possessing the property Φ and let $\mathcal{X}_2$ be the class of all functions $g : X \to \mathbb{R} \times \mathbb{R}$ possessing the same property Φ. Let the classes $\mathcal{X}_1$ and $\mathcal{X}_2$ fulfil the following conditions:*

(i) *if $f \in \mathcal{X}_2$ and $g \in \mathcal{C}$ ($g : \mathbb{R}^2 \to \mathbb{R}$), then $g \circ f \in \mathcal{X}_1$;*

(ii) *if $f \in \mathcal{X}_1$ and $g \in \mathcal{C}$ ($g : X \to \mathbb{R}$), then $h = (f,g) \in \mathcal{X}_2$, where $h : z \mapsto (f(x), g(x))$ for $x \in X$.*

Then $\mathcal{C} \subseteq \mathcal{M}_a(\mathcal{X}_1) \cap \mathcal{M}_m(\mathcal{X}_1) \cap \mathcal{M}_\ell(\mathcal{X}_1)$.

Lemma 7. *Let $\mathcal{X}$ be a subfamily of $\mathcal{U}_0$ and let the following conditions hold:*

(iii) *if $f : I \to \mathbb{R}$, $f \in \mathcal{X}$ and J is a subinterval of an interval I, then $f|J \in \mathcal{X}$;*

(iv) *if $h : (a,b) \to \mathbb{R}$, $h \in \mathcal{X}$, $y \in C^+(h,a)$ and $z \in C^-(h,b)$, then the functions $h_1 : [a,b) \to \mathbb{R}$, $h_2 : (a,b] \to \mathbb{R}$ and $h_3 : [a,b] \to \mathbb{R}$ belong* *to $\mathcal{X}$, where $h_1 = h \cup \{(a,y)\}$, $h_2 = h \cup \{(b,z)\}$, $h_3 = h_1 \cup h_2$;*

(v) *if $I \subset \mathbb{R}$ is an interval, $a \in I$ and $f|(I \cap (-\infty, a]) \in \mathcal{X}$, $f|(I \cap \cap[a,\infty)) \in \mathcal{X}$, then $f \in \mathcal{X}$;*

(vi) *$\mathcal{C}$onst $\subseteq \mathcal{M}_a(\mathcal{X})$ and $-1 \in \mathcal{M}_m(\mathcal{X})$.*

Then $\mathcal{M}_a(\mathcal{X}) \subseteq \mathcal{C}$, $\mathcal{M}_{\min}(\mathcal{X}) \subseteq \mathcal{X}$lsc and $\mathcal{M}_{\max}(\mathcal{X}) \subseteq \mathcal{X}$usc (hence $\mathcal{M}_\ell(\mathcal{X}) \subseteq \mathcal{C}$).

If moreover the class $\mathcal{X}$ fulfils the additional condition

(vii) *if $f : I \to (0,\infty)$ and $f \in \mathcal{X}$ then $1/f \in \mathcal{X}$,*

then also $\mathcal{M}_m(\mathcal{X}) \subseteq \mathcal{M}$.

Let us observe that the family $\mathcal{X} = \mathcal{QD}$ does not satisfy the assumptions of Lemma 6 but it satisfies all assumptions of Lemma 7. Thus

(a) $\mathcal{M}_a(\mathcal{QD}) \subseteq \mathcal{C}$,

(b) $\mathcal{M}_{\min}(\mathcal{QD}) \subseteq \mathcal{QD}$lsc and $\mathcal{M}_{\max}(\mathcal{QD}) \subseteq \mathcal{QD}$usc,

(c) $\mathcal{M}_m(\mathcal{QD}) \subseteq \mathcal{M}$.

Now we can prove the following theorem.

Theorem 2. *We have the following equalities:*

(1) $\mathcal{M}_a(\mathcal{QD}) = \mathcal{C}$onst,

(2) $\mathcal{M}_m(\mathcal{QD}) = \mathcal{C}$onst,

(3) $\mathcal{M}_{\min}(\mathcal{QD}) = \mathcal{QD}$lsc *and* $\mathcal{M}_{\max}(\mathcal{QD}) = \mathcal{QD}$usc.

Proof. Evidently, we have $\mathcal{C}$onst $\subseteq \mathcal{M}_a(\mathcal{QD}) \cap \mathcal{M}_m(\mathcal{QD})$. The inclusion $\mathcal{M}_a(\mathcal{QD}) \subseteq \mathcal{C}$onst follows from Lemma 7 and from Cor. 1 (1). Hence $\mathcal{M}_a(\mathcal{QD}) = \mathcal{C}$onst.

