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# SIMULTANEOUS EXTENSIONS OF PROXIMITIES, SEMI-UNIFORMITIES, CONTIGUITIES AND MEROTOPIES IV*

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**Abstract:** Given compatible merotopies in a semi-uniform or contiguity space (or semi-uniformities in a proximity space), we are looking for a common extension of these structures.

§§ 0 and 1 can be found in Part I [1], §§ 2 to 4 in Part II [2], §§ 5 and 6 in Part III [3]. See § 0 for terminology, notations and conventions.

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# 7. Extending a family of merotopies in a semi-uniform space

## A. WITHOUT SEPARATION AXIOMS

**7.1.** A family of merotopies in a semi-uniform space has a coarsest and a finest extension; we are going to construct both.
**Notation.** For an entourage $U$, let

$$\mathsf{c}^0(U)=\{C : C^2\subset U\},\qquad \mathsf{c}^1(U)=\Big\{\{x,y\} : xUy, yUx\Big\}.$$

Recall from 0.4 that $U(\mathsf{c}) = \bigcup_{C\in\mathsf{c}} C^2$ for a cover $\mathsf{c}$; this notation will be used for arbitrary collections $\mathsf{c}\subset \exp X$. ♢
**Lemma.**

a) $\mathsf{c}^0(U)$ *and* $\mathsf{c}^1(U)$ *are covers.* $U(\mathsf{c})$ *is an entourage iff* $\mathsf{c}$ *is a cover.*
b) $\mathsf{c}^k(U\cap V)=\mathsf{c}^k(U)\cap \mathsf{c}^k(V)\quad (k=1,2)$.
c) $U(\mathsf{c}^k(U))=U\cap U^{-1}\quad (k=1,2)$.
d) *For a cover* $\mathsf{c}$, $\mathsf{c}^1(U(\mathsf{c}))$ *refines* $\mathsf{c}$, *and* $\mathsf{c}\subset \mathsf{c}^0(U(\mathsf{c}))$.
e) *If* $c$ *is a topology on* $X$, *and* $U$ *is symmetric and open then* $U(\mathrm{int}_c\mathsf{c}^0(U))=U$.

**Proof.** e) $U(\mathrm{int}_c\mathsf{c}^0(U))\subset U(\mathsf{c}^0(U))=U$. Conversely, if $xUy$ then $V\times W\subset U$ for some $c$-open neighbourhoods $V$ of $x$ and $W$ of $y$. We may assume $V^2\subset U$, $W^2\subset U$, since $xUx$, $yUy$; $W\times V\subset U$ by the symmetry. Thus $C=V\cup W\in \mathsf{c}^0(U)$, and, $C$ being $c$-open, $C\in \mathrm{int}_c\mathsf{c}^0(U)$, $(x,y)\in C^2\subset U(\mathrm{int}_c(\mathsf{c}^0(U))$. ♢
**Remark.** Saying that $\mathsf{c}$ is *finer* than $\mathsf{d}$ instead of $\mathsf{c}$ refines $\mathsf{d}$ (which is, of course, in conflict with established terminology), the content of this trivial lemma can be interpreted as follows: any symmetric entourage $U$ can be induced by coverings; $\mathsf{c}^0(U)$ is the coarsest and $\mathsf{c}^1(U)$ the finest one (more precisely, one of the coarsest, respectively finest ones); if $U$ is open then $\mathrm{int}\,\mathsf{c}^0(U)$ is the coarsest open cover inducing $U$.

**7.2** Recall the following notations:

$$\mathsf{c}_i^0=\{C_i^0 : C_i\in \mathsf{c}_i\}\quad (i\in I,\ \mathsf{c}_i\in \mathsf{M}_i),$$
$$C_i^0=C_i\cup X_i^r,\quad X_i^r=X\setminus X_i;$$

$s\mathcal{U}$ denotes the collection of the symmetric elements of $\mathcal{U}$.
**Definition.** For a family of merotopies in a semi-uniform space,

a) Let $\mathsf{M}^0$ be the merotopy for which $\mathsf{c}_i^0$ $(i \in I,\ \mathsf{c}_i \in \mathsf{M}_i)$ and $\mathsf{c}^0(U)$ $(U \in \mathcal{U})$ form a subbase $\mathsf{B}^0$.

b) Let $\mathsf{M}^1$ consist of those covers $\mathsf{c}$ of $X$ for which

$$\mathsf{c}|X_i \in \mathsf{M}_i \quad (i \in I); \tag{1}$$

$$U(\mathsf{c}) \in \mathcal{U}. \tag{2}$$

The more precise notations $\mathsf{M}^k(\mathcal{U}, \mathsf{M}_i) = \mathsf{M}^k(\mathcal{U}, \{\mathsf{M}_i : i \in I\})$ will be used when necessary; $\mathsf{M}^k(\mathcal{U}) = \mathsf{M}^k(\mathcal{U}, \emptyset)$ $\quad (k = 0, 1)$. $\diamondsuit$

The elements of $\mathsf{B}^0$ are covers, so it is indeed a subbase for a merotopy. It does not change $\mathrm{B}^0$ if $\mathcal{U}$ is replaced by $s\mathcal{U}$ in the definition (since $\mathsf{c}^0(U)$ depends only on $U \cap U^{-1}$). Replacing $\mathcal{U}$ and each $\mathsf{M}_i$ by subbases, we still obtain a subbase for $\mathsf{M}^0$ (Lemma 7.1 b) and $\mathsf{c}_i^0(\cap)\mathsf{d}_i^0 = (\mathsf{c}_i(\cap)\mathsf{d}_i)^0$). If $I = \emptyset$ then $\mathsf{B}^0 = \{\mathsf{c}^0(U) : U \in s\mathcal{U}\}$ is a base, not just a subbase (Lemma 7.1 b)). $\mathsf{M}^1$ is clearly a merotopy. The next Lemma gives an alternative description of $\mathsf{M}^1$; in particular, if $I = \emptyset$ then $\mathsf{B}^1 = \{\mathsf{c}^1(U) : U \in s\mathcal{U}\}$ is a base for $\mathsf{M}^1$.

**Lemma** *The covers of the form*

$$\mathsf{c}_1(U) \cup \bigcup_{i \in 1} \mathsf{c}_i \quad (U \in s\mathcal{U}, \mathsf{c}_i \in \mathsf{M}_i \quad (i \in I)) \tag{3}$$

*make up a base* $\mathsf{B}^1$ *for* $\mathsf{M}^1$.

**Proof.** If $\mathsf{c}$ is as in (3) then $\mathsf{c}|X_i \supset \mathsf{c}_i$, thus (1) holds; $U(\mathsf{c}) \supset U(\mathsf{c}^1(U)) = U$, thus (2) holds, too. This means that $\mathsf{B}^1 \subset \mathsf{M}^1$. Conversely, any $\mathsf{c} \in \mathsf{M}^1$ is refined by (3) taken with $\mathsf{c}_i = \mathsf{c}|\mathsf{M}_i$ and $U = U(\mathsf{c})$. $\diamondsuit$

**Theorem.** *Any family of merotopies in a semi-uniform space has extensions;* $\mathsf{M}^0$ *is the coarsest and* $\mathsf{M}^1$ *the finest one.*

**Proof.** 1° $\mathsf{M}^0$ *is coarser than* $\mathsf{M}^1$. It is enough to show that $\mathsf{B}^0 \subset \mathsf{M}^1$, i.e. that (1) and (2) hold for the covers $\mathsf{c}_i^0$ and $\mathsf{c}^0(U)$. It follows from the accordance that $\mathsf{c}_i^0$ satisfies (1) (this fact was already used in the proof of Theorem 3.1). (2) is satisfied, too, since the compatibility implies that $U(\mathsf{c}_i) = U|X_i$ with some $U \in \mathcal{U}$, and from

$$C_i^{02} = C_i^2 \cup (C_i \times X_i^r) \cup (X_i^r \times C_i) \cup X_i^{r2}$$

we obtain $U(\mathsf{c}_i^0) = U|X_i \cup (X^2 \setminus X_i^2)$, so that $U \subset U(C_i^0)$. $\mathsf{c}^0(U)|X_i = \mathsf{c}^0(U|X_i)$ is clear from the definition, thus (1) holds for $\mathsf{c}^0(U)$ (since, assuming $U \in s\mathcal{U}$, $U|X_i = U(\mathsf{c}_i)$ for some $\mathsf{c}_i \in \mathsf{M}_i$, which refines $\mathsf{c}^0(U|X_i)$ by Lemma 7.1 d)); (2) follows from Lemma 7.1 c).

