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A CONCRETE ANALYSIS OF THE RADICAL CONCEPT

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Abstract: The notion of isolators, representing well–defined sets of proper ideals in an arbitrary ring, is employed to define preradicals through intersection. The various basic properties that a preradical may have are analysed through this technique. Also, isolators are emphasized in their role as "tangible mediators" between preradicals and their semisimple classes. Finally, the three classical nil radicals are characterized as radicals generated by a very natural sequence of isolators.

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1. Introduction

In describing the structure of certain types of algebraic systems, also such "radicals" which are not Kurosh–Amitsur radicals, may play an important role, for instance this is the case in the theory of near–rings, groups and lattice–ordered groups. Thus it seems to be useful to revisit the connection and interrelationship among the various notions of (not necessarily Kurosh–Amitsur) radicals.

The purpose of this paper is to analyse the concept of a radical of an arbitrary universal class $\mathcal{A}$ of not necessarily associative rings or Ω–groups, in particular near–rings. For the sake of simplicity we shall refer to the objects of $\mathcal{A}$ as *rings*. Let us recall that a *universal class* $\mathcal{A}$ of rings is one which is closed under taking ideals and homomorphic images. At this point we may mention that our investigation could be carried out on an even higher level of abstraction by taking arbitrary universal algebras for rings and congruence relations for ideals – this basis was selected by Hoehnke for his work in [3].

Our analysis is carried out progressively along the sequence:

$$\text{preradical} \longrightarrow \text{quasi–radical} \longrightarrow \text{radical}$$

A *Kurosh–Amitsur radical* r of $\mathcal{A}$ is a function $\mathcal{A} \to \mathcal{A}$, $A \mapsto rA \triangleleft A$ which satisfies the following conditions:

(A) for every homomorphism $f : A \to fA$, $(A \in \mathcal{A})$, $frA \subseteq rfA$;
(B) $r(A/rA) = 0$ for all $A \in \mathcal{A}$;
(C) for all $A \in \mathcal{A} : (I \triangleleft A \text{ and } rI = I) \Rightarrow I \subseteq rA$;
(D) for all $A \in \mathcal{A} : rrA = rA$.

A function $r : A \mapsto rA \triangleleft A$ which satisfies (A) is called a *preradical* (of $\mathcal{A}$). A preradical r which satisfies (B) is called a *quasi–radical* (also a *Hoehnke radical* due to [3]). A preradical r which satisfies (C) resp. (D) is said to be *complete* resp. *idempotent.* Connections between preradicals and (hereditary) quasi–radicals were first investigated by Michler in [8]. (Cf. also [9].) We follow a different line of approach.

The basic technique in our analysis is that of *definition through intersection of ideals* with the focus on basic properties of the *sets* of ideals to be intersected. For this purpose we employ the notion of an *isolator* (cf. our Def. 1, and [1]), and we impose various conditions on isolators, these conditions (stable, transferring, 0–extending) being generalizations of well–known properties of the set of semiprime ideals in an arbitrary associative ring (cf. definitions 2–4). We attempt to emphasize isolators as "concrete preradical definers" on the one hand, and "tangible mediators" between quasi–radicals and their semisimple classes, on the other. Our analyses culminate in three sequences of one–to–one correspondences (cf. corollaries 1–3) of the form:

$$r \longleftrightarrow \Lambda \longleftrightarrow \mathbf{S}$$

where r, Λ and $\mathbf{S}$ represent specific types of quasi–radicals, maximal stable isolators and subdirectly closed classes respectively.

As an application (section 5) we recover the three well-known nil-based radicals within our framework by giving new characterizations of these radicals in terms of natural cardinality conditions.

2. Quasi–radicals

We start off our analysis of the concept of a radical by defining our basic instrument.

Definition 1. An *isolator* Δ is a function which assigns to every ring A (in $\mathcal{A}$) a set $\Delta(A)$ of proper ideals of A satisfying the condition:

(α) if $f : A \to fA$ is any homomorphism, then for every $K \in \Delta(fA)$ there exists an $I \in \Delta(A)$ such that $fI \subseteq K$.

