

A EGOROFF-TYPE THEOREM FOR SET-VALUED MEASURABLE FUNCTIONS

T. Nishiura

*Department of Mathematics, Wayne State University, Detroit,
Michigan 48202, USA.*

F. J. Schnitzer

*Institut für Mathematik, Montanuniversität, 8700 Leoben, Franz-
Josef-Straße 18, Österreich.*

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Abstract: Results of the following type treated: If $\phi : \mathbf{S}^n \rightarrow C(\mathbf{S}^k)$, $k \geq 2$, is a Lebesgue measurable function it is shown that there exists a continuous function $f : \mathbf{B}_{n+1} \rightarrow \mathbf{S}^k \setminus \{\infty\}$ so that the radial cluster-set function f_R of f equals ϕ almost everywhere on \mathbf{S}^n .

In [6] and [7], question of interpolations by radial cluster set functions were addressed for functions of Baire class 1. The results of these papers can be used to prove Egoroff-type theorems and Lusin-type theorems for set-valued measurable functions. The present note illustrates one such Egoroff-type theorem. The construction found here can be used to establish other theorems of this type as well as Lusin-type theorems. Throughout the note, k and n are integers such that $k \geq 2$ and $n \geq 1$.

1. Statement of the theorem. We shall begin the statement of our Egoroff-type theorem. The notation used in its statement will be explained immediately after the statement and the proof of the theorem will be given in Sect. 3.

Theorem. *Let $\phi : \mathbf{S}^n \rightarrow C(\mathbf{S}^k)$ be a Lebesgue measurable function. Then, there is a continuous function $f : \mathbf{B}_{n+1} \rightarrow \mathbf{S}^k \setminus \{\infty\}$ and there is an increasing sequence of positive numbers $\{r_m\}$ converging to 1 such that*

- (i) *the radial cluster-set function f_R of f is equal to ϕ Lebesgue almost everywhere on \mathbf{S}^n , and*
- (ii) *for each positive number ε there is a Lebesgue measurable set E such that the continuous functions $\Gamma_m : \mathbf{S}^n \rightarrow C(\mathbf{S}^k)$, $m = 1, 2, \dots$, defined by*

$$\Gamma_m(z) = \{f(rz) : r_m \leq r \leq r_{m+1}\} \quad z \in \mathbf{S}^n,$$

converge uniformly to ϕ on E and $\mu(\mathbf{S}^n \setminus E) < \varepsilon$.

As usual, \mathbb{R}^{n+1} is the $(n+1)$ -dimensional Euclidean space. Its open unit ball and corresponding boundary are \mathbf{B}_{n+1} and \mathbf{S}^n , respectively. The Lebesgue measure on \mathbf{S}^n is denoted by μ . The point ∞ is the point $(0, \dots, 1)$ on the k -sphere \mathbf{S}^k of \mathbb{R}^{k+1} . By $C(\mathbf{S}^k)$ we mean the collection of all nonempty subcontinua of \mathbf{S}^k . When $C(\mathbf{S}^k)$ is endowed with the Hausdorff metric D , we have from a theorem of Curtis and Schori that $C(\mathbf{S}^k)$ is homeomorphic to the Hilbert cube I^{ω_0} , where $I = [0, 1]$ (see [10] and [11]).

Let us now turn to the radial cluster sets of a continuous function f defined on \mathbf{B}_{n+1} into \mathbf{S}^k . Assign to each point z of the boundary \mathbf{S}^n of \mathbf{B}_{n+1} the set

$$f_R(z) = \bigcap \{ \text{Cl}(\{f(rz) : \delta \leq r < 1\}) : 0 < \delta < 1 \}$$

called the *radial cluster set* of f at z , where Cl denotes the closure operator in \mathbf{S}^k . The set $f_R(z)$ is a nonempty subcontinuum of \mathbf{S}^k . The resulting function

$$f_R : \mathbf{S}^n \rightarrow C(\mathbf{S}^k)$$

is called the *radial cluster-set function* of f . It is proved in [6] that f_R is a Baire class 2 function and that there are continuous functions f for which f_R is not of Baire class 1. Of course, when $n = 1$ and $k = 2$, the classical complex analysis case results.