Now we shall prove that $\mathcal{M}_m(\mathcal{QD}) \subseteq \mathcal{C}$onst. It is enough to prove that $\mathcal{M}_m(\mathcal{QD}) \subseteq \mathcal{C}$ and to use Cor. 1(2). Fix $f \in \mathcal{M}_m(\mathcal{QD})$ and suppose that f is not continuous, i.e. $I \setminus C(f) \neq \emptyset$. Since $f \in \mathcal{M}$, the set $A = I \setminus C(f)$ is nowhere-dense, $f(x) = 0$ for $x \in \overline{A}$ and f is continuous on every component of the set $I \setminus \overline{A}$. Since f is not continuous, f is not constant. Since $f \in \mathcal{D}$, rng(f) has the cardinality equals the continuum and consequently there exists a component J of $I \setminus \overline{A}$ such that $f|J$ is continuous and not constant. We apply Cor. 1(2) and obtain some quasi-continuous function $g : J \to \mathbb{R}$ having the Darboux property for which $f \cdot g \notin \mathcal{D}$. Thus there exists a function h defined on the interval I such that $h \in \mathcal{QD}$ and $f \cdot h \notin \mathcal{D}$, which contradicts to $f \in \mathcal{M}_m(\mathcal{QD})$.

Now we shall prove (3). By Lemma 7 it follows that we need to prove the following two inclusions: $\mathcal{QD}$usc $\subseteq \mathcal{M}_{\max}(\mathcal{QD})$ and $\mathcal{QD}$lsc $\subseteq \subseteq \mathcal{M}_{\min}(\mathcal{QD})$. To prove that $\mathcal{QD}$usc $\subseteq \mathcal{M}_{\max}(\mathcal{QD})$ let $f \in \mathcal{QD}$usc and $g \in \mathcal{QD}$. Since $\mathcal{M}_{\max}(\mathcal{D}) = \mathcal{D}$usc, $\max(f,g) \in \mathcal{D}$. By Lemma 4 it follows that $\max(f,g) \in \mathcal{Q}$ and therefore $\max(f,g) \in \mathcal{QD}$. The proof that $\mathcal{QD}$lsc $\subseteq \mathcal{M}_{\min}(\mathcal{QD})$ is similar. $\Diamond$

Observe now observe that the family $\mathcal{Q}$ satisfies all assumptions of Lemma 6 (see [18]) and therefore,

$$\mathcal{C} \subseteq \mathcal{M}_a(\mathcal{Q}) \cap \mathcal{M}_m(\mathcal{Q}) \cap \mathcal{M}_{\max}(\mathcal{Q}) \cap \mathcal{M}_{\min}(\mathcal{Q}) \text{ ([11], [17])}.$$

We have also the inclusion

$$\mathcal{C} \subseteq \mathcal{M}_a(\mathcal{A}) \cap \mathcal{M}_m(\mathcal{A}) \cap \mathcal{M}_{\max}(\mathcal{A}) \cap \mathcal{M}_{\min}(\mathcal{A}) \text{ ([14])}$$

and consequently,

$$\mathcal{C} \subseteq \mathcal{M}_a(\mathcal{QA}) \cap \mathcal{M}_m(\mathcal{QA}) \cap \mathcal{M}_{\max}(\mathcal{QA}) \cap \mathcal{M}_{\min}(\mathcal{QA}).$$

Similarly,

$$\mathcal{C} \subseteq \mathcal{M}_a(\mathcal{QC}\mathrm{onn}) \cap \mathcal{M}_m(\mathcal{QC}\mathrm{onn}) \cap \mathcal{M}_{\max}(\mathcal{QC}\mathrm{onn}) \cap \mathcal{M}_{\min}(\mathcal{QC}\mathrm{onn}).$$

Moreover, the families $\mathcal{QA}$ and $\mathcal{QC}\mathrm{onn}$ satisfy all assumptions of Lemma 7. Thus we obtain the following theorem.

Theorem 3. *Let $\mathcal{K} = \mathcal{A}$ or $\mathcal{K} = \mathcal{C}\mathrm{onn}$. Then the following equalities hold:*

$$\mathcal{M}_a(\mathcal{QK}) = \mathcal{C}, \quad \mathcal{M}_\ell(\mathcal{QK}) = \mathcal{C} \text{ and } \mathcal{M}_m(\mathcal{QK}) = \mathcal{M}.$$

Proof. The first two equalities follow immediately from lemmata 6 and 7. In the third equality it is sufficient to prove the inclusion $\mathcal{M} \subseteq \mathcal{M}_m(\mathcal{QK})$. Fix $f \in \mathcal{M}$ and $g \in \mathcal{QK}$. Since $\mathcal{M}_m(\mathcal{K}) = \mathcal{M}$ ([14]), $f \cdot g \in \mathcal{K}$. By Lemma 5 we obtain that $f \cdot g \in \mathcal{Q}$. Hence $f \cdot g \in \mathcal{KQ}$ and consequently $\mathcal{M} \subseteq \mathcal{M}_m(\mathcal{QK})$. $\Diamond$

Problem. For $\mathcal{K} \in \{\mathcal{A}, \mathcal{C}\mathrm{onn}\}$ find $\mathcal{M}_{\max}(\mathcal{QK})$ and $\mathcal{M}_{\min}(\mathcal{QK})$.