It easily follows that the following sets of proper ideals in an arbitrary associative ring A define isolators:

$\pi(A)$: the prime ideals;
$\Pi(A)$: the prime maximal ideals;
$\mu(A)$: the maximal ideals;
$\sigma(A)$: the semiprime ideals;
$\kappa(A)$: the quasi–semiprime ideals, (cf. [2]).

Proposition 1. *If Δ is an isolator then the assignment $A \mapsto rA = \cap(I \in \Delta(A))$ is a preradical.*

Proof. Let $f : A \to fA$ be any homomorphism. Then $frA = f(\cap(I \in \Delta(A))) \subseteq \cap(fI : I \in \Delta(A))$. Using ($\alpha$) we get $frA \subseteq \cap(K \in \Delta(f(A))) = rfA$. $\Diamond$

The preradical defined by the assignment $A \mapsto rA = \cap(I \in \Delta(A))$ will be referred to as the preradical *generated* by Δ. As to a converse to Prop. 1 we note that every preradical is trivially generated by an isolator: Let r be a preradical and define the function Δ by $\Delta(A) := \{P \triangleleft A : rA \subseteq P\}$. Then $rA = \cap(P \in \Delta(A))$ for all A. Let $f : A \to fA$ be any homomorphism, and $K \in \Delta(fA)$. Then $frA \subseteq rfA \subseteq K$;

and we have that (α) is satisfied with $I = rA$. Hence Δ is an isolator which generates r.

Our first condition on isolators is the one in:

Definition 2. An isolator Δ may be called *stable* if it satisfies the condition:

(β) if $f : A \to fA$ is any homomorphism, then for every $I \in \Delta(A)$ with $\ker f \subseteq I$ there exists a $K \in \Delta(fA)$ such that $K \subseteq fI$.

There exist isolators which are *not* stable. We construct such an isolator: Let n be a positive integer ≥ 2. Then the assignment $\overline{n} : A \mapsto \mapsto \overline{n}A := \{a \in A : na = 0\}$ is a preradical. Define the function Δ by $\Delta(A) := \{\overline{m}A : m \geq n \text{ and } n|m\}$. Then Δ is an isolator, and it generates $\overline{n}$. However, Δ is not stable. For let us consider $f : A \to A/\overline{n}A$, and $I = \overline{n}A = \ker f$. Now if $K \in \Delta(fA)$ and $K \subseteq fI$, we must have that $K = \overline{s}(A/\overline{n}A) = \overline{n}A$ for some $s \geq n$ with $n|s$. This implies that $\{a + \overline{n}A : a \in \overline{sn}A\} = \overline{n}A$, i.e. $\overline{n}A = \overline{sn}A$, which is in general not true. Thus we have that Δ is not stable.

We note here (cf. [1]) that a function Δ assigning to every ring A a set $\Delta(A)$ of proper ideals of A, and satisfying:

(χ) for all A and every homomorphism $f : A \to fA$, the assignment $P \mapsto fP$ defines a bijection $\{P \in \Delta(A) : \ker f \subseteq P\} \to \Delta(fA)$,

satisfies (α) and (β), and hence it defines a stable isolator. In particular, if $\mathcal{P}$ is any abstract property of rings and the function Δ is defined by

$$\Delta(A) := \{P \triangleleft A : A/P \quad \text{has property} \quad \mathcal{P}\}$$

then Δ satisfies condition (χ), because by the isomorphism $A/P \cong \cong fA/fP$ there is a bijection between $\{P \in \Delta(A) : \ker f \subseteq P\}$ and $\Delta(fA)$. (It is easy to construct a stable isolator which does *not* satisfy (χ).) It now easily follows that the isolators listed above are stable. Moreover, every preradical gives rise to a stable isolator. This is:

Proposition 2. *If r is a preradical then the assignment $A \mapsto \Delta(A) = = \{I \triangleleft A : r(A/I) = 0\}$ is a stable isolator.*

Proof. Let f be a homomorphism, and $fK \in \Delta(fA)$. Then $r(A/K) \cong \cong r(fA/fK) = 0$ shows that $K \in \Delta(A)$, and (α) is satisfied. The validity of (β) follows in a similar way. $\Diamond$

Quasi–radicals and stable isolators stand in a very special relationship with one another. This is:

Theorem 1. *Let r be a preradical. Then r is a quasi–radical if and only if r is generated by a stable isolator.*

Proof. Suppose that r is a quasi–radical. Then $r(R/rR) = 0$ for all R. This shows that for an arbitrary ring A, $\cap(I \triangleleft A : r(A/I) = 0) \subseteq rA$. On the other hand, if $I \triangleleft A$ such that $r(A/I) = 0$ then the natural homomorphism $A \to A/I$ induces that $rA \to r(A/I) = 0$, so that $rA \subseteq I$. Thus we have that $rA = \cap(I \triangleleft A : r(A/I) = 0)$, and it remains to show that the function Λ defined by $\Lambda(A) = \{I \triangleleft A : r(A/I) = 0\}$ is a stable isolator. This follows by Prop. 2.

Conversely, suppose that r is generated by a stable isolator Δ. Let A be an arbitrary ring and consider $f : A \to A/rA$. Now $r(A/rA) = = rfA := \cap(K \in \Delta(fA))$. Using ($\beta$) and ($\alpha$) we get $r(A/rA) \subseteq \cap(fI : : I \in \Delta(A), rA \subseteq I) = \cap(fI : I \in \Delta(A)) = \cap(I/ \cap (I \in \in \Delta(A)) : I \in \Delta(A)) = 0$. $\Diamond$

The preradical r in Prop. 2 and the quasi–radical implied there in view of Th. 1, say s, are comparable:

Proposition 3. *$rA \subseteq sA$ for all rings A.*

Proof. Let $I \triangleleft A$ such that $r(A/I) = 0$ and f the natural homomorphism $A \to A/I$. Then $frA \subseteq r(A/I) = 0$, and this together with $frA = (rA + I)/I$ shows that $rA \subseteq I$. Thus we have that $rA \subseteq \cap(I \triangleleft A : r(A/I) = 0) = sA$. $\Diamond$

Referring back to our list of well–known stable isolators we recall the fact that $\cap(I \in \pi(A)) = \cap(I \in \sigma(A))$. This implies that a given quasi–radical may be generated by *different* stable isolators. In terms of the partial order on isolators defined by "$\Delta \leq \Delta' \Leftrightarrow \Delta(A) \subseteq \Delta'(A)$ for all A" we have:

Proposition 4. *If r is a quasi–radical then the function Λ defined by $\Lambda(A) = \{I \triangleleft A : r(A/I) = 0\}$ is a stable isolator such that Λ satisfies condition (χ) and the quasi-radical generated by Λ is r. Moreover, if Δ is any stable isolator generating the same quasi–radical r, then $\Delta \leq \Lambda$.*

Proof. The first claim was already verified in the proof of Th. 1. Let us therefore consider any stable isolator generating r. Let A be an arbitrary ring, and $I \in \Delta(A)$. Applying (β) to $A \to A/I$ we find that $K := 0 \in \Delta(A/I)$. Since $r(A/I) \subseteq X/I$ for all $X/I \in \Delta(A/I)$ we have that $r(A/I) = 0$, i.e. $I \in \Lambda(A)$. Hence $\Delta \leq \Lambda$. $\Diamond$

The unique maximal stable isolator Λ corresponding to the quasi–radical r will be referred to as the *maximal generating isolator* for r, and denoted by $\Lambda[r]$, or just by Λ where no ambiguity can occur. For any given quasi–radical r, $\Lambda[r]$ has another unique feature, as is exhibited in the following characterization.

Proposition 5. *Let Δ be a stable isolator. Then $\Delta = \Lambda[r]$ for some* *quasi–radical r if and only if Δ satisfies the condition:*

$$(\phi) \qquad \forall A((\Gamma \subseteq \Delta(A)) \Rightarrow \cap(I \in \Gamma) \in \Delta(A))$$

Proof. Suppose $\Delta = \Lambda$ for a quasi–radical r. Let A be an arbitrary ring and $\Gamma \subseteq \Delta(A)$; and consider $K := \cap(I \in \Gamma)$. Using (β) for Λ we get

$$r(A/K) := \cap(M/K \in \Lambda(A/K)) =$$
$$= \cap(M/K : K \subseteq M \in \Lambda(A)) \subseteq \cap(I/K \in \Gamma) = 0\,.$$

Thus we have that $\cap(I \in \Gamma) = K \in \Lambda(A) = \Delta(A)$.

