Mathematica Pannonica
3/1 (1992), 91 – 115

POWERS AND GENERALIZED CARDINAL NUMBERS FOR HCH-OBJECTS - BASIC NOTIONS

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Received August 1990

AMS Subject Classification: 03 E 70, 94 A 17, 03 B 50, 94 D 05

Keywords: Vaguely defined objects, Lukasiewicz logic, generalized cardinals.

Abstract: In this paper we present an introduction to a theory of powers and (generalized) cardinal numbers which is based on the infinite-valued Lukasiewicz logic and refers to so-called HCH-objects, i.e. to objects which in general cannot be mathematically modelled using the notion of a set. We focus here our attention on the notion of equipotency for HCH-objects and the construction of generalized cardinals and their basic properties. Problems related to order and operations on the generalized cardinals will be discussed in [24,25].

1. Introduction and notations

The purpose of this paper is to present mathematical base of a theory of powers and generalized cardinal numbers for hardly characterizable objects, shortly HCH-objects. By HCH-objects we mean here parts of some infinite universal set $\mathcal{U}$ which maybe are vaguely defined and do not need to be sets themselves, i.e. which in general cannot be mathematically modelled, without essential distortions, using the classical notion of a (sub)set (cf. semisets [18]). However, we assume that each HCH-object can be described, at least in a subjective way, by means of a function $\mathcal{U} \to \mathcal{L}$ or using a pair of such the functions ($\mathcal{L}$ denotes a suitable lattice). These functions will be called generalized characteristic functions or membership functions. This way sets, fuzzy sets ([27]), intuitionistic fuzzy sets ([16]) and generally $\mathcal{L}$-fuzzy sets ([4]; cf. Heyting algebra valued sets in [9]), twofold fuzzy sets ([3]), rough sets ([13]), and partial sets ([10]) become special cases of HCH-objects. HCH-objects which are not sets will be called proper HCH-objects.

So, although the given definition of an HCH-object is rather informal, it is sufficiently good for our purposes because in a way it makes possible to bring together those more or less different notions what is very convenient for the presentation of the theory (see e.g. Section 8).

If $A : \mathcal{U} \to \mathcal{L}$, then obj($A$) denotes the HCH-object 'embedded' in $\mathcal{U}$ and described (characterized) by means of A. Since obj(A) is not necessarily a set we shall write $x \epsilon \mathrm{obj}(A)$ instead of $x \in \mathrm{obj}(A)$; obviously, obj(A) is a set if $A(x) \in \{0,1\}$ for each x from $\mathcal{U}$. Then $[x \epsilon \mathrm{obj}(A)] := A(x)$, where $[s]$ denotes the truth value of a sentence s (obviously, $[s] \in \mathcal{L}$) and the symbol := stands always for 'equals by definition'. Each value $A(x)$ will be called membership grade of x in obj(A). Moreover, we accept the following definitions:

$$
\begin{aligned}
&[\neg s] := [s] \to 0,\\
&[r \& s] := [r] \wedge [s],\\
&[r \mid s] := [r] \vee [s],\\
&[r \Rightarrow s] := [r] \to [s],\\
&[r \Leftrightarrow s] := [r \Rightarrow s \;\&\; s \Rightarrow r],\\
&[\forall\!\!\!/ x \in \mathcal{U} : s(x)] := \bigwedge_{a \in \mathcal{U}} [s(x/a)],\\
&[\exists\!\!| x \in \mathcal{U} : s(x)] := \bigvee_{a \in \mathcal{U}} [s(x/a)],
\end{aligned}
$$

where

(a) $\neg$, &, |, $\Rightarrow$, $\Leftrightarrow$ are logical symbols of negation, conjunction, disjunction, implication, and equivalence, respectively;

(b) $\forall\!\!\!/$ and $\exists\!\!|$ denote general and existential many-valued quantifiers and $s(x/a)$ is the usual substitution notation (classical quantifiers will be denoted by $\forall$ and $\exists$);

(c) $\wedge$, $\bigwedge$ ($\vee$, $\bigvee$, resp.) denote the operation of the greatest lower

bound (least upper bound, resp.) for two arguments or their arbitrary number;
(d) $\rightarrow$ denotes many-valued implication operator; we additionally assume that it fulfills two properties: $b \rightarrow c = 1$ iff $b \leq c$,$1 \rightarrow b = b$ for each b, $c \in \mathcal{L}$.

Generalized inclusion obj $(A) \underset{\sim}{\subset}$ obj(B) and equality obj$(A) \approx$ $\approx$obj(B) of two HCH-objects are respectively defined by the conditions

$$\forall x \in \mathcal{U} : x \ \epsilon \ \mathrm{obj}(A) \Rightarrow x \ \epsilon \ \mathrm{obj}(B),$$

and

$$\mathrm{obj}(A) \underset{\sim}{\subset} \mathrm{obj}(B) \ \& \ \mathrm{obj}(B) \underset{\sim}{\subset} \mathrm{obj}(A).$$

Of course, the usual two-valued inclusion and equality one defines by

$$\mathrm{obj}(A) \subset \mathrm{obj}(B) \text{ iff } A \subset B \text{ with } A \subset B \text{ iff } \forall x \in \mathcal{U} : A(x) \leq B(x),$$

$$\mathrm{obj}(A) = \mathrm{obj}(B) \text{ iff } A = B \text{ with } A = B \text{ iff } \forall x \in \mathcal{U} : A(x) = B(x).$$

We at once see that

$$\mathrm{obj}(A) \subset \mathrm{obj}(B) \text{ iff } [\mathrm{obj}(A) \underset{\sim}{\subset} \mathrm{obj}(B)] = 1,$$
$$\mathrm{obj}(A) = \mathrm{obj}(B) \text{ iff } [\mathrm{obj}(A) \approx \mathrm{obj}(B)] = 1.$$

The conditions defining union and intersection of two HCH-objects are also quite natural, namely

$$\mathrm{obj}(A) \cup \mathrm{obj}(B) = \mathrm{obj}(C) \text{ iff } C = A \cup B, \text{ where } (A \cup B)(x) :=$$
$$:= A(x) \vee B(x),$$
$$\mathrm{obj}(A) \cap \mathrm{obj}(B) = \mathrm{obj}(D) \text{ iff } D = A \cap B, \text{ where } (A \cap B)(x) :=$$
$$:= A(x) \wedge B(x).$$

So, the sentence $x \ \epsilon \ \mathrm{obj}(A) \cup \mathrm{obj}(B)$ ($x \ \epsilon \ \mathrm{obj}(A) \cap \mathrm{obj}(B)$, resp.) has the same truth value as the sentence $x \ \epsilon \ \mathrm{obj}(A) \mid x \ \epsilon \ \mathrm{obj}(B)$ ($x \ \epsilon \ \mathrm{obj}(A)$ & & $x \ \epsilon \ \mathrm{obj}(B)$, resp. .

Nowadays (proper) HCH-objects play an important role in many branches of mathematics, computer and information sciences, social sciences, engineering, etc. It is quite clear that in many situations there is a necessity of having (as precise and adequate as possible) handy quantitative information about an HCH-object. So, it would be very useful to have for HCH-objects some counterpart of cardinal numbers. Such a reasonable counterpart will be constructed here and will be called generalized cardinal numbers (shortly gcn's).

In this paper we like to present a detailed discussion devoted to such basic questions as equipotency of HCH-objects, gcn's - their construction and elementary properties, the notion of finiteness for HCH-objects. Problems of comparing and ordering for gcn's will be presented in [24], operations on them are studied in [25]. We construct the theory for quite arbitrary HCH-objects and use here the infinite-valued Łukasiewicz logic; some results for HCH-objects with finite supports are already placed in [21,22]. So, we put $\mathcal{L} := \mathcal{J}$, where $\mathcal{J} := [0,1]$, whereas $\rightarrow$ is the Łukasiewicz implication operator, i.e. $b \rightarrow c := 1 \wedge 1 - b + c$; of course, $\wedge$, $\bigwedge$ and $\vee$, $\bigvee$ denote then usual operations of minimum, infimum, maximum, and supremum of numbers from the closed unit interval. It is however possible to construct an analogous intuitionistic theory of powers and gcn's for HCN-objects using triangular norms and φ-operators or applying intuitionistic logic with $\mathcal{L} :=$ complete Heyting algebra (see [26]).

As regards the notation and terminology, we decide to use troughout this paper the following additional rules:

(a) Sets are denoted by script capitals (e.g. $\mathcal{D}, \mathcal{J}, \mathcal{U}$) and some multiletter symbols defined in the sequel of the paper; as usual $\emptyset$ denotes the empty set.

(b) Capitals in italic denote the membership functions. The functions E and U are defined as follows: $\forall x \in \mathcal{U} : E(x) = 0,\ U(x) = 1$.