Finally, the Lebesgue measurability of $\phi : \mathbf{S}^n \rightarrow C(\mathbf{S}^k)$ is defined in the usual way, that is, $\phi^{-1}[F]$ is Lebesgue measurable for each closed set F of $C(\mathbf{S}^k)$.

2. Preliminary Lemmas. The proof of our theorem, which is given in Section 3, will rely heavily on the existence of certain homotopies. This section is devoted to these existence lemmas.

The first lemma is Lemma 5.7 of [6]. The statement of the lemma will require the use of the stereographic projection π in \mathbb{R}^{k+1} of $\mathbf{S}^k \setminus \{\infty\}$ onto \mathbb{R}^k . Here, \mathbb{R}^k is identified with the k -dimensional coordinate hyperplane of \mathbb{R}^{k+1} formed by setting the last coordinate equal to 0. We shall denote the Lipschitz constant of π^{-1} by M .

Lemma 1. *Suppose that $\varepsilon > 0$. If $g : \mathbf{S}^n \rightarrow \mathbb{R}^k$ and $\phi : \mathbf{S}^n \rightarrow C(\mathbf{S}^k)$ are continuous, then there exists a homotopy $\alpha : \mathbf{S}^n \times I \rightarrow \mathbb{R}^k$ such that, for all z in \mathbf{S}^n ,*

- (i) $\alpha(z, 0) = \alpha(z, 1) = g(z)$, and
- (ii) $D(\pi^{-1}[\alpha(z, I)], \phi(z)) < 2\text{dist}(\pi^{-1}(g(z)), \phi(z)) + \varepsilon M$.

From [6, Lemma 5.2] we infer the next lemma.

Lemma 2. *Suppose that $\varepsilon > 0$, that E is a compact, totally disconnected subset of \mathbf{S}^n and that h_0 and h_1 are continuous mappings of \mathbf{S}^n into $\mathbf{S}^k \setminus \{\infty\}$. Let*

$$K = \{z \in \mathbf{S}^n : |h_0(z) - h_1(z)| \leq 1\}.$$

Then, there exists a homotopy $\beta : \mathbf{S}^n \times I \rightarrow \mathbf{S}^k \setminus \{\infty\}$ such that

- (i) $\beta(z, 0) = h_0(z)$ and $\beta(z, 1) = h_1(z)$ for z in \mathbf{S}^n , and
- (ii) $|\text{diam}(\beta(z, I)) - |h_0(z) - h_1(z)|| < \varepsilon$ for z in $K \cap E$.

3. Proof of the Theorem. Let us begin with the homeomorphism H of $C(\mathbf{S}^k)$ onto I^{ω_0} given by the theorem of Curtis and Schori. The p -th coordinate H_p of H is a continuous function of $C(\mathbf{S}^k)$ into I . Also, a function ϕ from a space X into $C(\mathbf{S}^k)$ is continuous if and only if $H_p \circ \phi$ are continuous for all p . Consequently, we can prove the following lemma.

Lemma 3. *Let $\phi : \mathbf{S}^n \rightarrow C(\mathbf{S}^k)$ be a Lebesgue measurable function and $\varepsilon > 0$. Then, there exists a closed, totally disconnected subset E of \mathbf{S}^n such that the n -dimensional Lebesgue measure $\mu(\mathbf{S}^n \setminus E)$ does not exceed ε and ϕ restricted to E , is continuous.*

Proof. For each p , the function $H_p \circ \phi$ is real-valued. By classical

real function theory, there exists a closed subset E_p of \mathbf{S}^n such that $\mu(\mathbf{S}^n \setminus E_p) < 2^{-(p+1)}\varepsilon$ and $H_p \circ \phi$ restricted to E_p is continuous. Since μ is regular measure and \mathbf{S}^n is locally Euclidean, we may further assume that E_p is totally disconnected. The set $E = \bigcap \{E_p : p \geq 1\}$ is the required set. \diamond