Conversely, let Δ be a stable isolator satisfying (ϕ), and r the quasi–radical generated by Δ. Then $\Delta \leq \Lambda$. Let $P \in \Lambda(A)$. Then $r(A/P)=0$. This implies (since Δ generates r) that $\cap(I/P \in \Delta(A/P)) = 0$, and this in its turn implies that $\cap(I : I/P \in \Delta(A/P)) = P$. By (α), for each $I_\nu/P \in \Delta(A/P)$ there is an $M_\nu \in \Delta(A)$ such that $M_\nu/P \subseteq I_\nu/P$. Set $\Gamma := \{M_\nu\}$. Then we have $\cap(M_\nu \in \Gamma) \subseteq \subseteq \cap(I : I/P \in \Delta(A/P)) = P$. By (β), for each $M_\nu \in \Gamma$ there is an $L_\nu/P \in \Delta(A/P)$ such that $L_\nu/P \subseteq M_\nu/P$. Hence $P = = \cap(I : I/P \in \Delta(A/P)) \subseteq \cap L_\nu \subseteq \cap(M \in \Gamma)$. Thus we have shown that $P = \cap(M \in \Gamma)$; and condition (ϕ) yields $P \in \Delta(A)$. Hence $\Lambda \leq \Delta$; and the equality $\Delta = \Lambda[r]$ follows. $\Diamond$

A quasi–radical is, as mentioned in the introduction, just a Hoehnke radical. It is known from [3] that there is a one–to–one correspondence between quasi–radicals and subdirectly closed classes: if r is a quasi–radical then the class $\mathbf{S}_r := \{A \in \mathcal{A} : rA = 0\}$, (which is usually called the *semisimple class* of the quasi–radical r), is closed under taking subdirect sums; and if $\mathbf{S}$ is a subclass of $\mathcal{A}$ being closed under subdirect sums, then the assignment $r : A \mapsto rA$ defined by $rA = \cap(I \triangleleft A : A/I \in \mathbf{S})$ is a quasi–radical with semisimple class $\mathbf{S}$. In view of this and Th. 1 and Prop. 4 we have:

Corollary 1. *There exist one–to–one correspondences $r \longleftrightarrow \Lambda \longleftrightarrow \mathbf{S}$ between quasi–radicals r, maximal stable isolators Λ, and subdirectly closed classes $\mathbf{S}$.*

An example: $\beta : A \mapsto \beta A := \cap(P \in \pi(A)) = \cap(S \in \sigma(A))$ is a quasi–radical. (β is the well–known prime radical for associative rings.) Using Prop. 5 together with well–known properties of prime and semiprime ideals, we see that $\pi \neq \Lambda[\beta]$ while $\sigma = \Lambda[\beta]$.

3. Complete quasi–radicals

In this section we carry our analysis one step further : we consider those preradicals satisfying conditions (B) and (C) stated in the introduction, i.e. the complete quasi–radicals. For this purpose we shall need a further condition on isolators:

Definition 3. An isolator Δ may be called *transferring* if it satisfies the condition:

(γ) if $P \in \Delta(A)$ and $I \triangleleft A$ with $P \cap I \neq I$ then there exists a $K \triangleleft I$, $K \neq I$, such that $P \cap I \subseteq K$ and $K \in \Delta(I)$.

Of the five examples of isolators listed in section 2, the first four are transferring. In fact, in the case of any $\Delta \in \{\pi, \Pi, \mu, \sigma\}$, well–known properties of the ideals concerned show that $(P \in \Delta(A), I \triangleleft A, P \cap I \neq I)$ implies that $P \cap I \in \Delta(I)$, and (γ) is satisfied with $K = P \cap I$. The isolator κ, however, is not transferring, e.g. if A is a ring with identity and having a nilpotent ideal $N \neq 0$, the $\cap(P \in \kappa(A)) = 0$, so that $P \cap N \neq N$ for at least one $P \in \kappa(A)$, though $\kappa(N) = \emptyset$. (Cf. [2].)