(c) The letters i,j,.....,p,q denote both the finite and transfinite numbers.

(d) Small Greek letters with or without subscripts (e.g. α, $\beta_{f,g}$) will denote the generalized cardinal numbers related to HCH-objects.

(e) If $A : \mathcal{U} \rightarrow \mathcal{J}$, then $\text{supp}(\text{obj}(A)) := \text{supp}(A) := \{x \in \mathcal{U} : A(x) \neq \neq 0\}$; so, the so-called support of obj(A) and support of A are identically defined. Moreover $A_t := \{x \in \mathcal{U} : A(x) \geq t\}$for $t \in \mathcal{J}_o := (0,1]$; A_t will be called t-level set of A and obj(A).

(f) $PS(\mathcal{D}) := \{0,1\}^{\mathcal{D}}$, $GP(\mathcal{D}) := \mathcal{J}^{\mathcal{D}}$, $P_i(\mathcal{D}) := \{\mathcal{B} \subset \mathcal{D} : \text{card}\ \mathcal{B} = i\}$.

(g) $1_{\mathcal{D}}$ denotes the characteristic function of $\mathcal{D} \subset \mathcal{U}$, i.e. $1_{\mathcal{D}}(x) = 1$ if $x \in \mathcal{D}$ else $1_{\mathcal{D}}(x) = 0$. So, $E = 1_{\emptyset}$.

(h) CN denotes the set of all the cardinals i such that card $\mathcal{U} \geq i$, $\text{betw}(i,j) := \{k \in CN : i \leq k \leq j\}$ for $i, j \in CN$.

(i) i^+ denotes the succesor of i. Thus $i^+ = i + 1$ for finite i.

(j) For the simplicity of the presentation, in the examples placed in Section 8 we will accept the Continuum Hypothesis and use some special notation for elements $P \in GP(CN)$. Namely

$$P = (v_0, v_1, ..., v_r, (v) \parallel w_1, w_2)$$

means that $P(i) = v_i$ for $i \in \text{betw}(0, r), P(i) = v$ for each finite $i > r$, $P(\aleph_0) = w_1$, $P(\diamondsuit) = w_2$, $P(i) = 0$ for $i > \diamondsuit$. For instance, if $P = (1, 1, 0.5, (0.3) \parallel 0.3, 0.1)$, then we have $P(0) = P(1) = 1$, $P(2) = 0.5$, $P(i) = 0.3$ for $i \in$ betw $(3, \aleph_0)$, $P(\diamondsuit) = 0.1$, and $P(i) = 0$ for $i > \diamondsuit$.

2. Towards generalized cardinal numbers

In the earlier many-valued theories of cardinality presented in [1], [5,6], [11] one assumes at the beginning that the notion of cardinality is unknown even for sets; gcn's are then constructed via many-valued bijections,i.e. via direct adaptation of the classical construction of cardinals. Unfortunately, such an approach is not successful and appears not very useful in practice because the obtained theories become essentially dependent on the chosen definitions of such the bijections and, on the other hand, respective calculations of powers are extremely difficult even in the case of small finite supports (see also [2], [7], [19,20] for a review of some other early approaches). In the theory proposed here we use quite different approximative approach in which we try to make a good use of the already existing ordinary cardinals and apply some axiomatic method that generates various types of gcn's. So, we have then the possibility to choose such a type which is most suitable in a concrete application inside or outside mathematics. Moreover, we assume that our information about any membership function can be imprecise or incomplete.

Let us consider the family composed of all the subsets of $\mathcal{U}$. We define classical cardinals in the ordinary way. So, for any $\mathcal{A} \subset \mathcal{U}$ we have card $\mathcal{A} = i$ iff $\exists \mathcal{B} \in P_i(U) : \mathcal{A} = \mathcal{B}$, i.e. the power of $\mathcal{A}$ equals i iff $\mathcal{A}$ belongs to respective family of equipotent sets. It is quite clear that for each fixed $\mathcal{A}$ the sentence $\exists \mathcal{B} \in P_i(\mathcal{U}) : \mathcal{A} = \mathcal{B}$ is true (in other words: has positive truth value) for exactly one cardinal number i. However, if we deal with HCH-objects, then in general the many-valued counterpart $\exists | \mathcal{B} \in P_i(\mathcal{U}) : \text{obj}(A) \approx \text{obj}(1_{\mathcal{B}})$ attains positive truth values for different i's. So, one can say that obj(A) belongs "to a degree" to many families of equipotent sets. Thus the power of obj(A) cannot be represented by one cardinal number but should be expressed by means of an HCH-object 'embedded' in CN and having the membership grades identical with respective truth values of the above given many-valued sentence. Then it is quite natural to consider as equipotent such HCH-objects obj(A), obj(B) in $\mathcal{U}$ which are related to identical HCH-objects in CN. Since our information about A can be imprecise or incomplete we additionally assume that $f(A) \subset A \subset g(A)$, where f and g are some approximating functions (see Section 3). Therefore we finally use the condition

$$\exists \mathbb{B} \in P_i(\mathcal{U}): \quad \mathrm{obj}(f(A)) \underset{\sim}{\subset} \mathrm{obj}(1_{\mathbb{B}}) \quad \&$$
$$\& \ \exists \mathbb{C} \in P_i(\mathcal{U}): \quad \mathrm{obj}(1_{\mathbb{C}}) \underset{\sim}{\subset} \mathrm{obj}(g(A))$$

which in some cases can be rewritten in a simpler from (see Remark 6.5). In the main, in this paper we focus our attention on such properties of gcn's which are independent on the choice of (f, g) and on the power of supp(obj(A)). Simple proofs are given in outline. Although $\Rightarrow$, & and | are basically understood as many-valued connectives, in the sentences or conditions containing exclusively the classical (two-valued) quantifiers, relations or predicates they will be throughout interpreted as respective classical connectives.

3. Approximation of the membership functions

Let A denote a membership function characterizing some HCH-object in $\mathcal{U}$. As we mentioned in previous section, we suppose that in general A can be given imprecisely or incompletely. So, we approximate A by means of two other functions $f(A)$ and $g(A)$, i.e. we approximate obj(A) by obj($f(A)$) and obj($g(A)$), where $f, g : GP(\mathcal{U}) \to GP(\mathcal{U})$. However, we assume that either at least one of the functions f, g is a function to $PS(\mathcal{U}) \subset GP(\mathcal{U})$ (i.e. at least one of the HCH-objects obj($f(A)$) and obj($g(A)$) is in a way simpler than obj(A)) or $f = g =$id with id denoting the identity function (i.e. our information about A is assumed to be perfect). Moreover we accept the following additional axioms about f and g:

(A1) $\forall A \in GP(\mathcal{U}) : f(A) \subset A \subset g(A)$,
(A2) $\forall A, B \in GP(\mathcal{U}) \forall x, y \in \mathcal{U} : A(x) \leq B(y) \Rightarrow f(A)(x) \leq f(B)(y) \&$
$\quad \& \ g(A)(x) \leq g(B)(y)$,

(A3) $\forall A \in PS(\mathcal{U}) : f(A), g(A) \in PS(\mathcal{U})$.

The family of all the pairs (f,g) of approximating functions fulfilling these postulates, but excluding the trivial (E,U), will be denoted by $\mathcal{F}$. As regards some interpretation of the axioms, (A1) means that $f(A)$ and $g(A)$ are always the lower and upper approximations of A,(A2) says that both $f(A)(x)$ and $g(A)(x)$ depend only on $A(x)$. Finally, (A3) is also quite natural and states that if obj(A) is a set, then both obj$(f(A))$ and obj$(g(A))$ are sets too. As consequences of (A1)−(A3) we get some simple but useful properties which are listed in the following theorem and corollaries.

Theorem 3.1. *For each* $(f,g) \in \mathcal{F}$ *and each* $A, B \in GP(\mathcal{U})$ *we have*

(A2)′ $A(x) = B(y)$ *implies* $f(A)(x) = f(B)(y)$ *and* $g(A)(x) = g(B)(y)$;

(A4) $f(A \overset{\cup}{\cap} B) = f(A) \overset{\cup}{\cap} f(B)$, $g(A \overset{\cup}{\cap} B) = g(A) \overset{\cup}{\cap} g(B)$;

(A5) $A \subset B$ *implies* $f(A) \subset f(B)$ *and* $g(A) \subset g(B)$;

(A6) $A(x) = 0$ *implies* $f(A)(x) = 0$ *and* $g(A)(x) \in \{0,1\}$,
$A(x) = 1$ *implies* $f(A)(x) \in \{0,1\}$ *and* $g(A)(x) = 1$;

(A6)′ $f \equiv E$ *or* $(f(A)(x) = 1$ *iff* $A(x) = 1), g \equiv U$ *or* $(g(A)(x) = 0$ *iff* $A(x) = 0)$;

(A7) *If* $A \in PS(\mathcal{U})$, *then* $f(A) = A$ *or* $f(A) = E$ *and* $g(A) = A$ *or* $g(A) = U$;

(A7)′ $f(E) = E$, $g(U) = U$, $f(U), g(E) \in \{E, U\}$.

