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A GENERALIZATION OF CARISTI'S FIXED POINT THEOREM

Lj. Ćirić*

Matematički Institut, 11001–Beograd, Knez–Mihailova 35, Yugoslavia

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Abstract: General common fixed and periodic point theorems are proven for a class of selfmaps of a quasi–metric space which satisfy the contractive conditions (1), or (7), or (8), or (10) below. Presented theorems generalize and extend Caristi's Theorem [2]. Two examples are constructed to show that an introduced class of selfmaps is indeed wider than a class of selfmaps which satisfy Caristi's contractive definition (C) below.

1. Introduction.

Let X be a non–void set and $T : X \to X$ a selfmap. A point $x \in X$ is called a periodic point for T iff there exists a positive integer k such that $T^k x = x$. If $k = 1$, then x is called a fixed point for T.

J. Caristi [2] proved the following an important contraction fixed point theorem.

Theorem 1 (Caristi [2]). *Suppose $T : X \to X$ and $\phi : X \to [0, \infty)$, where X is a complete metric space and ϕ is lower semi–continuous. If for each x in X*

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$$\text{(C)} \qquad d(x,Tx) \le \phi(x) - \phi(Tx)\,,$$

then T has a fixed point.

Th. 1 is sometimes called a Caristi–Kirk–Browder theorem (see [5]). Recently A. Bollenbacker and T. Hicks [1] revisited Th. 1. Various proofs of Th. 1 were presented later in [11, 13, 15]. It is known that Caristi's theorem is essentially equivalent to Ekelend's variational principle [5]. Up to new many extensions of Caristi's result have been obtained [6, 7, 8, 9].

The purpose of this paper is to introduce and investigate a class of selfmaps which satisfy a contractive condition weaker than (C) and still have a fixed or periodic point.

2. Main results.

We begin with some notation needed in the sequel. A pair (X,d) of a set X and a mapping d from $X \times X$ into the real numbers is said to be a *quasi–metric space* iff for all $x, y, z \in X$:

(1) $d(x,y) \ge 0$ and $d(x,y) = 0$ iff $x = y$,
(2) $d(x,z) \le d(x,y) + d(y,z)$.

Let $d_x : X \to [0,+\infty)$ be defined by $d_x(y) = d(x,y)$. Let $\mathbb{N}$ denotes the set of all positive integers.

A sequence $\{x_n\}$ in X is said to be a *left k–Cauchy* sequence if for each $k \in \mathbb{N}$ there is one N_k such that $d(x_n, x_m) < 1/k$ for all $m \ge n \ge N_k$. A quasi–metric space is a *left k–sequentially complete* if each left k–Cauchy sequence is convergent (compare [12, 14]).

Now we are in position to state the following result.

Theorem 2.1. *Let (X,d) be a left k-complete quasi–metric space and let for each $x \in X$ a function d_x be lower semi–continuous (l.s.c) on X. Let F be a family of mappings $f : X \to X$. If there exists l.s.c. function $\phi : X \to [0,\infty)$ such that for each $x \in X$:*

$$(1) \qquad d(x,fx) \le \phi(x) - \phi(fx) \text{ for all } f \in F\,,$$

then for each $x \in X$ there is a common fixed point u of F such that

$$d(x,u) \le \phi(x) - s\,, \text{ where } s = \inf\{\phi(x) : x \in X\}\,.$$

Proof. For any $x \in X$ denote

$$S(x) = \{y \in X : d(x,y) \le \phi(x) - \phi(y)\},$$
$$a(x) = \inf\{\phi(y) : y \in S(x)\}.$$

As $x \in S(x)$, $S(x)$ is not empty and $0 \le a(x) \le \phi(x)$.

Let $x \in X$ be arbitrary. Put $x_1 = x$. Now we shall choose a sequence $\{x_n\}$ in X as follows: when $x_1, x_2, \ldots, x_n$ have been chosen, choose $x_{n+1} \in S(x_n)$ such that $\phi(x_{n+1}) \le a(x_n) + 1/n$. In doing so, one obtains a sequence $\{x_n\}$ such that

(2) $d(x_n, x_{n+1}) \le \phi(x_n) - \phi(x_{n+1}); \quad a(x_n) \le \phi(x_{n+1}) \le a(x_n) + 1/n.$

Then, as $\{\phi(x_n)\}$ is a decreasing sequence of reals, there is some $a \ge 0$ such that