A basic relationship between transferring isolators and complete quasi–radicals is exhibited in:

Theorem 2. *The following three conditions on a quasi–radical r are equivalent:*

(1) *r is complete;*

(2) *the maximal generating isolator $\Lambda = \Lambda\ [r]$ of r is transferring;*

(3) *the semisimple class $\mathbf{S}_r$ of r is regular, i.e.* $(0 \neq I \triangleleft A \in \mathbf{S}_r) \Rightarrow$
$\Rightarrow (\exists\ K \triangleleft I$ *such that* $0 \neq I/K \in \mathbf{S}_r)$.

Proof. (1) $\Rightarrow$ (2): Assume that Λ is not transferring. Then there is a $P \in \Lambda(A)$ and an $I \triangleleft A$ with $P \cap I \neq I$ such that $\Lambda(I/(P \cap I)) = \emptyset$. The latter implies that $r(I/(I \cap P)) = I/(I \cap P)$, and hence we get $r((I + P)/P) = (I + P)/P$. From the completeness of r it follows that $(I + P)/P \subseteq r(A/P) = 0$, giving the contradiction $I \subseteq P$. Hence Λ is transferring.

(2) $\Rightarrow$ (3): Let A be a ring with $rA = 0$, and $0 \neq I \triangleleft A$. From $rA = 0$ and the maximality of Λ it follows that $0 \in \Lambda(A)$. Since Λ is transferring, there exists a $K \triangleleft I$, $K \neq I$ such that $K \in \Lambda(I)$, i.e. $r(I/K) = 0$.

(3) $\Leftrightarrow$ (1): has been proved in Prop. 2.2 of [7]. $\Diamond$

In the structure theory of 0–symmetric near–rings the most important quasi–radical assignments are $\mathcal{J}_\nu : A \mapsto \mathcal{J}_\nu(A)$, $\nu = 0, 1, 2$. It is known that $\mathcal{J}_0 < \mathcal{J}_1 < \mathcal{J}_2$, that $\mathcal{J}_0$ and $\mathcal{J}_1$ are complete but not idempotent, (cf. [4]), and $\mathcal{J}_2$ is a Kurosh–Amitsur radical. For any *ring* A, $\mathcal{J}_0(A) = \mathcal{J}_1(A) = \mathcal{J}_2(A)$ is the usual Jacobson radical. For details we refer to [6]. By Th. 2 we have:

Corollary 2. *There exist one-to-one correspondences* $r \longleftrightarrow \Lambda \longleftrightarrow \mathbf{S}$ *between the complete quasi-radicals* r*, the maximal stable transferring isolators* Λ*, and the regular subdirectly closed classes* $\mathbf{S}$.

Remark: Applying our Th. 2 and Th. 1.2 of [7] to the property "the quasi-radical r is complete", and translating into the language of isolators, we get: If Δ is an isolator satisfying condition (χ) and generating the quasi-radical r, then Δ is transferring if and only if the maximal stable isolator $\Lambda[r]$ is transferring.

4. Idempotent complete quasi-radicals

We now proceed to isolating the (Kurosh–Amitsur) radicals from among the preradicals. First we shall briefly look at idempotent preradicals.

Proposition 6. *Let* Δ *be an isolator generating the preradical* r. r *is idempotent if and only if* $\Delta(rA) = \emptyset$ *for all* $A \in \mathcal{A}$. (The proof is straightforward.)

Proposition 7. *Let* r *be a quasi-radical and* Λ *the corresponding maximal stable isolator. The following conditions are equivalent:*

(1) r *is idempotent,*

(2) $\Lambda(rA) = \emptyset$ *for all* $A \in \mathcal{A}$,

(3) $r(rA/M) \neq 0$ *for all* $A \in \mathcal{A}$ *and all proper ideals* M *of* rA.

Proof. (1) $\Rightarrow$ (2): this follows from Prop. 6.

(2) $\Rightarrow$ (3): If $r(rA/M) = 0$ for a proper ideal M of A, then $M \in \Lambda(rA)$, contradicting (2).

(3) $\Rightarrow$ (1) If r is not idempotent, then $rrA \neq rA$ for some $A \in \in \mathcal{A}$. Since r is a quasi-radical, for $M = rrA$ we have $r(A/M) = 0$, contradicting (3). $\Diamond$

Another criterion for the idempotence of a quasi-radical has been given in Prop. 2.4 of [7].