2° $\mathsf{M}^0$ *and* $\mathsf{M}^1$ *are compatible.* According to 1°, it is enough to check that $\mathcal{U}(\mathsf{M}^1) \subset \mathcal{U} \subset \mathcal{U}(\mathsf{M}^0)$. The first inclusion is evident from (2). If $U \in s\mathcal{U}$ then $\mathsf{c}^0(\mathcal{U}) \in \mathsf{M}^0$, so $U(\mathsf{c}^0(U)) \in \mathcal{U}(\mathsf{M}^0)$; hence $U \in \mathsf{M}^0$ by Lemma 7.1 c).
3° $\mathsf{M}^0$ *and* $\mathsf{M}^1$ *are extensions.* By 1° and 2° , we have only to see that $\mathsf{M}^1|X_i \subset \mathsf{M}_i \subset \mathsf{M}_0|X_i$. The first inclusion is clear from (1), the second one from $\mathsf{c}_i^0|X_i = \mathsf{c}_i$.
4° $\mathsf{M}^0$ *is coarsest,* $\mathsf{M}_1$ *is finest.* Let $\mathsf{M}$ be an extension. Any $\mathsf{c} \in \mathsf{M}$ satisfies (1) and (2) by the definition of an extension, thus $\mathsf{M} \subset \mathsf{M}^1$. For $\mathsf{c}_i \in \mathsf{M}_i$, there is a $\mathsf{c} \in \mathsf{M}$ with $\mathsf{c}|X_i = \mathsf{c}_i$ ; $\mathsf{c}$ refines $\mathsf{c}_i^0$, thus $\mathsf{c}_i^0 \in \mathsf{M}$. Given a $U \in s\mathcal{U}$ , there is a $\mathsf{c} \in \mathsf{M}$ with $U = U(\mathsf{c})$ (see 0.4), and then $\mathsf{c}^0(U) \supset \mathsf{c}$ by Lemma 7.1 d), thus $\mathsf{c}^0(U) \in \mathsf{M}$, too. Hence $B^0 \subset \mathsf{M}$ , implying $\mathsf{M}^0 \subset \mathsf{M}$. ◇

## B. RIESZ MEROTOPIES IN A SEMI-UNIFORM SPACE

**7.3** If a family of merotopies in a semi-uniform space has a Riesz extension then the semi-uniformity is Riesz, and the trace filters are Cauchy (with respect to the merotopies). The merotopies are also Riesz, but this is included in the statement that the trace filters are Cauchy. The above conditions are sufficient, too.
**Definition.** For a family of merotopies in a semi-uniform space, let

$$\mathsf{M}_R^1 = \{\mathsf{c} \in \mathsf{M}^1 : \text{ int } \mathsf{c} \text{ is a cover of } X\}. \ \diamondsuit$$

(Compare with Definition 3.2.)
**Theorem.** *A family of merotopies in a Riesz semi-uniform space has a Riesz extension iff the trace filters are Cauchy; if so then* $\mathsf{M}^0$ *is the coarsest and* $\mathsf{M}_R^1$ *the finest Riesz extension.*
**Proof.** The necessity is obvious. Assume conversely that the trace filters are Cauchy. Now $\mathsf{M}^0$ is Riesz, since int $\mathsf{c}$ is a cover for each $\mathsf{c} \in B^0$. Indeed, int $\mathsf{c}_i^0$ is a cover by the Cauchy property, while if $U \in \mathcal{U}$ then $\Delta \subset \operatorname{int} U$ implies that for any $x \in X$, there is a $C \in \mathrm{v}(x)$ with $C^2 \subset U$, and it follows from $C \in \mathsf{c}^0(U)$ that int $\mathsf{c}^0(U)$is a cover, too.

$\mathsf{M}^0$ is the coarsest Riesz extension by Theorem 7.2. If $\mathsf{M}$ is a Riesz extension then $\mathsf{M} \subset \mathsf{M}^1$ (Theorem 7.2), therefore $\mathsf{M} \subset \mathsf{M}_R^1$. In particular, $\mathsf{M}^0 \subset \mathsf{M}_R^1$ ; this and the evident inclusion $\mathsf{M}_R^1 \subset \mathsf{M}^1$ imply that $\mathsf{M}_R^1$ is an extension (again Theorem 7.2). It follows from the definition that, being compatible, $\mathsf{M}_R^1$ is Riesz. ◇

**Remark.** Given a semi-uniformity $\mathcal{U}$ and a $U \in s\mathcal{U}$, there is in general no finest one (in the sense of Remark 7.1) among the covers $\mathsf{c}$ inducing $U$ for which int $\mathsf{c}$ is cover: take the Euclidean uniformity on $\mathbb{R}$ , and $U = \mathbb{R}^2$ ; observe that $U = U(\mathsf{c}(\varepsilon))$ $(\varepsilon > 0)$ where

$$\mathsf{c}(\varepsilon) = \{]x, x+\varepsilon[\cup\{y\} : x, y \in \mathbb{R}\}.$$

So we cannot hope for a characterization of $\mathsf{M}^1_R$ similar to Lemma 7.2.

## C. LODATO MEROTOPIES IN A SEMI-UNIFORM SPACE

**7.4** If a family of merotopies in a semi-uniform space has a Lodato extension then the semi-uniformity and the merotopies are Lodato, the trace filters are Cauchy, and 3.6 (1) holds. These conditions are not sufficient, see Examples 7.12.

**Definition.** For a family of Lodato merotopies in a Lodato semi-uniform space,

a) Let $\mathsf{M}^1_L = \{\mathsf{c} \in \mathsf{M}^1 : \text{int}\, \mathsf{c} \in \mathsf{M}^1\}$.

b) If the trace filters are Cauchy then let $\mathsf{M}^0_L$ be the merotopy for which $\{\text{int}\, \mathsf{c} : \mathsf{c} \in \mathsf{M}^0\}$ is a base. $\diamondsuit$

The open covers in $\mathsf{M}^1$ form a base for $\mathsf{M}^1_L$. In b), int $\mathsf{c}$ is a cover, because the trace filters are Cauchy and $\mathcal{U}$ is Lodato; these covers form a base for a merotopy, since int $\mathsf{c}$ $(\cap)$ int $\mathsf{d}$ = int $(\mathsf{c}(\cap)\mathsf{d})$. The following covers make up a subbase $\mathsf{B}^0_L$ for $\mathsf{M}^0_L$:

$$\text{int}\, \mathsf{c}^0_i \quad (i \in I,\ \mathsf{c}_i \in \mathsf{M}_i,\ \mathsf{c}_i \text{ is } c_i\text{-open});$$

$$\text{int}\, \mathsf{c}_0(U) \quad (U \in s\mathcal{U}, U \text{ is open}).$$

Observe that

(1) $$\text{int}\, \mathsf{c}^0(U) = \{C : C^2 \subset U, \quad C \text{ is open}\}.$$

**Remark.** There is a simple reason for the similarity with Definitions 3.4, 3.5 and 5.14: If $\mu$ is a collection of compatible merotopies in a topological space such that $\mathsf{M} \subset \mathsf{M}' \subset \mathsf{M}''$ and $\mathsf{M}, \mathsf{M}'' \in \mu$ imply $\mathsf{M}' \in \mu$, there is a coarsest $\mathsf{M}^0 \in \mu$ (a finest $\mathsf{M}^1 \in \mu$), and there exists a Lodato merotopy in $\mu$ then $\mathsf{M}^0_L$ $(\mathsf{M}^1_L)$ defined as above is the coarsest (finest) Lodato merotopy in $\mu$. (The proof is straightforward.) Analogous statements hold for contiguities and semi-uniformities.

**Lemma.** *A family of Lodato merotopies in a Lodato semi-uniform space has a Lodato extension iff the trace filters are Cauchy and* $\mathsf{M}^0_L \subset$

$\subset \mathsf{M}_L^1$ ; *if so then* $\mathsf{M}_L^0$ *is the coarsest and* $\mathsf{M}_L^1$ *the finest Lodato extension.*
**Proof.** The above remark applied to the collection of all extensions (Theorem 7.2) gives that if there are Lodato extensions then $\mathsf{M}_L^0$ is the coarsest and $\mathsf{M}_L^1$ the finest one; therefore $\mathsf{M}_L^0 \subset \mathsf{M}_L^1$. Assume conversely that the trace filters are Cauchy and $\mathsf{M}_L^0 \subset \mathsf{M}_L^1$. Then Theorem 7.2 and the trivial inclusions $\mathsf{M}^0 \subset \mathsf{M}_L^0$ and $\mathsf{M}_L^1 \subset \mathsf{M}^1$ yield that $\mathsf{M}_L^0$ and $\mathsf{M}_L^1$ are extensions. Being compatible, they are clearly Lodato. ◇

**7.5 Remark.** Lemma 7.4 remains valid if $\mathsf{M}_L^0 \subset \mathsf{M}_L^1$ is replaced by $\mathsf{M}_L^0 \subset \mathsf{M}^1$ (or $\mathsf{M}^0 \subset \mathsf{M}_L^1$). The proof is the same.