Proof. We get (A2)′ using twice (A2). (A4) is a direct consequence of (A2), (A2)′ and the definition of $\cup$ and $\cap$. (A5) is implied again by (A2). (A6) follows from (A1), (A2)′, (A3) and implies (A6)′. Finally, (A7) follows from (A6), (A6)′ and implies (A7)′. This completes the proof. ◇

Corollary 3.2 *For each* $(f,g) \in \mathcal{F}$ *and* $A \in GP(\mathcal{U})$ *we have*

(a) *if* $f : GP(\mathcal{U}) \to PS(\mathcal{U})$, *then* $f \equiv E$ *or* $f(A) = 1_{A_1}$;

(b) *if* $g : GP(\mathcal{U}) \to PS(\mathcal{U})$, *then* $g \equiv U$ *or* $g(A) = 1_{\mathrm{supp}(A)}$.

Proof. Both (a) and (b) are immediate consequences of (A6)′. ◇

The corollary given above is very useful when one proves other theorems because it shows how look the possible pairs $(f,g) \in \mathcal{F}$.

Corollary 3.3. *For each* $(f,g) \in \mathcal{F}$ *and* $A \in GP(\mathcal{U})$ *we have*

(a) $f(A) \supset 1_{A_1}$ *or* $f \equiv E$.

(b) $g(A) \subset 1_{\mathrm{supp}(A)}$ *or* $g \equiv U$.

Proof. Again, it follows directly from (A6)′. ◇

4. Equipotent HCH-objects

Now we are ready to introduce the notion of equipotency for HCH-objects. Let $f(A)_t$, $g(A)_t$ denote the t-level sets of $f(A)$ and $g(A)$, respectively.

Definition 4.1. We write $A \sim_{f,g} B$ and say that two HCH-objects obj(A) and obj(B) in $\mathcal{U}$ are *equipotent* (in other words: are of the *same power*) with respect to a pair $(f,g) \in \mathcal{F}$ iff the conditions

$$\bigwedge\{t : \text{card } f(A)_t \leq i\} = \bigwedge\{t : \text{card } f(B)_t \leq i\},$$

$$\bigvee\{t : \text{card } g(A)_t \geq i\} = \bigvee\{t : \text{card } g(B)_t \geq i\}$$

are fulfilled by each cardinal number i.

If $(f,g) \in \mathcal{F}$ is fixed, one can write simply $A \sim B$. It is quite obvious but very important that $\sim_{f,g}$ is an equivalence relation for each $(f,g) \in \in \mathcal{F}$. Using very puristic notation we should rather write obj$(A) \sim_{f,g}$ obj(B) but the form $A \sim_{f,g} B$ does not lead to misunderstanding and is more convenient in use. Also, it is justified by the fact that operations or relations for HCH-objects are often defined by means of respective operations or relations over membership functions.

As concerns the condition defining the equipotency of HCH-objects, we at once see that it is a weakened form of the following (both versions are equivalent for HCH-objects with finite supports):

$$\forall t \in (0,1) : \text{ card } f(A)_t = \text{ card } f(B)_t \& \text{ card } g(A)_t = \text{ card } g(B)_t.$$

But any definition describing the equipotency via equalities of powers of some t-level sets is dangerous since it makes the equipotency to much dependent on a finite number of membership values even if we deal with HCH-objects with infinite supports. So, we refuse it. By using infima, suprema and inequalities, the proposed definition reflects instead two facts: first that $f(A)$ and $g(A)$ are lower and upper approximations of A, and second that using an approximative approach we should accept as equipotent not only such HCH-objects whose respective t-level sets are equipotent but also such ones which for each $t \in \mathcal{J}_o$ have 'the same amount' of elements with membership values equal to t or lying as near to t as one likes.

5. Relativity of the equipotency for HCH-objects

Let $f_i(A) := \bigvee\{t : \text{card } f(A)_t \geq i\}$, $g_i(A) := \bigvee\{t : \text{card } g(A)_t \geq \geq i\}$ and $a_i := \bigvee\{t : \text{ card } A_t \geq i\}$ for $A \in GP(\mathcal{U})$ and $(f, g) \in \mathcal{F}$ (in the same way one defines for instance numbers r_i for some $R \in GP(\mathcal{U})$). One can easily check that $f_i(A)$ is nonincreasing with respect to i and $f_i(A) = 0$ for $i >$ card supp(A); of course, analogous properties are satisfied by $g_i(A)$ and a_i. Also, one can easily prove that for each $A, B \in GP(\mathcal{U})$, $(f, g) \in \mathcal{F}$ and any cardinal number i we have

$$A \subset B \Rightarrow f_i(A) \leq f_i(B) \,\&\, g_i(A) \leq g_i(B)$$

and

$$f_i(A) \leq a_i \leq g_i(A).$$

So, $A \subset B$ implies $a_i \leq b_i$ for each i. Moreover the following properties will be useful: $f_i(A) = 1$ for $i \leq$ card $f(A)_1$, $f_1(A) = \bigvee\{f(A)(x) : x \in \mathcal{U}\}$, and $g_o(A) = 1$.

5.1. Useful characterizations of the equipotency

We notice that the equipotency condition can be rewritten as

$A \sim_{f,g} B$ iff $g_i(A) = g_i(B)$ & $f_{i^+}(A) = f_{i^+}(B)$ for each $i \in CN$.

So,

$$A \sim_{f,g} B \quad \text{iff} \quad g(A) \sim_{E,id} g(B) \;\;\&\;\; f(A) \sim_{id,U} f(B)\ .$$

Hence

$$A \sim_{id,id} B \quad \text{iff} \quad A \sim_{E,\text{id}} B \;\&\; A \sim_{id,U} B;$$

we even have

$$A \sim_{id,id} B \;\text{ iff }\; A \sim_{E,id} B \;\text{ iff }\; A \sim_{id,U} B$$

i.e. if (f, g) equals (id,id), (E,id) or (id,U), then $A \sim_{f,g} B$ iff $a_i = b_i$ for each $i \in CN$. Moreover, the following implications hold:

(a) if $f(\cdot) = 1_{(\cdot)_1}$ and $g =$ id, then $A \sim_{f,g} B$ implies card $A_1 =$ card B_1;

(b) if $f =$ id and $g(\cdot) = 1_{\text{supp}(\cdot)}$, then $A \sim_{f,g} B$ implies card supp$(A) =$ card supp(B) ;

(c) if $f \equiv E$ and $g(\cdot) = 1_{\text{supp}(\cdot)}$, then $A \sim_{f,g} B$ iff card supp$(A) =$ = card supp(B);

(d) if $f(\cdot) = 1_{(\cdot)_1}$ and $g \equiv U$, then $A \sim_{f,g} B$ iff card $A_1 =$ card B_1 ;

(e) if $f(\cdot) = 1_{(\cdot)_1}$ and $g(\cdot) = 1_{\text{supp}(\cdot)}$, then $A \sim_{f,g} B$ iff card $A_1 =$ $=$ card B_1 & card supp$(A) =$ card supp(B) .

So, the spectrum of possible conditions characterizing the equipotency of HCH-objects is rather wide; of course, these and other characterizations can be enhanced for HCH-objects with finite supports. The most interesting postulate is however that $a_i = b_i$ for each $i \in CN$ since we like to have the equipotency independent on an order of elements in HCH-objects . Really, let us notice that if A and B have finite supports, then this condition means that functions A and B attain the same values (with regard to their repetitions) but maybe in different points. This follows from the observation that if supp(D) is finite, then d_i is the i-th value in the sequence of positive membership grades $D(x)$ (including their repetitions) ordered in a nonincreasing way with $d_o := 1$ and $d_i := 0$ for $i >$ card supp(D). Finally, let us notice that if $(f, g) = (E, U)$ were an element of $\mathcal{F}$, then all the HCH-objects in $\mathcal{U}$ are equipotent with respect to such (f, g).

It is possible that obj(A) and obj(B) are equipotent with respect to some $(f, g) \in \mathcal{F}$ but simultaneously they are not equipotent with respect to some other $(f^*, g^*) \neq (f, g)$ (one can easily give respective examples for instance for $(f, g) = (1_{(\cdot)_1}, 1_{\text{supp}(\cdot)})$ and $(f^*, g^*) = (\text{id}, \text{id})$). This fact is however not so surprising because we deal here with HCH-objects whose nature is vague. Using two different pairs of approximating functions we apply in essence two different criteria to evaluate the powers of those objects . This is analogous to the situation well-known in our common life when two persons compare two things which are vague in a way and they get different results.