4. In this section we shall prove that the family $\mathcal{QU}$ is the uniform closure of the class of all quasi-continuous functions having the Darboux property. Functions which we shall consider are defined on the unit interval $I = [0,1]$.

Lemma 8. *Assume that $f \in \mathcal{QU}$, $(J_n)_n$ is a sequence of pairwise disjoint open intervals and g is a function such that $g(x) = f(x)$ for $x \in \bigcup_n J_n$, $g|\bigcup_n J_n$ is continuous and $f(J_n) \subset C^+(g|J_n, a_n) \cap C^-(g|J_n, b_n)$, where $J_n = (a_n, b_n)$, $n \in \mathbb{N}$. Then $g \in \mathcal{QU}$.*

Proof. Note that the set $A = F(\bigcup_n J_n)$ is nowhere-dense and therefore $B = C(f) \setminus A$ is dense in I and $f|B$ is dense in f. Additionally g is continuous at each point $x \in B$. We shall verify that $g|B$ is dense in g. Let $U = U_1 \times U_2$ be a neighbourhood of $(x, g(x))$ (obviously it is sufficient to consider only $x \in A$). Then $g(x) = f(x)$ and since f is quasi-continuous, $(t, f(t)) \in U$ for some $t \in B$. If $t \notin \bigcup_n J_n$ then $g(t) = f(t)$ and $(t, g(t)) \in U$. Otherwise $t \in J_n$ for some n. Then $a_n \in U_1$ or $b_n \in U_1$. Let e.g. $a_n \in U_1$. Since $f(t) \in C^+(g|J_n, a_n)$, there exists $s \in U_1 \cap J_n \cap B$ such that $(s, g(s)) \in U$. Thus g is quasi-continuous.

Now we verify that $g \in \mathcal{U}$. By Lemma 2(b) it suffices to observe that for every $x \in I$ the sets $C^-(g,x)$ and $C^+(g,x)$ are intervals and $f(x) \in C^-(g,x) \cap C^+(g,x)$. Assume that g is not continuous at x e.g. from the right. Then $g(x) = f(x)$ and $C^+(f,x) \subset C^+(g,x)$. Moreover for $y \in C^+(g,x) \setminus C^+(f,x)$ there exists $t \in C^+(f,x)$ such that $[t,y] \subset \subset C^+(g,x)$. Indeed, since $y \notin C^+(f,x)$, there exist sequences $(k_n)_n$ of positive integers and $(y_n)_n$ such that $y_n \in J_{k_n}$, $\lim_{n\to\infty} y_n = y$ and the sequence $(g(a_{k_n}))_n$ converges to some limit $t \in \overline{R}$. Then $t \in C^+(f,x)$. Since $f|J_{k_n}$ is continuous, $(f(a_{k_n}), y_n) \subset g(J_{k_n})$. Therefore $[t,y] \subset \subset C^+(g,x)$. This proves that $C^+(g,x)$ is an interval and $g(x) \in \in C^+(g,x)$. ◊

Lemma 9. *For each $f \in \mathcal{QU}$ and positive ε there exists $g \in \mathcal{QU}$ which is constant on no interval and such that $||f-g|| \le \varepsilon$. Moreover, if f is of the Baire class α or measurable, then g may be taken from the same class.*

Proof. Let $\{J_n \subset I\colon n \in \mathbb{N}\}$ be the family of all maximal open intervals on which f is constant. Let $J_n = (a_n, b_n)$ and let $f(J_n) = \{y_n\}$ for each $n \in \mathbb{N}$. Since $f \in \mathcal{U}$, we obtain $f(a_n) = f(b_n) = y_n$. For every n we define a continuous surjection $g_n : \overline{J_n} \to [y_n - \varepsilon, y_n + \varepsilon]$ such that $g_n(a_n) = g_n(b_n) = y_n$ and g_n is constant on no subinterval of J_n. Then the function $g : I \to \mathbb{R}$ defined by $g(x) = g_n(x)$ for $x \in J_n$, $n \in \mathbb{N}$ and $g(x) = f(x)$ otherwise has the desired properties. Evidently $||f-g|| \le \varepsilon$ and g is constant on no subinterval of I. By Lemma 8, $g \in \mathcal{QU}$. Finally it is easy to verify that if f is of the Baire class α or measurable, then g is from the same class. ◊