Proof of the Theorem. By Lemma 3, there is a sequence $\{F_j\}$ of closed, totally disconnected subsets of \mathbf{S}^n such that $\phi|_{F_j}$, the restriction of ϕ to F_j , is continuous and $\mu(\mathbf{S}^n \setminus F) = 0$ where F is the union of $\{F_j\}$. Since the members of the collection $\{F_j\}$ are compact, totally disconnected sets, we may assume also that the collection is disjointed. By Michael's Theorem [14, Th. 2], the continuous set-valued function $\phi|_{F_j}$ has a continuous selection $s_j : F_j \rightarrow \mathbf{S}^k$, that is, s_j is a continuous function such that $s_j(z) \in \phi|_{F_j}(z)$ for $z \in F_j$. From [13, pp. 74–80], we infer for each j the existence of a sequence of continuous functions $s_{jm} : F_j \rightarrow \mathbf{S}^k \setminus \{\infty\}$, $m = 1, 2, \dots$, such that $|s_{jm}(z) - s_j(z)| < 1/(2m)$ for all z in F_j .

For each m , let $G_m = \bigcup \{F_j : j \leq m\}$. We have already mentioned that the Curtis-Schori Theorem gives us the fact that $C(\mathbf{S}^k)$ is homeomorphic to the Hilbert cube. Consequently, the Tietze Extension Theorem can be applied to get a continuous extension $\phi_m : \mathbf{S}^n \rightarrow C(\mathbf{S}^k)$ of $\phi|_{G_m}$ for each m . Next, for each m , let $h_m : G_m \rightarrow \mathbf{S}^k \setminus \{\infty\}$ be the continuous function defined by $h_m(z) = s_{jm}(z)$ for z in F_j and $1 \leq j \leq m$. As the set G_m is compact, by the Tietze Extension Theorem, $h_m : G_m \rightarrow \mathbf{S}^k \setminus \{\infty\}$ also has a continuous extension to \mathbf{S}^n which will be denoted again by h_m . Thus, for each m , there is a pair of continuous maps $\phi_m : \mathbf{S}^n \rightarrow C(\mathbf{S}^k)$ and $h_m : \mathbf{S}^n \rightarrow \mathbf{S}^k \setminus \{\infty\}$ with the properties:

$$\begin{aligned} \phi_m(z) &= \phi(z) \quad \text{for } z \text{ in } G_m, \\ \text{dist}(h_m(z), \phi(z)) &< 1/m \quad \text{for } z \text{ in } G_m, \\ |h_m(z) - h_{m+1}(z)| &< 1/m \quad \text{for } z \text{ in } G_m. \end{aligned}$$

We apply Lemma 1 to ϕ_m and $g_m = \pi \circ h_m$ to get a homotopy $\alpha_m : \mathbf{S}^n \times I \rightarrow \mathbb{R}^k$ such that, for all z in \mathbf{S}^n ,

$$\pi^{-1} \circ \alpha_m(z, 0) = \pi^{-1} \circ \alpha_m(z, 1) = h_m(z)$$

and

$$D(\pi^{-1} \circ \alpha_m(z, I), \phi_m(z)) < 2\text{dist}(h_m(z), \phi_m(z)) + M/m.$$

Next, we apply Lemma 2 to h_m and h_{m+1} to get a homotopy $\beta_m :$

$S^n \times I \rightarrow S^k \setminus \{\infty\}$ such that, for all z in S^n ,

$$\beta_m(z, 0) = h_m(z), \quad \beta_m(z, 1) = h_{m+1}$$

and

$$|\text{diam}(\beta_m(z, I)) - |h_m(z) - h_{m+1}(z)|| < 1/m.$$

Now we shall piece together the homotopies $\pi^{-1} \circ \alpha_m$ and β_m to get the desired function $f : \mathbf{B}_{n+1} \rightarrow S^k \setminus \{\infty\}$. Let $\{r_m\}$ and $\{r'_m\}$ be increasing sequences of positive numbers converging to 1 with $r_m < r'_m < r_{m+1}$. On the closed set $\{x \in \mathbf{B}_{n+1} : r_m \leq |x| \leq r'_m\}$ we define f by rescaling the homotopy $\pi^{-1} \circ \alpha_m$ in the obvious manner, and on the closed set $\{x \in \mathbf{B}_{n+1} : r'_m \leq |x| \leq r_{m+1}\}$ we define f by rescaling the homotopy β_m in the obvious manner. This defines f on the relatively closed set $\{x \in \mathbf{B}_{n+1} : r_1 \leq |x| < 1\}$ of \mathbf{B}_{n+1} . The Tietze Extension Theorem applied to the closed set $\{x \in \mathbf{B}_{n+1} : |x| \leq r_1\}$ will complete the definition of the continuous function f on \mathbf{B}_{n+1} .