5.2. Some criteria of choice for the approximating functions

Since the family $\mathcal{F}$ is rather rich and, on the other hand, the equipotency or nonequipotency of two HCH-objects depends in general case on the choice of $(f, g) \in \mathcal{F}$, it is essential to ask how to choose (f, g) in 'proper' way; there is no problem with $A, B \in PS(\mathcal{U})$ because then either $A \sim_{f,g} B$ for each (f, g) or the HCH-objects are nonequipotent with respect to each (f, g) from $\mathcal{F}$. Obviously, total instructions are not possible. However, we like to present some approaches starting from different motivations.

APPROACH 1. We choose (f,g) taking into account how looks the condition characterizing $\sim_{f,g}$.

APPROACH 2. If $(f,g) \neq (\mathrm{id},\mathrm{id})$, then $f(A)$ and $g(A)$ can be interpreted as components of a twofold fuzzy set (see Section 7). So, our choice of (f,g) from among pairs differing from (id,id) depends on that what elements in $\mathcal{U}$ are considered to be sure and possible elements of an HCH-object .

APPROACH 3. We choose $f = g = \mathrm{id}$ if A is known exactly. Otherwise we suppose that $F \subset A \subset G$ and that only F and G are given. The choice of (f,g) depends then on the form of F and G. For instance, if we know only all the points x such that $A(x) > 0$ and $A(x) = 1$, we choose $f(\cdot) = 1_{(\cdot)_1}$ and $g(\cdot) = 1_{\mathrm{supp}(\cdot)}$. If the lower approximation of A is given only, one can take $f = \mathrm{id}$, $g \equiv U$ (or $g(\cdot) = 1_{\mathrm{supp}\ (\cdot)}$ provided that we know all the points x such that $A(x) > 0$).

APPROACH 4. Some properties and the form of generalized cardinal numbers are dependent on the used pair (f,g). So, one can choose (f,g) so as to get such gcn's that have the most convenient form and properties from the viewpoint of a concrete application.

6. The operator GCN and its basic properties

Now we are going to define an operator which will be used in Section 7 to generate the generalized cardinal numbers. That is why it is denoted by **GCN**. More precisely, let

GCN: $GP(\mathcal{U}) \times GP(\mathcal{U}) \rightarrow GP(CN)$ and let $\mathbf{GCN}(F,G)(i)$ be equal to

$$[\exists \mathcal{Y} \in P_i(\mathcal{U}) : \mathrm{obj}(F) \subset \mathrm{obj}(1_{\mathcal{Y}})] \wedge [\exists \mathcal{Z} \in P_i(\mathcal{U}) : \mathrm{obj}(1_{\mathcal{Z}}) \underset{\sim}{\subset} \mathrm{obj}(G)]$$

provided that $F \subset G$. So, we have

Theorem 6.1. *For each $F, G \in GP(\mathcal{U})$ such that $F \subset G$ and each $i \in CN$*

$$\mathbf{GCN}(F,G)(i) = \bigvee_{\mathcal{Y} \in P_i(\mathcal{U})} \bigwedge_{x \in \mathcal{Y}} G(x) \ \wedge \bigvee_{\mathcal{Y} \in P_i(\mathcal{U})} \bigwedge_{x \notin \mathcal{Y}} 1 - F(x).$$

Proof. This equality is obvious because for each $i \in CN$ and $\mathcal{Y} \in P_i(\mathcal{U})$ we get

$$[\mathrm{obj}(F) \underset{\sim}{\subset} \mathrm{obj}(1_{\mathcal{Y}})] = \bigwedge_{x \in \mathcal{U}} F(x) \longrightarrow 1_{\mathcal{Y}}(x) = \bigwedge_{x \notin \mathcal{Y}} 1 - F(x)$$

and

$$[\text{obj}(1_{\mathcal{Y}}) \underset{\sim}{\subset} \text{obj}(G)] = \bigwedge_{x \in \mathcal{U}} 1_{\mathcal{Y}}(x) \longrightarrow G(x) = \bigwedge_{x \in \mathcal{Y}} G(x).$$

Remark 6.2. One can easily notice that using (instead of the Lukasiewicz implication operator) a φ-operator induced by a triangular norm or putting $\mathcal{L} :=$ complete Heyting algebra we obtain the formula

$$\mathbf{GCN}(F,G)(i) = \bigvee_{\mathcal{Y} \in P_i(\mathcal{U})} \bigwedge_{x \in \mathcal{Y}} G(x) \wedge \bigvee_{\mathcal{Y} \in P_i(\mathcal{U})} \bigwedge_{x \notin \mathcal{Y}} F(x) \longrightarrow 0.$$

We see that $i > \text{card}(\mathcal{U})$ implies $P_i(\mathcal{U}) = \emptyset$ and then $\mathbf{GCN}(F,G)(i) = = 0$. This is why we always restrict ourselves to cardinals belonging to CN. Moreover, it is quite clear that the following simplification is possible:

$$\mathbf{GCN}(F,G)(i) = \bigvee_{\mathcal{Y} \in P_i(\text{supp }(G))} \bigwedge_{x \in \mathcal{Y}} G(x) \wedge \bigvee_{\{\mathcal{Y} \in P_i(\mathcal{U}) : F_1 \subset \mathcal{Y}\}} \bigwedge_{x \notin \mathcal{Y}} 1 - F(x).$$

Hence $\mathbf{GCN}(F,G)(i) = 0$ for each $i \notin \text{betw}(\text{card} F_1, \text{card supp }(G))$. Finally if $i \geq \text{card supp }(\mathcal{F})$, there exists $\mathcal{Y} \in P_i(\mathcal{U})$ such that $F_1 \subset \subset \text{supp }(F) \subset \mathcal{Y}$. But then we get $\bigwedge\{1 - F(x) : x \notin \mathcal{Y}\} = 1$. So, for each $i \geq \text{card supp }(F)$ we obtain

$$\mathbf{GCN}(F,G)(i) = \bigvee_{\mathcal{Y} \in P_i(\text{supp }(G))} \bigwedge_{x \in \mathcal{Y}} G(x).$$

As a next corollary from Theorem 6.1. we have

Theorem 6.3. *$D \subset F \subset G \subset H$ implies* $\mathbf{GCN}(F,G) \subset \mathbf{GCN}(D,H)$.

Proof. $D \subset F \subset G \subset H$ implies $P_i(\text{supp }(G)) \subset P_i(\text{supp }(H))$ and $\{\mathcal{Y} \in P_i(\mathcal{U}) : F_i \subset \mathcal{Y}\} \subset \{\mathcal{Y} \in P_i(\mathcal{U}) : D_1 \subset \mathcal{Y}\}$. Using Th. 6.1 and the previous corollaries following therefrom, we at once obtain the final thesis. $\diamondsuit$

Corollary 6.4. $\mathbf{GCN}(A,A) \subset \mathbf{GCN}(f(A), g(A))$ *for each $A \in GP(\mathcal{U})$ and $(f,g) \in \mathcal{F}$.*

Proof. It follows directly from (A1) and Th. 6.3. $\diamondsuit$

So, if $A \in GP(\mathcal{U})$ is fixed and we consider $\mathbf{GCN}(f(A), g(A))$ with different pairs $(f,g) \in \mathcal{F}$, then the least possible energy measure (see e.g. [8], [12]) occurs when $f = g = \text{id}$. In other words, the least deviation of $\mathbf{GCN}(f(A), g(A))$ from a function of the form $1_{\{i\}}$ for some $i \in CN$, i.e. the least deviation from a membership function related to a classical cardinal number, is attained for $f = g = \text{id}$.

From now on, we shall always use $F = f(X)$ and $G = g(X)$ as arguments of the operator **GCN**, where f and g are some approximating functions defined in Section 3 and $X \in GP(\mathcal{U})$.

Remark 6.5. One can show that for each $A \in GP(\mathcal{U})$ if $(f,g) \neq \neq (\mathrm{id},\mathrm{id})$, then

$$\mathbf{GCN}(f(A),g(A))(i) = [\exists|\mathcal{Y} \in P_i(\mathcal{U}) : \mathrm{obj}(f(A)) \underset{\sim}{\subset} \mathrm{obj}(1_{\mathcal{Y}}) \underset{\sim}{\subset} \underset{\sim}{\subset} \mathrm{obj}(g(A))] = \bigvee_{\mathcal{Y}\in P_i(\mathcal{U})} (\bigwedge_{x\in\mathcal{Y}} g(A)(x) \wedge \bigwedge_{x\notin\mathcal{Y}} 1 - f(A)(x))\cdot$$

The same holds if $(f,g) \in \mathcal{F}$ is quite arbitrary and supp (A) is finite.

Now we like to express $\mathbf{GCN}(f(A),g(A))(i)$ in a form more simple and convenient than that following from Th. 6.1.