The quasi-radical r determined by κ is (as has already been indicated) not complete. It is, however, idempotent because: for any ring A and $I \triangleleft A$, $rI \supseteq I \cap rA$ (cf. [2], Lemma 4.6), and, setting $I := rA$, we have $rA \supseteq rrA \supseteq rA \cap rA = rA$.

For 0–symmetric near–rings the isolator π defines an idempotent quasi–radical, called the *prime* radical. As it has been shown in [5], this prime radical is not a Kurosh–Amitsur radical, so it is not complete, and π is not transferring.

The independence of being complete and being idempotent has been exhibited also in examples 1 and 2 of [7].

We shall need one more condition on isolators:

Definition 4. An isolator Δ may be called 0–*extending* if it satisfies the condition:

$$(\delta) \qquad \text{if} \quad I \in \Delta(A) \quad \text{and} \quad 0 \in \Delta(I) \quad \text{then} \quad 0 \in \Delta(A).$$

The best–known example of a 0–extending isolator is the isolator σ isolating the semiprime ideals in an arbitrary associative ring. As one easily sees π is not 0–extending, though it generates the same preradical as σ. Also the isolator μ is not 0–extending.

For the purposes of our next result we shall need a further construction. We consider an arbitrary fixed preradical r and define a function

$$\Psi : A \mapsto \Psi(A) := \{P \lhd A : (Q \lhd A \quad \text{and} \quad rQ = Q) \Rightarrow Q \subseteq P\};$$

and then a function $r' : A \mapsto r'A := \cap(P \in \Psi(A))$. It is easy to see that Ψ is an isolator, and hence r' is a preradical. Moreover, if Δ is a transferring isolator generating r, then $\Delta \leq \Psi$. In this notation we have:

Theorem 3. *The following three conditions on a complete quasi–radical r are equivalent:*

(1) *r is idempotent (and hence a Kurosh–Amitsur radical);*

(2) *for all A, $rrA \lhd A$; and $\Lambda = \Lambda[r]$ is 0–extending;*

(3) *for all A, $rrA \lhd A$ and $rA = r'A$.*

Proof. (1) $\Rightarrow$ (2): Since $rrA = rA$, $rrA \lhd A$. Let $I \in \Lambda(A)$ and $0 \in \Lambda(I)$, and assume that $0 \notin \Lambda(A)$, i.e. $rA \neq 0$. By the completeness of r it follows that $(rA + I)/I \subseteq r(A/I) = 0$. Hence $0 \neq rA \subseteq I$, contradicting $rI = 0$. It follows that Λ is 0–extending.

(2) $\Rightarrow$ (3): Assuming (2) we prove that $rA = r'A$. From $r((A/rrA)/(rA/rrA)) \cong r(A/rA) = 0$ we see that $rA/rrA \in \in \Lambda(A/rrA))$, and it is clear that $0 \in \Lambda(rA/rrA)$. Hence (2) implies that $0 \in \Lambda(rA/rrA)$. This gives the inclusion $rA \subseteq rrA$, and we have that $rrA = rA$. This equality shows that $rA \subseteq T$ for all $T \in \Psi(A)$, so that $rA \subseteq r'A$. On the other hand, if $P \in \Lambda(A)$ and $Q \lhd A$ with

$rQ = Q$, it follows by the completeness of r that $Q \subseteq rA \subseteq P$, so that $P \in \Psi(A)$. Hence $r'A \subseteq rA$; and now $rA = r'A$ follows.

(3) $\Rightarrow$ (1): Let $Q \triangleleft A$ such that $rQ = Q$. Since r is complete, $Q \subseteq rA$. Since now $Q \triangleleft rA$, it follows (again by the completeness of r) that $Q \subseteq rrA$. Hence it follows that $rrA \in \Psi(A)$. We now have that $r'A \subseteq rrA \subseteq rA = r'A$, showing that $rrA = rA$. $\Diamond$

Our final observation in this section comes in view of Th. 3 and known facts about the semisimple classes of Kurosh–Amitsur radicals (cf. [10]). This is:

Corollary 3. *There exist one–to–one correspondences $r \longleftrightarrow \Lambda \longleftrightarrow \mathbf{S}$ between Kurosh–Amitsur radicals r, those maximal stable transferring 0–extending isolators Λ for which $r_\Lambda r_\Lambda A \triangleleft A$ for all A, and those subdirectly closed, regular, extensionally closed classes $\mathbf{S}$ for which $r_\mathbf{S} r_\mathbf{S} A \triangleleft A$ for all A, where r_Λ and $r_\mathbf{S}$ represent the quasi–radicals associated with Λ and $\mathbf{S}$ respectively.*

5. An application — the nil radicals

In this final section — by way of an application — we use the isolator approach to construct a single formula function which yields the three classical nil radicals — the prime radical β, the locally nilpotent radical $\mathcal{L}$ and the nil radical $\mathcal{N}$. We confine our attention here to associative rings. We shall need the following:

Lemma. *A proper ideal S of a ring A is a semiprime ideal if and only if every nonzero ideal of A/S has a potent countably generated subring.*

Proof. Suppose S is a semiprime ideal of a ring A and let $0 \neq X/S \triangleleft \triangleleft A/S$. If every countably generated subring of X/S is nilpotent, then X/S itself is a nilpotent ring. (Suppose X/S is not nilpotent. Then for every natural number n we may select a sequence $r_{1n}, r_{2n}, \dots, r_{nn}$ in X/S such that $r_{1n} r_{2n} \dots r_{nn} \neq 0$. Thus we would have selected a countable subset of X/S which generates a countable *non*-nilpotent (i.e. potent) subring of X/S).** But then $X^m \subseteq S$ for some positive integer m, and this would imply that $X \subseteq S$.

Conversely, suppose that every nonzero ideal of A/S has a potent countably generated subring. Let $X \triangleleft A$ such that $X^2 \subseteq S$. If $X \not\subseteq S$,

**Professor Otto Kegel has on inquiry pointed out this subproof to the first author in a personal communication in 1981. We are indebted to professor Kegel.

then $(X+S)/S$ must have a potent countably generated subring T/S. But this is impossible since $(T/S)^2 \subseteq ((X+S)/S)^2 = 0$. $\Diamond$

We are now in a position to prove the main result of this section. This is:

Theorem 4. *Let Δ_i $(1 \leq i \leq 3)$ be the function defined by*
$\Delta_i(A) = \{P \triangleleft A, P \neq A : (P \subset X \triangleleft A) \Rightarrow X/P$ *has a potent* δ_i*–subring*$\}$
where the δ_i are defined as follows:
δ_1 : *"countably generated";*
δ_2 : *"finitely generated";*
δ_3 : *"singly generated".*
Then the Δ_i are stable transferring isolators, and they generate respectively the radicals:
$r_1 = \beta$: *the prime radical;*
$r_2 = \mathcal{L}$: *the locally nilpotent radical;*
$r_3 = \mathcal{N}$: *the nil radical.*

Proof. Fix any $i \in \{1,2,3\}$. The stableness of Δ_i is an easy consequence of the remarks following condition (χ). To show that Δ_i is tranferring we verify the stronger condition $(P \in \Delta_i(A),\ Q \triangleleft A,\ P \cap Q \neq Q) \Rightarrow P \cap Q \in \Delta_i(Q)$. Let $0 \neq X/(P\cap Q) \triangleleft Q/(P \cap Q)$. Then $Q/(P\cap Q) \cong (Q+P)/P$ gives us the existence of a nonzero ideal $Y/P \cong X/(P\cap Q)$ in $(Q+P)/P$. Let Y^*/P be the ideal generated in A/P by Y/P. Then Y^*/P has a potent δ_i–subring, and hence $(Y^*/P)^3 = (Y^{*^3}+P)/P \neq 0$. Hence, since $P \in \Delta_i(A)$, $(Y^*/P)^3$ has a potent δ_i–subring. Since by the Andrunakievich lemma $(Y^*/P)^3 \subseteq$ $\subseteq Y/P \cong X/(P\cap Q)$, it follows that $P \cap Q \in \Delta_i(Q)$.