**7.6 Lemma.** *A family of merotopies in a semi-uniform space has a Lodato extension iff*

(i) *the semi-uniformity and the merotopies are Lodato;*
(ii) $U(\operatorname{int} \mathsf{c}_i^0) \in \mathcal{U} \quad (i \in I, \mathsf{c}_i \in \mathsf{M}_i)$;
(iii) $(\operatorname{int} \mathsf{c}_i^0)|X_j \in \mathsf{M}_j \quad (i, j \in I, \mathsf{c}_i \in \mathsf{M}_i)$;
(iv) $(\operatorname{int} \mathsf{c}^0(U))|X_i \in \mathsf{M}_i \quad (U \in s\mathcal{U}, i \in I)$.

**Remarks.** a) (ii) implies that each $\operatorname{int} \mathsf{c}_i^0$ is a cover, i.e. that the trace filters are Cauchy.

b) In comparison with Lemmas 5.17 and 6.8, Condition (iv) is completely new; we shall later see that it is not superfluous.

**Proof.** 1° *Necessity.* (i) is clear. (iii) follows from Theorem 3.6. If there are Lodato extensions then $\mathsf{M}_L^0$ is one of them by Lemma 7.4, $\operatorname{int} \mathsf{c}_i^0 \in \mathsf{M}_L^0$ by definition, thus, $\mathsf{M}_L^0$ being compatible, (ii) holds; (iv) follows from $\mathsf{M}_L^0|X_i = \mathsf{M}_i$ and $\operatorname{int} \mathsf{c}^0(U) \in \mathsf{M}_L^0$.

2° *Sufficiency.* The assumptions of Definition 7.4 are fulfilled, so, according to Remark 7.5, it is enough to check that $\mathsf{M}_L^0 \subset \mathsf{M}^1$, i.e. that $B_L^0 \subset \mathsf{M}^1$. This means four conditions, from which three are just (ii), (iii) and (iv), and the fourth, namely $U(\operatorname{int} \mathsf{c}^0(U)) \in \mathcal{U}$ , holds by Lemma 7.1 e). ◇

**Corollary.** *A single Lodato merotopy* $\mathsf{M}_0$ *in a Lodato semi-uniform space has a Lodato extension iff* $U(\operatorname{int} \mathsf{c}_0^0) \in \mathcal{U}$ *for each* $c_0$*-open* $\mathsf{c}_0 \in \mathsf{M}_0$, *and* $(\operatorname{int} \mathsf{c}^0(U))|X_0 \in \mathsf{M}_0$ *for each open* $U \in s\mathcal{U}$. ◇

The first assumption cannot be replaced by the Cauchy property of the trace filters, and the second one cannot be dropped either, see Examples 7.12.

**7.7 Corollary.** *Any Lodato semi-uniformity* $\mathcal{U}$ *can be induced by Lodato merotopies;* $\mathsf{M}_L^0(\mathcal{U})$ *is the coarsest and* $\mathsf{M}_L^1(\mathcal{U})$ *the finest one.* ◇

$B_L^0$ (consisting in this special case of the covers given in 7.4 (1)) is a base for $\mathsf{M}_L^0(\mathcal{U})$).

It can happen that $\mathsf{M}_L^0(\mathcal{U}) \neq \mathsf{M}^0(\mathcal{U})$ for a Lodato semi-uniformity $\mathcal{U}$. (In a proximity space, $I \neq \emptyset$ was needed for an analogous example, see Lemma 5.15 and Example 5.17.)

**Example.** On $X = \mathbb{R}$, take the semi-uniformity $\mathcal{U}$ for which $\{U(k) : k \in \mathbb{N}\}$ is a base, where

$$U(k) = \{(x,y) : |x-y| < 1/k\} \cup \bigcup \{Q_{mn} : m,n > k\},$$

$$Q_{mn} = ]m - \frac{1}{m+n}, m + \frac{1}{m+n}[ \times ] n - \frac{1}{m+n}, n + \frac{1}{m+n}[.$$

$\mathsf{c}$ is the Euclidean topology, thus $U(k)$ is open, and $\mathcal{U}$ is Lodato. We claim that

$$\mathsf{c} = \operatorname{int} \mathsf{c}^0(u(1)) \in \mathsf{M}_L^0(\mathcal{U}) \backslash \mathsf{M}^0(\mathcal{U}).$$

Indeed, if $\mathsf{c}$ belonged to $\mathsf{M}^0(\mathcal{U})$ then there were a $k \in \mathbb{N}$ with $\mathsf{d} = \mathsf{c}^0(U(k))$ refining $\mathsf{c}$. This is, however, impossible since $A = \{n \in \mathbb{N} : n > k\} \in \mathsf{d}$, but there is no open set $G \supset A$ such that $G^2 \subset U(1)$. ◇

**7.8** $\mathsf{M}_L^0(\mathcal{U})$, $\mathsf{M}_L^1(\mathcal{U})$ and $\mathsf{M}_R^1(\mathcal{U})$ can be different:

**Example.** Take the Euclidean uniformity $\mathcal{U}$ on $X = \mathbb{R}$, and let $f(x) = x + (1+|x|)^{-1}$. Then

(1) $$\mathsf{d} = \{]x, f(x)[ \cup ]y, f(y)[ : x, y \in X\} \in \mathsf{M}_L^1(\mathcal{U}) \backslash \mathsf{M}_L^0(\mathcal{U}),$$

$$\{\{x,y\} : x, y \in X\} \cup \{]x, f(x)[ : x \in X\} \in \mathsf{M}_R^1(\mathcal{U}) \backslash \mathsf{M}_L^1(\mathcal{U}). \ \diamond$$

**7.9** Condition (iii) is not superfluous in Lemma 7.6:

**Example.** Let $\mathcal{U}$ be the Euclidean uniformity on $\times = \mathbb{R} \times [0,1[$, $X_0 = \mathbb{R} \times \{0\}$, $X_1 = X_0^r$, $\mathsf{M}_i = \mathsf{M}_L^i(\mathcal{U})|X_i$. 7.5 (ii) and (iv) are satisfied, since $\mathsf{M}_0$ and $\mathsf{M}_1$ separately have extensions. But (iii) fails for $i = 1$, $j = 0$ ,

$$\mathsf{c}_1 = \{D \times ]0, 1[ : D \in \mathsf{d}\} \in \mathsf{M}_1$$

with $\mathsf{d}$ from 7.8 (1). ◇

**7.10 Corollary.** *A family of merotopies in a Lodato semi-uniform space has a Lodato extension iff* $\{\mathsf{M}_i, \mathsf{M}_j\}$ *has a Lodato extension for any* $i, j \in I$. ◇

**7.11 Corollary.** *A family of merotopies in a Lodato semi-uniform space has a Lodato extension iff it has a Lodato extension in* $(X, \mathsf{c})$, *and each* $\mathsf{M}_i$ *has a Lodato extension in* $(X, \mathcal{U})$.

**Proof.** Theorem 3.6 and Lemma 7.6. ◊

**7.12 Theorem.** *A family of Lodato merotopies given on open-closed subsets in a Lodato semi-uniform space has Lodato extensions.*
**Proof.** By Corollaries 3.8 and 7.11 it is enough to check that each $\mathsf{M}_i$ separately has a Lodato extension, i.e. that

(1) $$U(\text{int } \mathsf{c}_i^0) \in \mathcal{U} \quad (\mathsf{c}_i \in \mathsf{M}_i \quad \text{is} \quad c_i\text{-open}),$$

(2) $$(\text{int } \mathsf{c}^0(U))|X^i \in \mathsf{M}_i \quad (U \in s\mathcal{U} \quad \text{is open}).$$

$X_i$ being closed, we have int $\mathsf{c}_i^0 = \mathsf{c}_i^0$; $U(\mathsf{c}_i^0) \in \mathcal{U}$ , because $\mathsf{c}_i^0 \in \mathsf{M}^0$, which is compatible. Thus (1) holds indeed. On the other hand, the openness of $X_i$ implies that

$$(\text{int } \mathsf{c}_0(U))|X_i = \text{int}_i\,(\mathsf{c}^0(U)|X_i)$$

(see 7.4 (1)). Now $\mathsf{c}^0(U)|X_i \in \mathsf{M}_i$, since $\mathsf{c}^0(U)$ belongs to the extension $\mathsf{M}^0$. Thus, $\mathsf{M}_i$ being Lodato, (2) is satisfied, too. ◊

It is not enough to assume that the sets are open and the trace filters Cauchy, or that the sets are closed. The next examples (with $|I| = 1$) have the additional property that there exists a Lodato extension in $(X, \delta(\mathcal{U}))$.