Theorem 6.6. *For each* $(f,g) \in \mathcal{F}$, $A \in GP(\mathcal{U})$ *and* $i \in CN$ *we have*

$$\mathbf{GCN}(f(A),g(A))(i) = g_i(A) \wedge 1 - f_{i^+}(A).$$

Proof. Let $L_{g(A),i} := \bigvee_{\mathcal{Y}\in P_i(\mathrm{supp}\ (g(A)))} \bigwedge_{x\in\mathcal{Y}} g(A)(x)$. We shall prove that $L_{g(A),i} = g_i(A)$. Let us fix $i \in CN$ and suppose that $L_{g(A),i} < g_i(A)$. Then there exists t^* such that card$g(A)_{t^*} \geq i$ and $L_{g(A),i} < t^*$. But one can choose $\mathcal{Y}^* \in P_i(\mathrm{supp}\ (g(A)))$ such that $\mathcal{Y}^* \subset g(A)_{t^*}$. Hence $\bigwedge\{g(A)(x) : x \in \mathcal{Y}^*\} \geq t^*$ what leads to a contradiction.

Now, suppose that $L_{g(A),i} > g_i(A)$. Then, again, there exists $\mathcal{Y}^*$ such that card $\mathcal{Y}^* = i$ and $g_i(A) < \bigwedge\{g(A)(x) : x \in \mathcal{Y}^*\}$. Let $t^* := g_i(A)$. If card $g(A)_t \geq i$ for each t, then $t^* = 1$ and the previous inequality cannot be true. So, we can assume that there exists t such that card $g(A)_t < i$. Moreover, card $g(A)_{t_*} < i$ for each $t_* > t^*$. But $g(A)(x) > t^*$ for each $x \in \mathcal{Y}^*$. Hence $g(A)(x) \geq t_* > t^*$ for each $x \in \mathcal{Y}^*$ and some $t_* > t^*$, i.e. $\mathcal{Y}^* \subset g(A)_{t_*}$ what implies that card $\mathcal{Y}^* \leq$ card $g(A)_{t_*} < i$ and gives this way a contradiction. So, $L_{g(A),i} = g_i(A)$. The equality $\bigvee_{\{\mathcal{Y}\in P_i(\mathcal{U}):f(A)_1\subset\mathcal{Y}\}} \bigwedge_{x\notin\mathcal{Y}} 1 - f(A)(x) = = 1 - f_{i^+}(A)$ can be proved in an analogous way. This completes the proof. ◇

Remark 6.7. In the proof of Th. 6.6. we obtained two important equalities which imply that

$$[\exists|\mathcal{Y} \in P_i(\mathcal{U}) : \mathrm{obj}(1_{\mathcal{Y}}) \underset{\sim}{\subset} \mathrm{obj}(g(A))] = \bigvee\{t : \mathrm{card}\ g(A)_t \geq i\}$$

and

$$[\exists|\mathcal{Y} \in P_i(\mathcal{U}) : \mathrm{obj}(f(A)) \underset{\sim}{\subset} \mathrm{obj}(1_{\mathcal{Y}})] = 1 - \bigwedge\{t : \mathrm{card}\ f(A)_t \leq i\}.$$

Thus, again, the equipotency condition could be rewritten in another equivalent form. Using it one can formulate a generalized (i.e. many-valued) version of the equipotency definition for HCH-objects and introduce this way a notion of HCH-objects equipotent "to a degree $a \in \mathcal{J}$". However, we shall use here only the sharp two-valued Def. 4.1 which is quite sufficient if one likes to construct an applicable and useful theory. On the other hand, this definition accepts also some vagueness and subjectivity of the equipotency by the presence of the approximating functions which after all can be chosen from $\mathcal{F}$ quite arbitrary (cf. Section 5.2).

Applying Cor. 3.2 and Th. 6.6. one can express the membership values to obj($\mathbf{GCN}(f(A), g(A))$ in more explicit way. It suffices to consider the following variants of pairs $(f,g) \in \mathcal{F} : f = g = \text{id}$, g is arbitrary and $f \equiv E$ or $f(\cdot) = 1_{(\cdot)_1}$, f is arbitrary and $g \equiv U$ or $g(\cdot) = 1_{\text{supp }(\cdot)}$. We easily notice that

$$\mathbf{GCN}(f(A), g(A))(i) = \begin{cases} 1 - f_{i^+}(A) & \textit{if } i < z_{A,f,g} , \\ g_i(A) & \textit{otherwise}, \end{cases}$$

where $z_{A,f,g} := \bigwedge\{i \in CN : g_i(A) + f_{i^+}(A) \leq 1\}$.

Theorem 6.8. *For each $A \in GP(\mathcal{U})$ the following properties are fulfilled:*

(a) $$\mathbf{GCN}(E, g(A))(i) = g_i(A) \quad \textit{for each } i \in CN.$$

(b) $$\mathbf{GCN}(1_{A_1}, g(A))(i) = \begin{cases} 0 & \textit{if } i < \text{card } A_1, \\ 1 & \textit{if } i = \text{card } A_1, \\ g_i(A) & \textit{otherwise}. \end{cases}$$

(c) $$\mathbf{GCN}(f(A), U)(i) = 1 - f_{i^+}(A) \textit{ for each } i \in CN.$$

(d) $$\mathbf{GCN}(f(A), 1_{\text{supp }(A)})(i) = \begin{cases} 1 - f_{i^+}(A) & \textit{if } i < \text{card supp }(A), \\ 1 & \textit{if } i = \text{card supp }(A), \\ 0 & \textit{otherwise}. \end{cases}$$

(e) $\mathbf{GCN}(A, A)(i) = a_i \wedge 1 - a_{i^+}$ *for each $i \in CN$ with a_i defined in Section 5.*

Proof. It is an immediate consequence of Th. 6.6 and definitions of $f_i(A), g_i(A)$ and a_i. ◊

Now we are going to present very specific property of the operator **GCN** which holds exclusively for $f = g = \text{id}$ (cf. Cor. 6.4). One can check that for each $A \in GP(\mathcal{U})$ there exists such i that $\mathbf{GCN}(A, A)(i) \geq$

≥ 0.5. It follows from Th. 6.8 that this holds for any other pair (f,g) from $\mathcal{F}$, too. However

Theorem 6.9. *For each* $A \in GP(\mathcal{U})$ *there exists at most one cardinal number* i *such that* $\mathbf{GCN}(A,A)(i) > 0.5$.

Proof. It suffices to observe that $\mathbf{GCN}(A,A)(i) > 0.5$ only if $t = 0.5$ is an internal point of $\{t : \text{card } A_t = i\}$. Such the cardinal number is unique if exists. ◇

Using Th. 6.8 we notice that the property described in Th. 6.9 does not hold for pairs $(f,g) \neq (\text{id},\text{id})$. Finally, we like to formulate some decomposition theorem which will be useful in proving other facts.

Theorem 6.10. *For each* $(f,g) \in \mathcal{F}$ *and* $A \in GP(\mathcal{U})$ *we have*

$$\mathbf{GCN}(f(A),g(A)) = \mathbf{GCN}(E,g(A)) \cap \mathbf{GCN}(f(A),U).$$

Proof. This is quite clear since from Th. 6.8 and Th. 6.6 it follows that for each $i \in CN$ we get $\mathbf{GCN}(f(A),g(A))(i) = g_i(A) \wedge \wedge\, 1 - f_{i^+}(A)$, $\mathbf{GCN}(E,g(A))(i) = g_i(A)$, and $\mathbf{GCN}(f(A),U)(i) = = 1 - f_{i^+}(A)$. ◇

So, we have for instance

Corollary 6.11. $\mathbf{GCN}(A,A) = \mathbf{GCN}(E,A) \cap \mathbf{GCN}(A,U)$, $\mathbf{GCN}(1_{A_1},A) = \mathbf{GCN}(E,A) \cap \mathbf{GCN}(1_{A_1},U)$, $\mathbf{GCN}(1_{A_1},1_{\text{supp }(A)}) = = \mathbf{GCN}(E,1_{\text{supp }(A)}) \cap \mathbf{GCN}(1_{A_1},U)$, and $\mathbf{GCN}(A,1_{\text{supp }(A)}) = = \mathbf{GCN}(E,1_{\text{supp }(A)}) \cap \mathbf{GCN}(A,U)$.