Having established that Δ_i is stable and transferring we now prove that the complete quasi–radical r_i generated by Δ_i is a radical. We apply Th. 3. Let once again A be an arbitrary ring. From the lemma we infer that every $P \in \Delta_i(A)$ is a semiprime ideal of A. This ensures that $r_i r_i A \triangleleft A$. Suppose that $r_i' A \subset r_i A$. Then there exists a $T \in \Psi(A)$ such that $r_i A \not\subseteq T$. This shows that $\Delta_i(r_i A) \neq \emptyset$, i.e., there is a proper ideal U of $r_i A$ such that every nonzero ideal of $r_i A/U$ has a potent δ_i–subring. It follows by the lemma that U is a semiprime ideal of $r_i A$, and we know that $r_i A := \cap(P \in \Delta_i(A))$ is a semiprime ideal of A. Hence U is a semiprime ideal of A. Let $0 \neq X/U \triangleleft A/U$. If $r_i A \cap X \neq U$ then there is a potent δ_i–subring in $(r_i A \cap X)/U$ and hence in X/U. If $r_i A \cap X = U$ then $(X + r_i A)/r_i A \cong X/U$. Now $X + r_i A \neq r_i A$ and since $r_i((X+r_i A)/r_i A) = 0$ there exists a $T = r_i A \in (X + r_i A)/r_i A$, so that there is a potent δ_i-subring in $((X + r_iA)/r_iA)/(T/r_iA)$, and consequently there is a potent δ_i–subring in $(X+r_iA)/r_iA$. Hence X/U has a potent δ_i–subring. It follows that $U \in \Delta_i(A)$ — a contradiction.

By our construction $r_iA := \cap(P \in \Delta_i(A))$ it follows that $r_iA = A$ if and only if $\Delta_i(A) = \emptyset$. In the case $i = 1$ this means (by the lemma) that $r_1A = A$ if and only if A has no semiprime ideals, i.e. A is a β–radical ring. In the case $i = 2$ we note that a locally nilpotent ring A clearly has $\Delta_2(A) = \emptyset$. On the other hand, let B be a ring with $\Delta_2(B) = \emptyset$. If B has a potent finitely generated subring $\langle x_1, \dots, x_n\rangle$, (we may assume all x_i are potent elements), we may, in view of the finiteness, select a maximal element in the set $\{C \triangleleft B : \{x_1, \dots, x_n\} \not\subseteq C\}$, say M. But then clearly M must be in $\Delta_2(B)$, contradicting $\Delta_2(B) = \emptyset$. Thus we have that $r_2 = \mathcal{L}$. The same argument applied in the case $i = 3$, together with the fact that a singly generated subring $\langle x\rangle$ of a ring is nilpotent if and only if x is a nilpotent element, ensures that $r_3 = \mathcal{N}$. ◊

Corollary 4. *The function* Γ_i *defined by*

$$\Gamma_i(A) := \{I \in \Delta_i(A) : I \quad \textit{is a prime ideal}\}$$

is a stable isolator generating the radical r_i *for* $i = 1, 2, 3$.

Proof. One readily sees that Γ_i satisfies condition (χ) and therefore Γ_i is a stable isolator.

Let B be a prime ring such that $r_iB = 0$. This means exactly that every nonzero ideal of B has a potent δ_i–subring. Since $r_i = \beta, \mathcal{L}, \mathcal{N}$ are special radicals, every ring A with $r_iA = 0$ is a subdirect sum of prime rings B_α with $r_iB_\alpha = 0$. Thus for a ring A the condition

$$s_iA := \cap(P \in \Gamma_i(A)) = 0$$

is equivalent to $r_iA = 0$. Moreover, by $\Gamma_i \leq \Delta$ it follows that $r_iX \subseteq s_iX$ for every ring X. Now suppose that $r_iX \neq s_iX$ for a ring X. Then we have

$$s_i(s_iX/r_iX) = r_i(s_iX/r_iX) = 0,$$

which implies that r_iX is an element of the maximal stable isolator generating the quasi–radical s_i. Hence $s_iX \subseteq r_iX$ follows, contradicting the assumption. ◊

The natural cardinality considerations in Th. 4 seem to explain the imperturbable monopoly that β, $\mathcal{L}$ and $\mathcal{N}$ have maintained on the lower end of the chain of useful concrete radicals.

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