**Examples.** a) With $X$ , $X_0$ and $\mathsf{M}_0$ from Example 5.20, $\mathsf{M}_0$ is compatible with $\mathcal{U}|X_0$, where $\mathcal{U}$ is the Euclidean uniformity on $X$. $\mathcal{U}$ and $\mathsf{M}_0$ are Lodato, and $X_0$ is open. The trace filters are Cauchy; in fact, $\mathsf{M}_0$ has a Lodato extension in $(X, \delta(\mathcal{U}))$ (see 5.20 and Corollary 5.17). The second condition of Corollary 7.6 holds (because $X_0$ is open), but the first one fails for $\mathsf{c}_0(1)$: no set of the form $(]-\varepsilon, \varepsilon[\times\{0\})^2 \cap x$ is contained by $U(\text{int } \mathsf{c}_0(1)^0)$.

b) Let $X$ and $\mathcal{U}$ be as in Example 7.7, $X_0 = \mathbb{N}$, $\mathsf{M}_0 = \mathsf{M}^0(U_0)$ $\mathcal{U}_0 = \mathcal{U}|X_0$. Now $\mathcal{U}$ and $\mathsf{M}_0$ are Lodato (the latter because $c_0$ is discrete), and $X_0$ is closed. $\mathsf{M}_0$ has a Lodato extension in $(X, \delta(\mathcal{U}))$ (Theorem 5.22), but it does not have one in $(X,\mathcal{U})$: $(\text{int } \mathsf{c}^0(U(1)))|X_0 \notin \mathsf{M}_0$, since this cover consists of finite sets, while $\mathsf{M}_0$ is contigual. ◊

# 8. Extending a family of semi-uniformities in a proximity space

## A. WITHOUT SEPARATION AXIOMS

**8.1** Results are, and proofs could be, analogous to those for merotopies in a proximity space (§ 5). The foll]owing simple observation will save us from doing all over again:

**Lemma.** *For a family of semi-uniformities in a proximity space,* $\{\mathsf{M}^0(U_i) : i \in I\}$ *is a family of merotopies in the same space. The trace filters are* $U_i$*-Cauchy iff they are* $\mathsf{M}^0(U_i)$*-Cauchy.*

**Proof.** The accordance follows from $C^0(U|X_i) = \mathsf{c}^0(U)|X_i$. ◇

**8.2** Definition. For a family of merotopies in a proximity space, let:

$$U^0 = \mathcal{U}(\mathsf{M}^0(\delta, \mathsf{M}^0(U_i))). \ \diamond$$

The following entourages constitute a subbase $\mathcal{B}$ for $\mathcal{U}^0$:

$$U_i^0 = U_i \cup (X^2 \backslash X_i^2) = U((\mathsf{c}^0(U_i)^0) \qquad (i \in I, U_i \in \mathcal{U}_i);$$

$$U_{A,B} = A^{r2} \cup B^{r2} = U(\mathsf{c}_{A,B}) \qquad (A \,\bar{\delta}\, B).$$

**Theorem.** *A family of semi-uniformities in a proximity space can always be extended;* $\mathcal{U}^0$ *is the coarsest extension.*

**Proof.** It follows from Theorem 5.4 and Lemma 8.1 that $\mathcal{U}^0$ is an extension. Let $\mathcal{U}$ be another extension; then $\mathsf{M}^0(\mathcal{U})$ is an extension of the merotopies $\mathsf{M}^0(\mathcal{U}_i)$, thus $\mathsf{M}^0 \subset \mathsf{M}^0(\mathcal{U})$ (Theorem 5.4), implying $\mathcal{U}^0 = \mathcal{U}(\mathsf{M}^0) \subset \mathcal{U}(\mathsf{M}^0(\mathcal{U})) = \mathcal{U}$. ◇

It follows from Example 5.3 that there is in general no finest compatible (Riesz/Lodato) semi-uniformity in a (Riesz/Lodato) proximity space.

## B. RIESZ SEMI-UNIFORMITIES IN A PROXIMITY SPACE

**8.3 Theorem.** *A family of semi-uniformities in a Riesz proximity space has a Riesz extension iff the trace filters are Cauchy; if so then* $\mathcal{U}^0$ *is the coarsest Riesz extension.*

**Proof.** If the conditions are fulfilled then $\mathcal{U}^0$ is Riesz by Lemma 8.1 and Theorem 5.9. ◇

## C. LODATO SEMI-UNIFORMITIES IN A PROXIMITY SPACE

**8.4** Although the results are analogous to those for Lodato merotopies, we cannot keep on applying the results of § 5, since $\mathsf{M}^0(\mathcal{U}_i)$ is in general not Lodato (Example 7.7), while it can occur that $\{\mathsf{M}^0_L(\mathcal{U}_i) : i \in I\}$ is not a family of merotopies (it is not accordant):

**Example.** With $X$ and $\mathcal{U}$ from Example 7.7, let $\delta = \delta(\mathcal{U})$, $X_0 = \mathbb{N}$, $X_1 = X$, $\mathcal{U}_i = \mathcal{U}|X_i$. Now $\{\mathcal{U}_0, \mathcal{U}_1\}$ is a family of semi-uniformities having a Lodato extension (namely $\mathcal{U}$), but $\mathsf{M}^0_L(\mathcal{U}_0)$ and $\mathsf{M}^0_L(\mathcal{U}_1)$ are not accordant: if they were then $\mathsf{M}^0_L(\mathcal{U})$ would be a Lodato extension of $\mathsf{M}^0_L(\mathcal{U}_0)$, contradicting Example 7.12 b). $\diamond$

**Remark.** An open filter (in particular, a trace filter) is $\mathcal{U}_i$-Cauchy iff it is $\mathsf{M}^0_L(\mathcal{U}_i)$-Cauchy. This observation makes it possible to apply the results of § 5 C in the special case $|I| \leq 1$.

**8.5 Definition.** The entourage $U$ is a *$\delta$-entourage* if $A \,\delta\, B$ implies that there are $x \in A$, $y \in B$ with $xUy$. $\diamond$

$U$ is a $\delta$-entourage iff $A \,\bar{\delta}\, U[A]^r \quad (A \subset X)$.

**Lemma.** *A cover* $\mathsf{c}$ *is a $\delta$-cover iff* $U(\mathsf{c})$ *is a $\delta$-entourage.* $\diamond$

**8.6 Lemma.** *For a semi-uniformity* U *on* $X$, $\delta(\mathcal{U})$ *is coarser than* $\delta$ *iff every* $U \in \mathcal{U}$ *is a $\delta$-entourage iff* $\mathcal{U}$ *has a base consisting of $\delta$-entourages.* $\diamond$

**8.7 Lemma.** *If* $U$ *and* $V$ *are $\delta$-entourages and* $V = U(\mathsf{f})$ *with a finite cover* $\mathsf{f}$ *then* $U \cap V$ *is a $\delta$-entourage.*

**Proof.** Take a cover $\mathsf{c}$ such that $U \cap U^{-1} = U(\mathsf{c})$, and use Lemmas 5.2 and 8.5. $\diamond$

**8.8 Definition.** For a family of Lodato semi-uniformities in a Lodato proximity space with Cauchy trace filters, let $\{\operatorname{Int} U : U \in \mathcal{B}\}$ be a subbase for $\mathcal{U}^0_L$ (with $\mathcal{B}$ from 8.2). $\diamond$

The Cauchy property implies that Int $U$ is indeed an entourage. Copying the argument from 5.14 to 5.17 and 5.22, we obtain:

**Lemma.** *A family of semi-uniformities in a proximity space has a Lodato extension iff*

(i) *the proximity and the semi-uniformities are Lodato;*

(ii) $\bigcap_{i \in F} \operatorname{Int} U^0_i$ *is a $\delta$-entourage whenever* $\emptyset \neq F \subset I$ *is finite, and* $U_i \in \mathcal{U}_i \quad (i \in F)$;

(iii) $(\operatorname{Int} U^0_i)|X_j \in \mathcal{U}_j \quad (i, j \in I, U_i \in \mathcal{U}_i)$.