7. The generalized cardinal numbers

First of all, we like to formulate a property which is a key-stone of the presented theory, namely

Theorem 7.1. *For each* $(f,g) \in \mathcal{F}$ *and* $A, B \in GP(\mathcal{U})$ *the following equivalence holds*

$$\mathbf{GCN}(f(A),g(A)) = \mathbf{GCN}(f(B),g(B)) \textit{ iff } A \sim_{f,g} B.$$

Proof. Let us fix some arbitrary (f,g) from $\mathcal{F}$ and A, B from $GP(\mathcal{U})$. It is quite obvious that $A \sim_{f,g} B$ implies $\mathbf{GCN}(f(A),g(A)) = \mathbf{GCN}$ $(f(B),g(B))$. So, assume $\mathbf{GCN}(f(A),g(A)) = \mathbf{GCN}(f(B),g(B))$. If $f \equiv E$, then from Th. 6.8 we obtain $g_i(A) = g_i(B)$ for each $i \in CN$. Obviously, $f_{i^+}(A) = f_{i^+}(B) = 0$ for all i from CN. Thus $A \sim_{f,g} B$. If $f(\cdot) = 1_{(\cdot)_1}$, then from Th. 6.8 we get again $g_i(A) = g_i(B)$ for $i >$

card A_1 = card B_1. From the definitions of $g_i(D)$ and $f_i(D)$ it follows however that $g_i(A) = g_i(B) = 1$ for $i \leq$ card A_1. On the other hand, $f_{i^+}(D) = 1$ if $i <$ card D_1 else $f_{i^+}(D) = 0$. So, $f_{i^+}(A) = f_{i^+}(B)$ for each $i \in CN$. Hence $A \sim_{f,g} B$. Our thesis for $g \equiv U$ and $g(\cdot) = 1_{\text{supp }(\cdot)}$ can be proved in quite similar way. Finally, if $f = g =$ id, it suffices to show that $a_i = b_i$ for each $i \in CN$ what is however again a simple exercise and therefore omitted. $\diamondsuit$

Thus the values of the operator **GCN** fulfill the axiomatic definition of cardinal numbers proposed by A.Tarski ([17], see also [14]). These values will be just called generalized cardinal numbers (gcn's) and denoted by small Greek letters equipped with indexing pair f, g emphasizing which approximating functions have been used. If $\mathbf{GCN}(f(A), g(A)) = \alpha_{f,g} \in GP(CN)$, then we shall write $\text{Gcard}_{f,g}(A) = \alpha_{f,g}$ and say that the power of obj(A) equals $\alpha_{f,g}$ with respect to $(f, g) \in \mathcal{F}$. Obviously, then $\text{Gcard}_{f,g}(A)(i) = \alpha_{f,g}(i) = g_i(A) \wedge 1-$ $-f_{i^+}(A)$.

Let us observe that the Tarski's definition gives us in essence two equivalent possibilities: the first one has been already described, the second and in fact more proper variant is instead to consider the HCH-object obj$(\mathbf{GCN}(f(A), g(A)))$ in CN as a gcn, i.e. as a tool describing the power of obj(A). Then we should rather write $\text{Gcard}_{f,g}(A) = \text{obj}(\alpha_{f,g})$; moreover, this would be in a way a generalization of the idea of S.Gottwald from [7] who proposed to express the power of a fuzzy set by means of a set composed of some cardinal numbers. However, obj(A) = obj(B) iff $A = B$. This in fact gives us free hand to choose any of those two variants. We have chosen the first one which is more convenient from the practical viewpoint. On the other hand, operations and relations on HCH-objects resolve themselves anyway to operations and relations on respective generalized characteristic functions .

It follows from Th. 7.1 that $\text{Gcard}_{f,g}(A) = \text{Gcard}_{f,g}(B)$ iff $A \sim_{f,g} B$. Moreover the following equivalence is quite obvious

$$\alpha_{f,g} = \beta_{f,g} \text{ iff } \exists A, B \in GP(\mathcal{U}) : \text{Gcard}_{f,g}(A) = \alpha_{f,g} \ \&$$
$$\& \ \text{Gcard}_{f,g}(B) = \beta_{f,g} \ \& \ A \sim_{f,g} B.$$

So, the equality of gcn's can be defined quite naturally by

$$\alpha_{f,g} = \beta_{f,g} \text{ iff } \alpha_{f,g}(i) = \beta_{f,g}(i) \text{ for each } i \in CN.$$

If the pair $(f, g) \in \mathcal{F}$ is fixed, we write simply Gcard $(A) = \alpha$. From Th. 6.10 we get at once

$$\text{Gcard}_{f,g}(A) = \text{Gcard}_{E,g}(A) \cap \text{Gcard}_{f,U}(A)$$

and

$$\text{Gcard}_{f,g}(A) = \text{Gcard}_{E,id}(g(A)) \cap \text{Gcard}_{id,U}(f(A)).$$

Hence for instance

$$\text{Gcard}_{\text{id,id}}(A) = \text{Gcard}_{\text{E,id}}(A) \cap \text{Gcard}_{\text{id,U}}(A),$$
$$\text{Gcard}_{1_{(\cdot)_1},\text{id}}(A) = \text{Gcard}_{\text{E,id}}(A) \cap \text{Gcard}_{\text{id},U}(1_{A_1}),$$
$$\text{Gcard}_{1_{(\cdot)_1},1_{\text{supp }(\cdot)}}(A) = \text{Gcard}_{E,\text{id}}(1_{\text{supp (A)}}) \cap \text{Gcard}_{\text{id},U}(1_{A_1}),$$
$$\text{Gcard}_{\text{id},1_{\text{supp }(\cdot)}}(A) = \text{Gcard}_{E,\text{id}}(1_{\text{supp }(A)}) \cap \text{Gcard}_{\text{id},U}(A),$$
$$\text{Gcard}_{E,1_{\text{supp }(\cdot)}}(A) = \text{Gcard}_{E,id}(1_{\text{supp }(A)}) \cap \text{Gcard}_{\text{id},U}(E).$$

Obviously, using Th. 6.8 one can automatically express $\text{Gcard}_{f,g}(A)(i)$ for the five basic groups of pairs $(f, g) \in \mathcal{F}$. Let

$$\mathbf{GCN}_{f,g} := \{\alpha \in GP(CN) : \text{Gcard}_{f,g}(D) = \alpha \text{ for some } D \in GP(\mathcal{U})\}.$$

Theorem 7.2. (a) *For each* $(f, g) \in \mathcal{F}$ *all the elements* $\alpha \in \mathbf{GCN}_{f,g}$ are convex, i.e. $\alpha(j) \geq \alpha(i) \wedge \alpha(k)$ *for* $i \leq j \leq k$.
(b) *If* $f \equiv E(f(\cdot) = 1_{(\cdot)_1}$, *resp.*), *then each* $\alpha \in \mathbf{GCN}_{f,g}$ *is antitonic (is antitonic on its support, resp.).For* $g \equiv U(g(\cdot) = 1_{supp\ (\cdot)}$, *resp.*) *each* $\alpha \in \mathbf{GCN}_{f,g}$ *is isotonic (is isotonic on its support, resp.)* .
(c) *If* $(f, g) \neq (\text{id}, \text{id})$, *then each element* $\alpha \in \mathbf{GCN}_{f,g}$ *is normal, i.e. there exists* $i \in CN$ *such that* $\alpha(i) = 1$.
Proof. All the results are simple corollaries of Th. 6.8. ◇

It is quite obvious that $\alpha \in \mathbf{GCN}_{f,g}$ is in general case nonmonotonic for $f = g = \text{id}$ (see however the formula preceding Th. 6.8). Also, it is not normal in general but, on the other hand, fulfills an interesting property described by Th. 6.9. Cleary, if α is normal for $f = g = \text{id}$, then its support has exactly one element.

If $(f, g) \neq (\text{id}, \text{id})$, then obviously $f(D) \subset 1_{g(D)_1}$ for each $D \in GP(\mathcal{U})$. So, in that case $\text{Gcard}_{f,g}(D)$ is simultaneously equal to the power of the twofold fuzzy set $\Omega = (f(D), g(D))$ (see [3]). Thus gcn's constructed by means of pairs $(f, g) \neq (\text{id}, \text{id})$ refer not only to fuzzy sets but also to twofold fuzzy sets (see Section 8.1).