(3) $$a = \lim_n \phi(x_n) = \lim_n a(x_n).$$

Let now $k \in \mathbb{N}$ be arbitrary. From (3) there exists some N_k such that $\phi(x_n) < a + 1/k$ for $n = N_k$. Thus, by monotonocity of $\{\phi(x_n)\}$ for $m \ge n \ge N_k$ we have $a \le \phi(x_m) \le \phi(x_n) < a + 1/k$ and hence

(4) $$\phi(x_n) - \phi(x_m) < 1/k \text{ for all } m \ge n \ge N_k.$$

From (ii) and (2) we get

(5) $$d(x_n, x_m) \le \sum_{s=n}^{m-1} d(x_s, x_{s+1}) \le \phi(x_n) - \phi(x_m).$$

Then by (4) we have

$$d(x_n, x_m) < 1/k \text{ for all } m \ge n \ge N_k.$$

Therefore, $\{x_n\}$ is a left k–Cauchy sequence and, by completeness of X, it converges to some $u \in X$.

Since d_x and ϕ are l.s.c. functions, by (5) we have

$$\begin{aligned} d(x_n, u) &\le \lim_m \inf d(x_n, x_m) \le \lim_m \sup d(x_n, x_m) \le \\ &\le \phi(x_n) + \lim_m \sup[-\phi(x_m)] = \phi(x_n) - \lim_m \inf \phi(x_m) \le \\ &\le \phi(x_n) - \phi(u). \end{aligned}$$

Thus $u \in S(x_n)$ for all $n \in \mathbb{N}$ and hence $a(x_n) \le \phi(u)$. So by (3), $a \le \phi(u)$. On the other hand, by l.s.c. of ϕ and (3), we have $\phi(u) \le \le \lim_n \inf \phi(x_n) = a$. Therefore, $\phi(u) = a$.

Now we shall show that $fu = u$ for all $f \in F$. Suppose not and let $f \in F$ be such that $fu \neq u$. Then (1) implies $\phi(fu) < \phi(u) = a$. Hence, by (3), there is a $n \in \mathbb{N}$ such that

$$\phi(fu) < a(x_n)\,. \tag{6}$$

Since $u \in S(x_n)$ for all $n \in \mathbb{N}$, we have

$$d(x_n, fu) \leq d(x_n, u) + d(u, fu) \leq [\phi(x_n) - \phi(u)] + [\phi(u) - \phi(fu)] = \\ = \phi(x_n) - \phi(fu)\,.$$

Hence we conclude that $fu \in S(x_n)$. Hence $\phi(fu) \geq a(x_n)$, which is a contradiction with (6). Therefore, $fu = u$ for all $f \in F$. Since $u \in S(x_n)$ implies

$$d(x_n, u) \leq \phi(x_n) - \phi(u) \leq \phi(x) - \inf\{\phi(y) : y \in X\} = \phi(x) - s\,. \quad \Diamond$$

The following result contains the above theorem.

Theorem 2.2. *Let E be a set, (X,d) as in Th. 2.1, $g : E \to X$ a surjective mapping and $F = \{f\}$ a family of arbitrary mappings $f : : E \to X$. If there exists a l.s.c. function $\phi : X \to [0,\infty)$, such that*

$$d(ga, fa) \leq \phi(ga) - \phi(fa) \text{ for all } f \in F \tag{7}$$

and each $a \in E$, then g and F has a common coincidence point, that is, for some $v \in E$ $gv = fv$ for all $f \in F$.

Proof. Let $x \in X$ be arbitrary and $u \in X$ as in Th. 2.1. Since g is surjective, for each $x \in X$ there is some $a = a(x)$ such that $ga = x$. Let $f \in F$ be a fixed mapping. Define by f a mapping $h = h(f)$ of X into itself such that $hx = fa$, where $a = a(x)$, that is, $ga = x$. Let H be a family of all mappings $h = h(f)$. Then (7) implies

$$d(x, hx) \leq \phi(x) - \phi(hx) \text{ for all } h \in H\,. \tag{8}$$

Thus, by Th. 2.1, $u = hu$ for all $h \in H$. Hence $gv = fv$ for all $f \in F$, where $v = v(u)$ is such that $gv = u$. $\Diamond$

The following result is related to periodic points.