*If these conditions are satisfied then* $\mathcal{U}_L^0$ *is the coarsest Lodato extension.* ◇

(When showing that $\bigcap_{i\in F} \mathrm{Int}U_i^0 \cap \bigcap_{k=1}^{n} U_{A_k,B_k}$ is a $\delta$-entourage, apply Lemma 8.7 $n$ times.)

**Corollary.** *A single Lodato semi-uniformity* $\mathcal{U}_0$ *in a Lodato proximity space has a Lodato extension iff* Int $U_0^0$ *is a* $\delta$*-entourage for each* ($c_0 \times c_0$*-open*) $U_0 \in \mathcal{U}_0$. ◇

**Theorem.** *A family of Lodato semi-uniformities given on closed subsets in a Lodato proximity space has Lodato extensions;* $\mathcal{U}^0 = \mathcal{U}_L^0$ *is the coarsest one.* ◇

**8.9** The condition in Corollary 8.8 cannot be replaced by the weaker assumption that the trace filters are Cauchy:

**Examples.** a) Let

$$X_0 = \{(1/k, 1/n) : k, n \in \mathbb{N}, k \leq n\},\ X = X_0 \cup \{(1/k, 0) : k \in \mathbb{N}\}.$$

With the Euclidean proximity $\delta$ on $X$, $X_0$ is open. For $x = (x', x''), y = (y', y'')$, $x, y \in X$ and $\varepsilon > 0$, define

$$(1) \qquad xU_0(\varepsilon)y \text{ iff } |x' - y'| < \varepsilon,\ |x'' - y''| < \varepsilon,\ (x' \neq y' \Rightarrow x'' \neq y''),$$

and let $\{U_0(\varepsilon) : \varepsilon > 0\}$ be a base for $\mathcal{U}_0$. Each $U_0(\varepsilon)$ is an open $\delta_0$-entourage, and $\mathcal{U}_0$ is clearly finer than the Euclidean semi-uniformity on $X_0$, thus $\mathcal{U}_0$ is a compatible Lodato semi-uniformity. The trace filters are Cauchy, but Int $U_0(1)^0$ is not a $\delta$-entourage (let $A$ and $B$ be disjoint infinite subsets of $X_0^r$).

b) Let everything be as above, but replace the last condition in (1) by

$$(x' = x'',\ y' \neq y'' \Rightarrow x'' < y''),\ (x' \neq x'',\ y' = y'' \Rightarrow y'' < x'').$$

Now the sets $A = X_0^r$ and $B = \{(1/n, 1/n) : n \in \mathbb{N}\}$ show that Int $U_0(1)^0$ is not a $\delta$-entourage. ◇

Similarly to 5.18 the condition of Corollary 8.8 can be split into two parts. The above examples show that neither of these parts is sufficient in itself.

**8.10** Condition (iii) cannot be dropped from Lemma 8.8, see Example 2.10; (ii) cannot be replaced by the weaker assumption that each Int $U_i^0$ is a $\delta$-entourage:

**Example.** Taking $X$, $X_0$, $X_1$ and $\delta$ from Example 5.20, let $\{U_i(\varepsilon) : \varepsilon > 0\}$ be a base for $\mathcal{U}_i$ on $X_i$, where, with $x = (x', x'')$ and $y = (y', y'')$,

$$\begin{aligned} xU_1(\varepsilon)y \text{ iff } & |x' - y'| < \varepsilon,\ |x'' - y''| < \varepsilon, \\ & (x'', y'' < \varepsilon, x' < 0 < y' \Rightarrow -x' < y'), \\ & (x'', y'' < \varepsilon, y' < 0 < x' \Rightarrow -y' < x'), \\ xU_0(\varepsilon)y \text{ iff } & (-x', -x'')U_1(\varepsilon)(-y', -y'') \end{aligned}$$

The reasoning from 5.20 can be easily adapted. $\diamondsuit$

## 9. Extending a family of merotopies in a contiguity space

### A. WITHOUT SEPARATION AXIOMS

**9.1** In the problems investigated so far, a family of structures always had an extension if no separation property was required; this is not the case for merotopies in a contiguity space. It will be easier to describe the counterexample after some definitions and lemmas.

**Definition.** In a contiguity space $(X, \Gamma)$,

a) A cover $\mathsf{c}$ of $X$ is a $\Gamma$-*cover* if any finite cover refined by $\mathsf{c}$ belongs to $\Gamma$.

b) (See e.g. [4].) A collection $\mathsf{n} \subset \exp X$ is $\Gamma$-*near* if it is finite and $\mathsf{n}^r = \{N^r : N \in \mathsf{n}\} \notin \Gamma$. $\diamondsuit$

A finite cover is a $\Gamma$-cover iff it belongs to $\Gamma$. It follows easily from the axioms that Co2 could be replaced by

Co2$''$. if $\mathsf{n}$ is $\Gamma$-near and each $N \in \mathsf{n}$ is the union of a finite collection $\mathsf{a}(N)$ then there are $A(N) \in \mathsf{a}(N)$ such that $\{A(N) : N \in \mathsf{n}\}$ is $\Gamma$-near.

(Compare with P5 in 0.2, or rather with its more complicated form that can be obtained by induction. Observe that $A\,\delta(\Gamma)\,B$ iff $\{A, B\}$ is $\Gamma$-near.)

**Lemma.** *A cover* $\mathsf{c}$ *is a* $\Gamma$*-cover iff* $\mathsf{c} \cap \sec \mathsf{n} \neq \emptyset$ *for each* $\Gamma$*-near collection* $\mathsf{n}$.

**Proof.** $\mathsf{c}$ is not a $\Gamma$-cover iff it refines some finite $\mathsf{f} \notin \Gamma$, i.e. iff there is a $\Gamma$-near collection $\mathsf{n}$ such that each $C \in \mathsf{c}$ is the subset of some

$N^r \in \mathsf{n}^r$. $\diamond$

Compare this lemma with the definition of a $\delta$-cover (5.1). By the observation made before the lemma, any $\Gamma$-cover is a $\delta(\Gamma)$-cover. Conversely, any $\delta$-cover is a $\Gamma^1(\delta)$-cover (indeed, if $\mathsf{c}$ is a $\delta$-cover then any finite cover refined by $\mathsf{c}$ is a $\delta$-cover, too, so it belongs to $\Gamma^1(\delta)$ by definition).

**9.2 Lemma.** *For a merotopy* $\mathsf{M}$ *on* $X$, $\Gamma(\mathsf{M})$ *is coarser than* $\Gamma$ *iff each element of* $\mathsf{M}$ *is a* $\Gamma$*-cover iff* $\mathsf{M}$ *has a base consisting of* $\Gamma$*-covers.* $\diamond$

**9.3 Lemma.** *If* $\mathsf{c}$ *is a* $\Gamma$*-cover and* $\mathsf{f} \in \Gamma$ *then* $\mathsf{C}\,(\cap)\,\mathsf{f}$ *is a* $\Gamma$*-cover.*
**Proof.** Given a $\Gamma$-near collection $\mathsf{n}$, we need $C \in \mathsf{c}$ and $D \in \mathsf{f}$ such that $C \cap D \in \sec \mathsf{n}$ (Lemma 9.1). By Co$''$2, it can be assumed that each element of $\mathsf{n}$ is contained by some element of the partition generated by $\mathsf{f}$. As $\mathsf{f}$ is a $\Gamma$-cover, there is a $D \in \mathsf{f} \cap \sec \mathsf{n}$, implying $\cup \mathsf{n} \subset D$. Taking now a $C \in \mathsf{c} \cap \sec \mathsf{n}$, we have $C \cap D \in \sec \mathsf{n}$. $\diamond$

For $\Gamma$-covers $\mathsf{c}$ and $\mathsf{d}$, $\mathsf{c}\,(\cap)\,\mathsf{d}$ is not necessarily a $\Gamma$-cover: in Example 5.2, take $\Gamma = \Gamma^1(\delta)$.

**9.4 Definition.** For a family of merotopies in a contiguity space, let $\mathsf{M}^0$ be the merotopy for which $\Gamma$ and the covers $\mathsf{c}_i^0$ $(i \in I, \mathsf{c}_i \in \mathsf{M}_i)$ form a subbase $\mathsf{B}$. $\diamond$

$\Gamma$ could be replaced here by a subbase.