8. Examples and comments

8.1. Let $n :=$ card supp (A), $m :=$ card A_1 and $\mathrm{Gcard}_{f,g}(A) = \alpha$ for some fixed $(f,g) \in \mathcal{F}$. As previously, $a_i := \bigvee\{t : \text{card } A_t \geq i\}$. Using Cor. 3.2 and Th. 6.8 one can present how looks the gcn representing the power of obj(A) with respect to eight main pairs of approximating functions . Then we obtain the following formulae:
(Pair#1: $f = g = \mathrm{id}$)

$$\alpha(i) = a_i \wedge 1 - a_{i+} = \begin{cases} 1 - a_{i+} & \text{if } i < z_{A,f,g}, \\ a_i & \text{otherwise} \end{cases}.$$

(Pair#2: $f \equiv E,\ g = \mathrm{id}$)

$$\alpha(i) = a_i, \text{ where } a_0 = 1 \text{ and } a_i = 0 \text{ for } i > n.$$

(Pair#3: $f(\cdot) = 1_{(.)_1},\ g = \mathrm{id}$)

$$\alpha(i) = \begin{cases} 1 & \text{if } i = m, \\ a_i & \text{if } m < i \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

(Pair#4: $f = \mathrm{id},\ g(\cdot) = 1_{\mathrm{supp}\,(\cdot)}$)

$$\alpha(i) = \begin{cases} 1 - a_{i+} & \text{if } m \leq i < n, \\ 1 & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

(Pair#5: $f = \mathrm{id},\ g \equiv U$)

$$\alpha(i) = \begin{cases} 0 & \text{if } i < m, \\ 1 - a_{i+} & \text{if } m \leq i < n, \\ 1 & \text{otherwise.} \end{cases}$$

(Pair#6: $f \equiv E,\ g(\cdot) = 1_{\mathrm{supp}\,(\cdot)}$)

$$\alpha(i) = \begin{cases} 1 & \text{if } i \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

(Pair#7: $f(\cdot) = 1_{(\cdot)_1},\ g(\cdot) = 1_{\mathrm{supp}\,(.)}$)

$$\alpha(i) = \begin{cases} 1 & \text{if } i \in \text{ betw } (m,n), \\ 0 & \text{otherwise.} \end{cases}$$

(Pair#8: $f(\cdot) = 1_{(\cdot)_1},\ g \equiv U$)

$$\alpha(i) = \begin{cases} 0 & \text{if } i < m, \\ 1 & \text{otherwise.} \end{cases}$$

From Th. 6.3 we get at once some inclusions, for instance

$$\alpha_{\#1} \subset \alpha_{\#3} \subset \alpha_{\#2} \subset \alpha_{\#6},$$

$$\alpha_{\#1} \subset \alpha_{\#4} \subset \alpha_{\#5} \subset \alpha_{\#8};$$

$$\alpha_{\#3} \subset \alpha_{\#7} \subset \alpha_{\#8}, \alpha_{\#4} \subset \alpha_{\#7} \subset \alpha_{\#6}.$$

One can easily formulate different simple conditions for having $\alpha_{\#i} = = \alpha_{\#j}$. Let us notice that if the pair (E, U) were an element of $\mathcal{F}$, then $\alpha(i) = 1$ for each $i \in CN$.

Using Pair#2 we get gcn's defined for fuzzy sets by L.A.Zadeh ([28]; cf.[15] and see also [20] for a review of early approaches). Pair#3 generates instead gcn's of the type introduced by D.Dubois and H.Prade ([2]) also for fuzzy sets. Finally, $(f, g) = (\mathrm{id}, \mathrm{id})$ gives gcn's defined for fuzzy sets by the author in [20]. Pair#7 generates gcn's identical to the partial cardinal numbers of D.Klaua ([10]) and seems to be suitable (like Pair#6 and #8) for rough sets (see [13],[23]). So, the presented theory brings together a lot of early approaches to gcn's although they have been started from different motivations and have been proposed for different kinds of HCH-objects such as fuzzy sets, twofold fuzzy sets, partial sets and rough sets.

8.2. Let $B \in PS(\mathcal{U})$, $q :=$ card supp (B) and $\mathrm{Gcard}_{f,g}(B) = \beta_{f,g}$ for $(f, g) \in \mathcal{F}$. Then

$$\beta_{f,g} = \begin{cases} 1_{\{q\}} & \text{if } f \not\equiv E \text{ and } g \not\equiv U, \\ 1_{\{i \in CN: i \leq q\}} & \text{if } f \equiv E, \\ 1_{\{i \in CN: i \geq q\}} & \text{if } g \equiv U. \end{cases}$$

Hence $\mathrm{Gcard}_{f,g}(B) = 1_{\mathrm{betw}(\mathrm{card}\ f(B)_1, \mathrm{card}\ g(B)_1)}$ for each $(f, g) \in \mathcal{F}$. These results suggest some interpretation of the values $\mathrm{Gcard}_{f,g}(A)(i)$ for $A \in GP(\mathcal{U})$, namely: if respectively $f \equiv E, g \equiv U, f \not\equiv E$ and $g \not\equiv U$, then one can consider $\mathrm{Gcard}_{f,g}(A)(i)$ to be the degree to which obj(A) has at least, at most, exactly i elements, respectively. As a second corollary we obtain the following formulae ($k :=$ card $\mathcal{U}$):

$$\mathrm{Gcard}_{f,g}(E) = \begin{cases} 1_{\{0\}} \\ 1_{\{0\}} \\ 1_{CN} \end{cases} \text{ and } \mathrm{Gcard}_{f,g}(U) = \begin{cases} 1_{\{k\}} & \text{if } f \not\equiv E \text{ and } g \not\equiv U, \\ 1_{CN} & \text{if } f \equiv E, \\ 1_{\{k\}} & \text{if } g \equiv U. \end{cases}$$

8.3. Let $\mathcal{U} = \mathbb{R}$, i.e. $CN = \{i : i \leq \diamondsuit\}$. Moreover, let

$$A(x) = \begin{cases} 1-x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise,} \end{cases} \qquad B(x) = 0.8A(x),$$

$$C(x) = \begin{cases} 1-1/x & \text{if } x = 2,3,4,\dots, \\ 0 & \text{otherwise,} \end{cases}$$

$$D(x) = \begin{cases} 1 & \text{if } x = 0,1 \\ 0.9 & \text{if } x = 2 \\ 0.7 & \text{if } x = 3 \\ 0.3 & \text{if } 4 \leq x \leq 5 \\ 0 & \text{otherwise,} \end{cases} \qquad S(x) = \begin{cases} 1 & \text{if } x = 2,3,4 \\ 0.2 & \text{if } x = 5 \\ 0.3 & \text{if } x = 6 \\ 0.9 & \text{if } x = 7 \\ 0.6 & \text{if } x = 8 \\ 0 & \text{otherwise.} \end{cases}$$

Further, let $\mathrm{Gcard}_{f,g}(A) = \alpha_{f,g}$, $\mathrm{Gcard}_{f,g}(B) = \beta_{f,g}$, $\mathrm{Gcard}_{f,g}(C) = \gamma_{f,g}$, $\mathrm{Gcard}_{f,g}(D) = \delta_{f,g}$, and $\mathrm{Gcard}_{f,g}(S) = \sigma_{f,g}$. So, we have $a_i = 1$ for each $i \in CN$, $b_i = 1$ for $i = 0$ and $b_i = 0.8$ if $i \in$ betw $(1, \diamondsuit)$, $c_i = 1$ if $i \leq \aleph_0$ and $c_i = 0$ for $i = \diamondsuit$, and

$$d_i = \begin{cases} 1 & \text{if } i = 0,1,2 \\ 0.9 & \text{if } i = 3 \\ 0.7 & \text{if } i = 4 \\ 0.3 & \text{if } i \in \text{ betw } (5,\diamondsuit), \end{cases} \qquad s_i = \begin{cases} 1 & \text{if } i = 0,1,2,3 \\ 0.9 & \text{if } i = 4 \\ 0.6 & \text{if } i = 5 \\ 0.3 & \text{if } i = 6 \\ 0.2 & \text{if } i = 7 \\ 0 & \text{if } i \in \text{ betw } (8,\diamondsuit). \end{cases}$$

Then using the notational rule (j) from Section 1 we get

$\alpha_{\#1} = ((0) \parallel 0,1)$, $\beta_{\#1} = ((0.2) \parallel 0.2, 0.8)$, $\gamma_{\#1} = ((0) \parallel 1,0)$,
$\delta_{\#1} = (0,0,0.1,0.3,0.7,(0.3) \parallel 0.3,0.3)$,
$\sigma_{\#1} = (0,0,0,0.1,0.4,0.6,0.3,0.2,(0) \parallel 0,0)$,

$\alpha_{\#2} = ((1) \parallel 1,1)$, $\beta_{\#2} = (1,(0.8) \parallel 0.8,0.8)$, $\gamma_{\#2} = ((1) \parallel 1,0)$,
$\delta_{\#2} = (1,1,1,0.9,0.7,(0.3) \parallel 0.3,0.3)$,
$\sigma_{\#2} = (1,1,1,1,0.9,0.6,0.3,0.2,(0) \parallel 0,0)$,

$\alpha_{\#3} = (0,(1) \parallel 1,1)$, $\beta_{\#3} = (1,(0.8) \parallel 0.8,0.8)$, $\gamma_{\#3} = ((1) \parallel 1,0)$,
$\delta_{\#3} = (0,0,1,0.9,0.7,(0.3) \parallel 0.3,0.3)$,
$\sigma_{\#3} = (0,0,0,1,0.9,0.6,0.3,0.2,(0) \parallel 0,0)$,