Let us verify that $\Gamma_m(z) = \{f(rz) : r_m \leq r \leq r_{m+1}\}$, $m = 1, 2, \dots$, converges uniformly to $\phi(z)$ on G_j for each j . To this end, let $m > j$ and $z \in G_j$. Since G_j is contained in G_m , we obtain from the identity

$$\Gamma_m(z) = \pi^{-1} \circ \alpha_m(z, I) \cup \beta_m(z, I)$$

and the definition of the Hausdorff metric D the inequality

$$D(\Gamma_m(z), \pi^{-1} \circ \alpha_m(z, I)) \leq \text{diam}(\beta_m).$$

Consequently,

$$\begin{aligned} D(\Gamma_m(z), \phi(z)) &\leq D(\Gamma_m(z), \pi^{-1} \circ \alpha_m(z, I)) + D(\pi^{-1} \circ \alpha_m(z, I), \phi(z)) \leq \\ &\leq \text{diam}(\beta_m(z, I)) + 2\text{dist}(h_m(z), \phi(z)) + M/m \leq \\ &\leq |h_m(z) - h_{m+1}(z)| + 1/m + 2\text{dist}(h_m(z), \phi(z)) + \\ &+ M/m < (4 + M)/m. \end{aligned}$$

Thus, we have that $\Gamma_m(z)$ converges to $\phi(z)$ uniformly on G_j . Finally, let us show $f_R(z) = \phi(z)$ for each z in F . Each z in F is a member of G_j for some j . Clearly, for $p \geq m > j$, we have from the definition of the Hausdorff metric D that

$$D(\cup\{\Gamma_q(z) : m \leq q \leq p\}, \phi(z)) \leq (4 + M)/m.$$

Therefore,

$$D(\text{Cl}(\{f(rz) : r_m \leq r < 1\}), \phi(z)) \leq (4 + M)/m,$$

from which we conclude that $f_R(z) = \phi(z)$. Since $\mu(\mathbf{S}^n \setminus F) = 0$, we have that f_R is equal to ϕ Lebesgue almost everywhere on \mathbf{S}^n . \diamond

Remark. The convergence of Γ_m to ϕ in the theorem is closely related to the concept of uniform convergence defined by Bagemihl and McMillan in [2]. Further investigations of this type of convergence can be found in [6] and [7]. The references [3], [4], [5], [8] and [9] contain discussions on radial limit behavior of continuous functions defined on an open ball.

Finally, consider the setting of classical complex variables. That is, \mathbb{R}^2 is identified with the set \mathbb{C} of complex numbers and the unit disk and the unit circle are \mathbf{B}_2 and \mathbf{S}^1 , respectively. Moreover, the set of extended complex numbers $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ becomes \mathbf{S}^2 . By employing the Arakeljan Approximation Theorem [1] in the same manner as in [7], [8], [9] and [12], we can establish the following corollary. Since its proof is a straightforward modification of those in the above references, we shall not prove the corollary.

Corollary. *Let $\phi : \mathbf{S}^1 \rightarrow C(\hat{\mathbb{C}})$ be a Lebesgue measurable function. Then, there is an analytic function f from the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ into \mathbb{C} and there is an increasing sequence of real numbers $\{r_m\}$ converging to 1 such that*

- (i) *the radial cluster-set function f_R of f is equal to ϕ Lebesgue almost everywhere on \mathbf{S}^1 , and*
- (ii) *for each positive number ε there is a measurable set E such that the continuous functions $\Gamma_m : \mathbf{S}^1 \rightarrow C(\hat{\mathbb{C}})$, $m = 1, 2, \dots$, defined by*

$$\Gamma_m(\xi) = \{f(r\xi) : r_m \leq r \leq r_{m+1}\}, \quad \xi \in \mathbf{S}^1,$$

converge uniformly to ϕ on E and $\mu(\mathbf{S}^1 \setminus E) < \varepsilon$.

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