**Lemma.** *A family of merotopies in a contiguity space has an extension iff*

(1) $(\bigcap_{i \in F})\, \mathsf{c}_i^0$ *is a* $\Gamma$*-cover whenever* $\emptyset \neq F \subset I$ *is finite and* $\mathsf{c}_i \in \mathsf{M}_i$ $(i \in F)$; *if so then* $\mathsf{M}^0$ *is the coarsest extension.*

**Remark.** Compare (1) with (ii) of Lemma 5.17.

**Proof.** 1° *Necessity.* Let $\mathsf{M}$ be an extension. Then $\mathsf{c}_i \in \mathsf{M}_i = \mathsf{M}|X_i$, thus $\mathsf{c}_i^0 \in \mathsf{M}$, and $(\bigcap_{i\in F})\mathsf{c}_i^0 \in \mathsf{M}$, hence it is a $\Gamma$-cover by Lemma 9.2.

2° *Sufficiency.* We show that $\mathsf{M}^0$ is an extension. Each element of $\mathsf{M}^0$ is refined by a cover of the form $\mathsf{c} = ((\bigcap_{i\in F} \mathsf{c}_i^0)\,(\cap)\,\mathsf{f}$, where $\mathsf{c}_i \in \mathsf{M}_i$ and $\mathsf{f} \in \Gamma$. It follows from (1) and Lemma 9.3 that $\mathsf{c}$ is a $\Gamma$-cover; hence $\Gamma(\mathsf{M}^0) \subset \Gamma$ by Lemma 9.2. On the other hand, $\Gamma \subset \mathsf{B} \subset \mathsf{M}^0$ implies $\Gamma \subset \Gamma(\mathsf{M}^0)$. As $\mathsf{M}^0|X_i \supset \mathsf{M}_i$ is evident, we have only to check that $\mathsf{M}^0|X_i \subset \mathsf{M}_i$, i.e. that $\mathsf{B}|X_i \subset \mathsf{M}_i$. It was already used in other proofs that, in consequence of the accordance, $\mathsf{c}_j^0|X_i \in \mathsf{M}_i$ ; if $\mathsf{f} \in \Gamma$ then $\mathsf{f}|X_i \in \Gamma_i \subset \mathsf{M}_i$.

3° $\mathsf{M}^0$ *is the coarsest extension.* It is clear that any extension has to contain $\mathsf{B}$. ◊

**Theorem.** *A family of merotopies given on disjoint subsets in a contiguity space can always be extended.* $\mathsf{M}^0$ *is the coarsest extension.*

**Proof.** To prove that (1) holds, it is enough to show (by Lemma 9.1) that if $\mathsf{n}$ is $\Gamma$-near then there are $C_i \in \mathsf{c}_i$ such that $\bigcap_{i\in F} C_i^0 \in \sec \mathsf{n}$. Take an index $k \notin I$, and define $X_k = (\bigcup_{i\in F} X_i)^r$, $J = F\{\cup k\}$. By Co″2, we may assume that each $N \in \mathsf{n}$ is the subset of some $X_{j(N)}$ with $j(N) \in J$. For any $i \in F$ fixed, take a $C_i \in \mathsf{c}_i$ that meets each $N \in \mathsf{n}$ lying in $X_i$; this is possible because a subcollection of a $\Gamma$-near collection is $\Gamma$-near, a $\Gamma$-near collection in $X_i$ is $\Gamma_i$-near, and $\mathsf{c}_i$ is a $\Gamma_i$-cover. Now $\bigcap_{i\in F} C_i^0 = X_k \cup \bigcup_{i\in F} C_i$ meets each $N \in \mathsf{n}$. ◊

There is, in general, no finest compatible merotopy in a contiguity space: replace $\delta$ by $\Gamma^1(\delta)$ in Example 5.3 (if there existed a finest merotopy compatible with $\Gamma^1(\delta)$ then it would be the finest one among the merotopies compatible with $\delta$).

**9.5** Disjointness is essential in Theorem 9.4. In fact, for $n = 2, 3, \ldots$, there is a family of $n$ merotopies in a contiguity space that has no extension, although any subfamily of cardinality $n-1$ has one:

**Example.** Let $2 \leq n \in \mathbb{N}$, $Y_s = \mathbb{N} \times \{s\}$ $(1 \leq s \leq 2n)$, $I = \{1, \ldots, n\}$, $K = \{n+1, \ldots, 2n\}$, $X = \bigcup_{s=1}^{2n} Y_s$, $X_i = Y_i \cup \bigcup_{k\in K} Y_k$. Take the proximity $\delta$ on $X$ for which $A\delta B$ iff either $A \cap B \neq \emptyset$ or both $A$ and $B$ are infinite. For $i \in I$, let $\mathsf{M}^0(\delta_i) \cup \{\mathsf{d}_i\}$ be a subbase for $\mathsf{M}_i$ on $X_i$, where

$$\mathsf{d}_i = \Big\{\{(m_s, s) : s \in \{i\} \cup K\} : m_s \in \mathbb{N}, m_{n+i} < m_{n+i+1}\Big\} \cup$$

$$\cup \Big\{\bigcup_{k\in K} Y_k\Big\} \cup \{Y_i\},$$

and, in the definition of $\mathsf{d}_n$, $\mathsf{m}_{2n+1}$ is identified with $m_{n+1}$. $\mathsf{d}_i$ is a $\delta_i$-cover, thus $\delta(\mathsf{M}_i) = \delta_i$ by Lemmas 5.2 and 5.1 (because $\mathsf{M}^0(\delta_i)$ is compatible with $\delta_i$, and it has a base consisting of finite covers). If $i, j \in I$, $i \neq j$ then $X_{ij} = \bigcup_{s\in K} Y_s \in d_i$, thus

$$\mathsf{M}_i|X_{ij} = \mathsf{M}^0(\delta_i)|X_{ij} = (\mathsf{M}^0(\delta)|X_i)|X_{ij} = \mathsf{M}^0(\delta)|X_{ij} = \mathsf{M}^0(\delta|X_{ij})$$

(Lemma 5.3 c)). Hence $\{\mathsf{M}_i : i \in I\}$ is a family of merotopies in $(X, \delta)$. Define $\Gamma_i = \Gamma(\mathsf{M}_i)$. Now $\{\Gamma_i : i \in I\}$ is clearly a family of contiguities in $(X, \delta)$, so we can take the coarsest extension $\Gamma = \Gamma_0$ (Definition and Theorem 6.2). $\{\mathsf{M}_i : i \in I\}$ is a family of merotopies in $(X, \Gamma)$. We claim that $\mathsf{y} = \{Y_s : s \in I \cup K\}$ is $\Gamma$-near but $\mathsf{c} \cap \sec \mathsf{y} = \emptyset$ for $\mathsf{c} = (\bigcap_{i \in I})\mathsf{d}_i^0$ ; according to Lemmas 9.1 and 9.4, this implies that the family of merotopies cannot be extended.

To prove that $\mathsf{y}$ is $\Gamma$-near, it is enough to check that $\mathsf{f} \cap \sec \mathsf{y} \neq \emptyset$ for each $\mathsf{f} \in \Gamma$ (because this condition does not hold for $\mathsf{f} = \mathsf{y}^r$). $\mathsf{f}$ is refined by a cover $\mathsf{g}(\cap)(\bigcap_{i \in I})\mathsf{f}_i^0$ with $\mathsf{g} \in \Gamma^0(\delta)$ and $\mathsf{f}_i \in \Gamma_i$ (see the definition of $\Gamma^0$; it is enough to take only one $\mathsf{f}_i$ from each $\Gamma_i$, because the operations $(\cap)$ and $^0$ commute). $\mathsf{f}_i \in \mathsf{M}_i$, thus there is a finite $\mathsf{g}_i \in \mathsf{M}^0(\delta_i)$, i.e. a $\mathsf{g}_i \in \Gamma^0(\delta_i)$, such that $\mathsf{g}_i(\cap)\mathsf{d}_i$ refines $\mathsf{f}_i$. If $A\bar{\delta}B$ then either $A$ or $B$ is finite, thus $\mathsf{c}_{A,B}$ contains a cofinite set; hence there is a cofinite $H \in \mathsf{g}$ (see the definition of $\Gamma^0(\delta)$). Similarly, there are sets $H_i \in \mathsf{g}_i$ cofinite in $X_i$. Pick a $\nu \in \mathbb{N}$ such that

$$H^r \cup \bigcup_{i \in I}(X_i \backslash H_i) \subset \{1, \ldots, \nu\} \times (I \cup K).$$

Consider the sets

$$D_1(\mu) = \{(\nu+1, 1)\} \cup \{(\mu+k, k) : k \in K\} \in \mathsf{d}_1 \quad (\mu > \nu).$$

$D_1(\mu) \subset H_1 \in \mathsf{g}_1$, thus $D_1(\mu) \in \mathsf{g}_1(\cap)\mathsf{d}_1$. As this cover refines the finite $\mathsf{f}_1$, there are $\mu > \nu, \eta > \mu+n$ and $E_1 \in \mathsf{f}_1$ such that $D_1(\mu), D_1(\eta) \subset E_1$. For $1 \neq i \in I$, define

$$D_i = \{(\nu+1, i), (\eta+n+1)\} \cup \{(\mu+k, k) : n+1 \neq k \in K\}.$$

$D_i \in \mathsf{d}_i$, and also $D_i \subset H_i \in \mathsf{g}_i$, thus $D_i \in \mathsf{g}_i(\cap)\mathsf{d}_i$ ; hence $D_i \subset E_i$ with some $F_i \in \mathsf{f}_i$. Now

$$(\nu+1, 1), \ldots, (\nu+1, n), (\eta+n+1, n+1), (\mu+n+2, n+2), \ldots,$$
$$\ldots (\mu+2n, 2n) \in H \cap \bigcap_{i \in I} E_i^0 \in \mathsf{g}(\cap)(\bigcap_{i \in I})\mathsf{f}_i^0.$$

So there is indeed an element of $\mathsf{f}$ meeting each $Y_s$.