$\alpha_{\#4} = ((0) \parallel 0,1)$, $\beta_{\#4} = ((0.2) \parallel 0.2,1)$, $\gamma_{\#4} = ((0) \parallel 1,0)$,
$\delta_{\#4} = (0,0,0.1,0.3,(0.7) \parallel 0.7,1)$,
$\sigma_{\#4} = (0,0,0,0.1,0.4,0.7,0.8,1,(0) \parallel 0,0)$,

$\alpha_{\#5} = ((0) \parallel 0,1)$, $\beta_{\#5} = ((0.2) \parallel 0.2,1)$, $\gamma_{\#5} = ((0) \parallel 1,1)$,
$\delta_{\#5} = (0,0,0.1,0.3,(0.7) \parallel 0.7,1)$,
$\sigma_{\#5} = (0,0,0,0.1,0.4,0.7,0.8,(1) \parallel 1,1)$,

$\alpha_{\#6} = ((1) \parallel 1,1)$, $\beta_{\#6} = ((1) \parallel 1,1)$, $\gamma_{\#6} = ((1) \parallel 1,0)$,
$\delta_{\#6} = ((1) \parallel 1,1), \sigma_{\#6} = (1,1,1,1,1,1,1,1,(0) \parallel 0,0)$,

$\alpha_{\#7} = (0,(1) \parallel 1,1)$, $\beta_{\#7} = ((1) \parallel 1,1)$, $\gamma_{\#7} = ((1) \parallel 1,0)$,
$\delta_{\#7} = (0,0,(1) \parallel 1,1)$, $\sigma_{\#7} = (0,0,0,1,1,1,1,1,(0) \parallel 0,0)$,

$\alpha_{\#8} = (0,(1) \parallel 1,1)$, $\beta_{\#8} = ((1) \parallel 1,1)$, $\gamma_{\#8} = ((1) \parallel 1,1)$,
$\delta_{\#8} = (0,0,(1) \parallel 1,1)$, $\sigma_{\#8} = (0,0,0,(1) \parallel 1,1)$.)

Worth noticing is that for instance $\alpha_{\#1} \in PS(CN)$ although $A \notin$ $\notin PS(\mathcal{U})$. This is because A has continuum of values which lie as near to 1 as one likes.

9. Further properties of the generalized cardinal numbers

One of the most fundamental requirements concerning gcn's is the coincidence with cardinal numbers if we deal with HCH-objects being sets. This is fulfilled.

Theorem 9.1. *For each* $(f,g) \in \mathcal{F}$ *there exists a bijection* $\Psi_{f,g}$: $CN \to PS(CN)$ *such that* $\mathrm{Gcard}_{f,g}(1_{\mathcal{D}}) = \Psi_{f,g}(q)$ $\quad (q := \mathrm{card}\ \mathcal{D})$, *i.e. respective diagram is commutative.*

Proof. It suffices to use Ex. 8.2 and to define

$$\Psi_{f,g}(i) = \begin{cases} 1_{\{i\}} & \text{if } f \not\equiv E \text{ and } g \not\equiv U, \\ 1_{\{j \in CN:\ j \leq i\}} & \text{if } f \equiv E, \\ 1_{\{j \in CN:\ j \geq i\}} & \text{if } g \equiv U. \ \diamondsuit \end{cases}$$

Corollary 9.2. An immediate consequence of Ex. 8.2 is also that for each pair $(f,g) \in \mathcal{F}$ if $B \in PS(\mathcal{U})$ and $\mathrm{Gcard}_{f,g}(B) = \beta_{f,g}$, then $\beta_{f,g} \in PS(CN)$. It is quite clear that the property $\mathrm{Gcard}_{f,g}(A) = \alpha \in PS(CN)$ holds for each $A \in GP(\mathcal{U})$ iff both f and g are functions to $PS(\mathcal{U})$.

10. Finite HCH-objects

Sets are divided into two disjoint classes: finite sets and infinite ones. Some properties of powers and cardinal numbers refer either to finite or to infinite sets. The others refer instead to all the sets and those are in fact real properties of powers and cardinals, for instance the monotonicity $\mathcal{A} \subset \mathcal{B} \Rightarrow \text{card } \mathcal{A} \leq \text{card } \mathcal{B}$. Let us try to extend the notions of finiteness and infiniteness to HCH-objects.

As we pointed out, the power of an HCH-object depends in general on the choice of $(f, g) \in \mathcal{F}$. However, it seems to be reasonable to accept the following postulates:

(a) The finiteness/infiniteness of an HCH-object does not depend on $(f, g) \in \mathcal{F}$.

(b1) An HCH-object of power less than or equal to power of a finite HCH-object has to be finite, too.

(b2) An HCH-object of power greater than or equal to power of an infinite HCH-object has to be infinite, too.

On the other hand, if we like to define a relation $\preceq$ ordering gcn's and powers of HCH-objects , then $\preceq$ should be monotonic, i.e. $\forall (f, g) \in \mathcal{F} : \text{obj}(A) \subset \text{obj}(B) \Rightarrow \text{Gcard}_{f,g}(A) \preceq \text{Gcard}_{f,g}(B)$; such a relation is defined and investigated in [24]. So, HCH-objects with finite supports must be considered to be finite ones. Really, if obj(A) has a finite support, then obj$(1_{\text{supp }(A)})$ is a finite set and obj$(A) \subset$ obj$(1_{\text{supp }(A)})$. Thus the monotonicity condition and (b1) imply that obj(A) is finite. Therefore the problem how to define finite HCH-objects resolves itself to the following question: which HCH-objects besides those with finite supports (if any) should be considered to be finite. To answer it let us recall the Dedekind's definition of an infinite set: $\mathcal{A}$ is infinite iff $\mathcal{A}$ is equipotent to its proper subset. This definition seems to be suitable for extending it to HCH-objects because it operates only with the notions of equipotency and proper containment and does not go into the nature of the notion of a set. So, let us test the following tentative definition: obj(A) is infinite iff it is equipotent with respect to any $(f, g) \in \mathcal{F}$ to an HCH-object obj(A^*) properly contained in obj(A). Now the problem is how to define the proper containment (denoted here by $\subset\subset$) of two HCH-objects. Let us consider two variants of definitions:

(v1) obj$(A^*) \subset\subset$ obj(A) iff $A^* \subset A \& \exists x \in \mathcal{U} : A^*(x) < A(x)$.

But then if we put e.g. $f \equiv E$ and $g(\cdot) = 1_{\text{supp }(\cdot)}$, even an HCH-object supported by one element would be infinite. So, we reject this variant.

(v2) $\text{obj}(A^*) \subset\subset \text{obj}(A)$ iff $f(A^*) \subset f(A) \& g(A^*) \subset g(A) \& \exists x \in \mathcal{U}$: $f(A^*)(x) < f(A)(x) \mid g(A^*)(x) < g(A)(x)$.

Let us consider then an example with $\mathcal{U} := \{2, 3, 4, \ldots\}, A(i) := 1/i$, $f(\cdot) = 1_{(\cdot)_1}$, $g = \text{id}$. So, $\text{Gcard}_{f,g}(A)(i) = 1/i$ for $i > 0$. obj(A) is not equipotent to any $\text{obj}(A^*) \subset\subset \text{obj}(A)$. Thus obj$(A)$ is finite although its support is infinite. But putting $f \equiv E, g(\cdot) = 1_{\text{supp }(\cdot)}$ we however get that obj(A) is infinite what contradicts (a).

So, we cannot use simultaneously (a) and the extended Dedekind's definition. Moreover, using the last example one can point out that if we like to consider some HCH-object with infinite support to be finite and even if we apply another definition of the infiniteness, then that HCH-object will be always infinite with respect to $f \equiv E$ and $g(\cdot) = 1_{\text{supp }(\cdot)}$. So, either we reject (a) or we consider an HCH-object to be finite iff its support is finite. We choose the second possibility; clearly, HCH-objects which are not finite will be called infinite. This definition is convenient and, moreover, it appears that just the transition from finite to infinite supports causes the same change of properties of gcn's and powers for HCH-objects as the change of properties of cardinal numbers and powers of sets caused by a transition from sets nonequipotent to their proper subsets to sets equipotent to such subsets. The gcn's related to finite (infinite, resp.) HCH-objects will be called finite (transfinite, resp.) gcn's.

11. Final remarks

In this paper our attention has been focused on the construction and basic properties of powers and gcn's for HCH-objects. It appears that a lot of these properties can be enhanced or even quite new properties can be formulated if we restrict ourselves to finite HCH-objects (see [25] for details) which seem to be important from the viewpoint of applications. However the aim of this presentation was to emphasize some general properties, i.e. properties which are independent on the powers of supports and on the choice of $(f, g) \in \mathcal{F}$.

Another two basic subjects have to be discussed: order and operations. Detailed solutions of these problems are given in [24,25]. For instance, it appears that gcn's form a commutative semiring.

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