Theorem 2.3. *Let (X,d) and ϕ be as in Th. 2.1. Let $T : X \to X$ be an arbitrary mapping. If for each $x \in X$ there is $n(x)$ in $\mathbb{N}$ such that*

$$d(x, T^{n(x)}x) \leq \phi(x) - \phi(T^{n(x)}x)\,, \tag{9}$$

then T has a periodic point.

Proof. Define $f : X \to X$ by $fx = T^{n(x)}x$. Then by Th. 2.1 (with F singleton) $fu = u$ for some $u \in X$. Hence $T^{n(x)}u = u$ that is, u is a periodic point of T. ◊

Remark 2.1. Example 2 below shows that a periodic point in Th. 2.3 need not be a fixed point. Therefore, one must add some hypothesis in order to ensure that T possesses a fixed point.

Theorem 2.4. *Let (X, d) and ϕ be as in Th. 2.1 and let $T : X \to X$ be a mapping. If for each $x \in X$, with $Tx \neq x$, there is $n(x) \in \mathbb{N}$ and a real number $C(x) > 0$ such that*

$$\max\{d(x, T^{n(x)}x), C(x) \cdot d(x, Tx)\} \leq \phi(x) - \phi(T^{n(x)}x), \tag{10}$$

then T has a fixed point.

Proof. If we suppose that $T^n x \neq x$ for all $n \in \mathbb{N}$, then we can choose $C(x)$ such that (10) reduces to (9). Then by the proof of Th. 2.3 $T^{n(x)}u = u$ for some $u \in X$. Therefore, from (10) we have

$$\max\{0, C(u) \cdot d(u, Tu)\} \leq \phi(u) - \phi(u) = 0 .$$

If we suppose that $u \neq Tu$, then $C(u) > 0$ and so we have $C(u) \cdot d(u, Tu) \leq 0$, a contradiction. Therefore $Tu = u$. ◊

Remark 2.2. It is clear that if T satisfies (C), then T satisfies (10) with $n(x) = 1$ and, for instance, $C(x) = 1$. Therefore, Th. 1 is a special case of Th. 2.1, even if (X, d) in Th. 2.1 is a metric space. Example 1 below shows that Th. 2.1 is a proper generalization of Caristi's Th. 1.

Remark 2.3. In [14] is given an example of a quasi–metric space (X, d) with d_x continuous for each x that is not metrizable.

3. Examples.

1. Let $X = \{0\} \cup \{\pm 1/n : n = 1, 2, \ldots\}$ with the usual metric. Define $T : X \to X$ by $T(1/n) = -1/(n+1)$, $T(-1/n) = 1/(n+1)$ and $T(0) = 0$. Define $\phi : X \to [0, +\infty)$ by $\phi(x) = d(x, Tx)$. Then for $x = \pm 1/n$ we have

$$d(x, Tx) = 1/n + 1/(n+1) : \ d(x, T^2 x) = 1/n = 1/(n+2) .$$

Hence

$$d(x, T^2 x) = 1/n - 1/(n+2) < 1/n + 1/(n+1) - \\ -[1/(n+2) + 1/(n+3)] = \phi(x) - \phi(T^2 x) .$$

Since for each $x = \pm 1/n$ we can choose $C(\pm 1/n) \leq 2(n+1)/(n+2)^2$, we conclude that T satisfies (10) for each x in X with $n(x) = 2$ (and $n(0) = 1$). As X is a complete metric space and $\phi(x) = |x| + |x|/(1+|x|)$ is continuous on X, we conclude that Th. 2.4 can be applied and $x = 0$ is a fixed point.

To show that Caristi's theorem is not applicable, we shall show that there is not a function $\phi : X \to [0, \infty)$ such that T satisfies (C). We pointed put [4] that such a function exists if and only if the series $\sum_{n=0}^{\infty} d(T^n x, T^{n+1} x)$ converges for all $x \in X$. Since in our example for any fixed $x = \pm 1/m_0$ we have

$$d(T^n x, T^{n+1} x) = 1/(n + m_0) + 1/(n + 1 + m_0) > 2/(n + 1 + m_0),$$

we conclude that the above series is divergent and hence there is no function ϕ such that (C) holds for any $x = \pm 1/n$ in X.

2. Let $X = [-2, -1] \cup [1, 2]$ with the usual metric. Define $T : X \to X$ by $Tx = -x$. Then T satisfies (9) with $n(x) = 2$ for any (continuous) function $\phi : X \to [0, +\infty)$.

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