On the other hand, $\mathsf{c} \cap \sec \mathsf{y} = \emptyset$ is evident: any $C \in \mathsf{c}$ is of the form $\bigcap_{i \in I} D_i^0$ with suitable $D_i \in \mathsf{d}_i$; if $D_i = \bigcup_{k \in K} Y_k$ for some $i$ then $C \cap Y_i = \emptyset$; if $D_i = Y_i$ for some $i$ then $C \cap Y_k = \emptyset (k \in K)$; otherwise, $C \cap Y_k \neq \emptyset$ $(k \in K)$ would lead to $n$ inequalities that cannot hold at the same time.

So we have proved that the merotopies cannot be extended. Any $n-1$ have, however, an extension; for reasons of symmetry, it is enough to show that this holds for $\mathsf{M}_1, \ldots, \mathsf{M}_{n-1}$, i.e. that, with $I_0$ denoting $\{1, \ldots, n-1\}, \mathsf{b} = (\bigcap_{i \in I_0})\mathsf{c}_i^0$ is a $\Gamma$-cover if $\mathsf{c}_i \in \mathsf{M}_i$ (Lemma

9.4). $\mathsf{c}_i$ is refined by $\mathsf{f}_i(\cap)\mathsf{d}_i$ with some finite $\mathsf{f}_i \in \mathsf{M}_i$ ; therefore $(\bigcap_{i\in I_0})\mathsf{f}_i^0(\cap)(\bigcap_{i\in I_0})\mathsf{d}_i^0$ refines b. The covers $\mathsf{f}_i^0$ and $\mathsf{d}_i^0$ are $\Gamma$-covers by Theorem and Lemma 9.4 (applied to $\mathsf{M}_i$). $\mathsf{f}_i^0$ being finite, it belongs to $\Gamma$, so we have only to prove that $(\bigcap_{i\in I_0})\mathsf{d}_i^0$ is a $\Gamma$-cover as well (Lemma 9.3).

Let $\mathsf{n}$ be $\Gamma$-near; sets $D_i \in \mathsf{d}_i$ have to be chosen such that $\bigcap_{i\in I_0} D_i^0 \in \sec \mathsf{n}$ (Lemma 9.1). According to Co2$''$ we may assume that for $N \in \mathsf{n}, N \subset Y_{s(N)}$ with some $s(N) \in I \cup K$ . Consider the set $S = \{s(N) : N \in \mathsf{n}\}$ of indices. If $S \cap I_0 = \emptyset$ or $S \cap K = \emptyset$ then $D_i = \bigcup_{k\in K} Y_k$ $(i \in I_0)$, respectively $D_i = Y_i$ $(i \in \in I_0)$ will do. So we may assume that $S \cap I_0 \neq \emptyset \neq S \cap K$ . Define

$$Z_s = \bigcap\{N \in \mathsf{n} : N \subset Y_s\} \quad (s \in S).$$

We claim that $Z_s \neq \emptyset$.

Indeed, let $j = s$ if $s \in I$, and $j \in S \cap I$ arbitrary if $s \in K$; $\mathsf{d}_j^0$ being a $\Gamma$-cover, there is an $E_j \in \mathsf{d}_j$ with $E_j^0 \in \sec \mathsf{n}$. Clearly $Y_j \neq \neq E_j \neq \bigcup_{k\in K} Y_k$ (as $E_j$ has to meet both sets). Hence $E_j$ (so also $E_j^0$) meets $Y_s$ in a single point, which lies necessarily in $Z_s$. We can deduce from this that $Z_s$ is in fact infinite for $s \in S \cap K$ :

Assume it is finite, and apply Co2$''$with $\mathsf{a}(N) = \{Z_s, N\backslash Z_s\}$ for $N \subset Y_s$ and $\mathsf{a}(N) = \{N\}$ otherwise. $A(N) = Z_s$ is impossible, since $\mathsf{c}^* = \{Z_s, Z_s^r\} \in \Gamma^0(\delta) \subset \Gamma$, so it is a $\Gamma$ -cover; but $Z_s^r \cap A(N) = = Z_s^r \cap Z_s = \emptyset$, and $Z_s \cap A(\mathsf{M}) = Z_s \cap \mathsf{M} = \emptyset$ for $\mathsf{M} \subset Y_i$ , $\mathsf{M} \in \mathsf{n}$, $i \in S \cap I$ (there is such an $\mathsf{M}$ because $S \cap I_0 \neq \emptyset$); hence $\mathsf{c}^* \cap \sec\{A(N) : N \in \mathsf{n}\} = \emptyset$, contradicting Lemma 9.1. Therefore $A(N) = N\backslash Z_s$ for $N \subset Y_s$, and the result of the foregoing paragraph, applied to $\{A(N) : N \in \mathsf{n}\}$ instead of $\mathsf{n}$, yields $\bigcap\{N\backslash Z_s : N \in \mathsf{n}, N \subset Y_s\} \neq \emptyset$, a contradiction.

Pick now points $x_s = (\mu_s, s)$ for $s \in S \cup K$ such that $x_s \in Z_s$ if $s \in S$, and $\mu_s < \mu_{s+1}$ if $2n \neq s \in K$. The requires sets $D_i$ $(i \in I_0)$ can be defined as follows: $D_i = \bigcup_{k\in K} Y_k$ if $i \notin S$ ; $D_i = \{x_s : s \in \{i\} \cup K\}$ if $i \in S$. $\diamondsuit$

## B. RIESZ MEROTOPIES IN A CONTIGUITY SPACE

**9.6 Lemma.** *A family of merotopies in a contiguity space has a Riesz extension iff the trace filters are Cauchy, the contiguity is Riesz, and 9.4 (1) holds; if so then* $\mathsf{M}^0$ *is the coarsest Riesz extension.*

**Proof.** The necessity of the conditions is clear. Conversely, if they are fulfilled then $\mathsf{M}^0$ is an extension by Lemma 9.4, and it is Riesz, since $\operatorname{int}\mathsf{c}$ is a cover whenever $\mathsf{c} \in \mathsf{B}$ (for $\mathsf{c} \in \Gamma$ because $\Gamma$ is Riesz, for $\mathsf{c}_i^0$ because the trace filters are Cauchy). ◇

**Theorem.** *A family of merotopies given on disjoint subset in a Riesz contiguity space has a Riesz extension iff the trace filters are Cauchy.* ◇

## C. LODATO MEROTOPIES IN A CONTIGUITY SPACE

**9.7** If a family of merotopies in a contiguity space has a Lodato extension then the contiguity and the merotopies are Lodato, (ii) and (iii) from Lemma 5.17 hold, as well as 9.4 (1). We shall see that these conditions are sufficient if "$\Gamma$-cover" is substituted for "$\delta$-cover" in (ii) (and then 9.4 (1) is superfluous), but not otherwise.

**Definition.** For a family of Lodato merotopies in a Lodato contiguity space with Cauchy trace filters, let $\{\operatorname{int}\mathsf{c} : \mathsf{c} \in \mathsf{B}\}$ be a subbase for $\mathsf{M}_L^0$ (with $\mathsf{B}$ from Definition 9.4). ◇

Cf. Definition 5.14. $\{\operatorname{int}\mathsf{c} : \mathsf{c} \in \mathsf{M}^0\}$ is a base for $\mathsf{M}_L^0$ ; the following covers form a subbase $\mathsf{B}_L$ for $\mathsf{M}_L^0$ : the open elements of $\Gamma$ , and $\operatorname{int}\mathsf{c}_i^0$ $(i \in I, \mathsf{c}_i \in \mathsf{M}_i, \mathsf{c}_i$ is $c_i$-open). $\mathsf{M}_L^0$ is a Lodato merotopy compatible with $c$ (just like in Lemma 5.14).