Lemma 10. *For every $f \in \mathcal{QU}$ and $\varepsilon > 0$ there exists a function $g \in \mathcal{QD}$ such that $||f-g|| < \varepsilon$. Moreover, if f is of the Baire class α or measurable then g may be taken from the same class.*

Proof. By Lemma 9 we can assume that $f : I \to \mathbb{R}$ is constant on no subinterval of I. Fix $n \in \mathbb{N}$ with $1/n < \varepsilon$. Since $f \in \mathcal{U}$, $T = \overline{f(I)}$ is an interval. Assume that $T = (-\infty, \infty)$ (the proof is similar when $T = [a, b]$, $T = [a, \infty)$ or $T = (-\infty, a]$). Put $a_k = k/n$, $J_k = (a_k, a_{k+1})$, $A_k = f^{-1}(J_k)$ and $B_k = f^{-1}(a_k)$ for each integer k. Since f is quasi-continuous, $f|C(f)$ is bilaterally dense in f and therefore we obtain the following conditions (for each k):

(1) $A_k = G_k \cup K_k$, where G_k is a non-empty, open set, K_k is nowhere-dense, $G_k \cap K_k = \emptyset$ and $K_k \subset \overline{G_k \cap (x, \infty)} \cap \overline{G_k \cap (-\infty, x)}$,

(2) B_k is a nowhere-dense subset of $\overline{(G_{k-1} \cup G_k) \cap (x, \infty)} \cap \cap \overline{(G_{k-1} \cup G_k) \cap (-\infty, x)}$.

Fix an integer k. Let $(I_{k,m})_m$ be a sequence of all components of G_k. For every m we define a continuous surjection $g_{k,m} : I_{k,m} \to \overline{J_k}$ such that:

(3) the end-points of $I_{k,m}$ belong to $\overline{g_{k,m}^{-1}(y)}$ for each $y \in \overline{J_k}$.

Now we define the function $g : I \to \mathbb{R}$ by $g(x) = g_{k,m}(x)$ for $x \in I_{k,m}$ (for each k, m) and $g(x) = f(x)$ otherwise. Evidently $||f - g|| \leq \leq 1/n < \varepsilon$. By Lemma 8, $g \in \mathcal{QU}$. To show that g has the Darboux property fix $a < b$ with $g(a) \neq g(b)$ (e.g. $g(a) < g(b)$) and $y \in (g(a), g(b))$. Let $J = (a, b)$. Obviously it is sufficient to consider the case when J is included in no interval $I_{k,m}$. Because $g \in \mathcal{U}$, $[g(a), g(b)] \subset \overline{g(J)}$. Let k be an integer such that $y \in \overline{J_k}$. Then $(g(a), g(b)) \cap J_k \neq \emptyset$ and consequently, $J \cap g^{-1}(J_k) \neq \emptyset$. Since $g^{-1}(J_k) \subset \subset f^{-1}(J_k)$, the condition (1) implies $I_{k,m} \cap J \neq \emptyset$ for some $m \in \mathbb{N}$. Let $I_{k,m} = (c, d)$. Since J is not a subset of $I_{k,m}$, $c \in J$ or $d \in J$. Let e.g. $c \in J$. Then $g(x) = y$ for some $x \in (c, b)$.

Finally let us assume that f is of the Baire class α and let $G \subset R$ be an open set. Then $g^{-1}(G) = \bigcup_{k,m} g_{k,m}^{-1}(G) \cup (f^{-1}(G) \setminus \bigcup_{k,m} I_{k,m})$ is clearly a Borel set of the additive class α. Hence g is of the Baire class α. Similarly we can prove that g is measurable if so is f. ◊

Theorem 4. *A necessary and sufficient condition for f to belong to $\mathcal{QU}$ is that f be the uniform limit of a sequence of quasi-continuous functions having the Darboux property. Moreover, if f is of the Baire class α or measurable then the approximating functions may be taken to be Baire class α or measurable.*

Proof. Because the families of all quasi-continuous, of the Baire class α, measurable functions are closed with respect to uniform limits (see [6] and e.g. [3]) and the uniform limits of sequences of Darboux functions belong to the class $\mathcal{U}$ [4], we obtain the sufficiency. The necessity is proved by applying Lemma 10. ◊

Corollary 2. *The class $\mathcal{QU}$ is closed with respect to uniform limits.*

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