**Lemma.** *A family of merotopies in a contiguity space has a Lodato extension iff*

(i) *the contiguity and the merotopies are Lodato;*

(ii) $(\bigcap_{i\in F})\operatorname{int}\mathsf{c}_i^0$ *is a $\Gamma$-cover whenever $\emptyset \neq F \subset I$ is finite and* $\mathsf{c}_i \in \mathsf{M}_i \quad (i \in F)$;

(iii) $(\operatorname{int}\mathsf{c}_i^0)|X_j \in \mathsf{M}_j \quad (i, j \in I, \mathsf{c}_i \in \mathsf{M}_i)$.

*If these conditions are satisfied then $\mathsf{M}_L^0$ is the coarsest Lodato extension.*

**Remark.** See Remarks 5.17.

**Proof.** 1° *Necessity.* (i) is clear. If $\mathsf{M}$ is a Lodato extension then $\mathsf{c}_i^0 \in \mathsf{M}$ and $\operatorname{int}\mathsf{c}_i^0 \in \mathsf{M}$, implying $\mathsf{c} = (\bigcap_{i\in F})\operatorname{int}\mathsf{c}_i^0 \in \mathsf{M}$, thus $\mathsf{c}$ is a $\Gamma$-cover by Lemma 9.2. (iii) follows from Theorem 3.6.

2° *Sufficiency.* We are going to show that $\mathsf{M}_L^0$ is a Lodato extension (the conditions in its definition are satisfied, since the Cauchy property follows from (ii)). $\mathsf{M}^0$ is an extension by Lemma 9.4 (as 9.4 (1) follows from (ii)). $\mathsf{M}^0 \subset \mathsf{M}_L^0$ , so $\Gamma(\mathsf{M}_L^0) \supset \Gamma$ and $\mathsf{M}_L^0|X_i \supset \mathsf{M}_i$. It follows from (ii) and Lemma 9.3 that the elements of $\mathsf{B}_L$ are $\Gamma$-covers; hence

$\Gamma(\mathsf{M}_L^0) \subset \Gamma$ (Lemma 9.2). $\mathsf{B}_L|X_i \subset \mathsf{M}_i$ (for int $\mathsf{c}_j^0$ by (iii), for the others by the compatibility), thus $\mathsf{M}_L^0|X_i \subset \mathsf{M}_i$, too. $\mathsf{M}_L^0$ is clearly Lodato; it is the coarsest one, see Remark 7.4. ◊

The following three weaker conditions together cannot stand in lieu of (ii): 9.4 (1), 5.7 (ii), and each int $\mathsf{c}_i^0$ is a $\Gamma$-cover (Example a) below). Condition (iii) cannot be dropped either (Example b)).

**Examples.** a) (A modification of Example 5.20.) Let $T = \{-1/n, 1/n : n \in \mathbb{N}\}, X = T\times]-1,1[, X_0 = T\times]-1,0[, X_1 = T\times]0,1[$, and take the Euclidean contiguity $\Gamma$ on $X$, i.e. the one induced by the Euclidean merotopy (whose definition was given at the end of Example 3.8). Denoting the Euclidean closure on $\mathbb{R}^2$ by $\mathsf{c}^*$, $\mathsf{n}$ is $\Gamma$-near iff $\bigcap_{N\in\mathsf{n}} \mathsf{c}^*(N) \neq \emptyset$ (because $X$ is bounded in $\mathbb{R}^2$). Let $\{\mathsf{c}_i(\varepsilon) : 0, \varepsilon \leq 1\}$ be a base for $\mathsf{M}_i$ on $X_i$, where

$$\mathsf{c}_1(\varepsilon) = \{(]p,p+\varepsilon[\times]q,q+\varepsilon[)\cap X_1 : (p\in\mathbb{R}, q>0) \text{ or } (0\notin]p,p+\varepsilon[, q=0)\}\cup$$

$$\cup\{C_1(k,n) : k,n\in\mathbb{N},\ k,n>1/\varepsilon\}\cup\{D_1(\varepsilon',\varepsilon'') : 0<\varepsilon''<\varepsilon'<\varepsilon\},$$

$$C_1(k,n) = \{-1/k, 1/n\}\times]0,\varepsilon[,$$

$$D_1(\varepsilon',\varepsilon'') = ((]-\varepsilon'',0[\cup]\varepsilon'',\varepsilon[)\times]0,\varepsilon[)\cap X_1,$$

$$\mathsf{c}_0(\varepsilon) = \{-C_1 : C_1 \in \mathsf{c}_1(\varepsilon)\},\ -C_1 = \{(-p,-q) : (p,q)\in C_1\}.$$

$\mathsf{M}_1$ is compatible, because if $\mathsf{f}$ is a finite cover refined by $\mathsf{c}_1(\varepsilon)$ then there is an $E \in \mathsf{f}$ that contains infinitely many of the sets

$$(]-\varepsilon/2,\varepsilon/2[\times]1/m, 1/m+\varepsilon[)\cap X_1 \qquad (m\in\mathbb{N}),$$

therefore $Q(\varepsilon) = (]-\varepsilon/2,\varepsilon/2[\times]0,\varepsilon[)\cap X_1 \subset E$, i.e. the merotopy $\mathsf{N}_1$ with the base $\{\mathsf{c}_1(\varepsilon)\cup\{Q(\varepsilon)\} : 0<\varepsilon<1\}$ induces the same contiguity as $\mathsf{M}_1$ ; one can, however, easily see that $\mathsf{N}_1$ is the Euclidean merotopy on $X_1$. $\mathsf{c}_1(\varepsilon)$ is $c_1$-open, thus $\mathsf{M}_1$ is Lodato. Analogously, $\mathsf{M}_0$ is compatible and Lodato, too. The merotopies are evidently accordant; (iii) holds bacause int $\mathsf{c}_i(\varepsilon)^0|X_{1-i} = \{X_{1-i}\}$.

To check that $\mathsf{c} = \text{int}\, \mathsf{c}_i(\varepsilon)^0$ is a $\Gamma$-cover, take a near collection $\mathsf{n}$; we need a $C \in \mathsf{c}\cap \sec \mathsf{n}$. Pick a $z \in \bigcap_{N\in\mathsf{n}} \mathsf{c}^*(N)$. If $z=(0,0)$ then $C = \text{int}\, D_i(\varepsilon',\varepsilon'')^0$ will do with suitable $\varepsilon'$ and $\varepsilon''$ ; the other cases are trivial. $\mathsf{c}_0(\varepsilon)^0\ (\cap)\ \mathsf{c}_1(\varepsilon)^0$ is a $\Gamma$-cover by Lemma and Theorem 9.4. The sets $C_1(k,n)$ and $-C_1(n,k)$ guarantee that int $\mathsf{c}_0(\varepsilon)^0\ (\cap)$ int $\mathsf{c}_1(\varepsilon)^0$ is a $\delta$-cover. Thus all the three weaker versions of (ii) are fulfilled.

Nevertheless, (ii) fails for $\mathsf{c}_0(1)$ and $\mathsf{c}_1(1)$: take $\mathsf{n} = \{N_1, N_2, N_3\}$,

$$N_1 = \{(-1/n, 0) : n \in \mathbb{N}\}, N_2 = \{(1/2n, 0) : n \in \mathbb{N}\},$$
$$N_3 = \{(1/(2n+1), 0) : n \in \mathbb{N}\}.$$

b) (A modification of Example 5.19.) Let $X, X_0, X_1$ and $\mathsf{M}_0$ be as in Example 3.8, but replace $\mathsf{c}_1(\varepsilon)$ in the definition of $\mathsf{M}_1$ by

$$\mathsf{d}_1(\varepsilon) = \mathsf{c}_1(\varepsilon) \cup \{(H \times ]0, \varepsilon[) \cap X_1 : H \subset ]0, \varepsilon[ \text{ is finite }\}.$$

(In 5.19, we did the same with $|H| = 2$.) $\{\mathsf{M}_0, \mathsf{M}_1\}$ is a family of Lodato merotopies in the Euclidean contiguity space on $X$; the modification was needed to make $\mathsf{M}_1$ compatible. (ii) holds, but (iii) fails, just like in 5.19. (Use Lemma 9.3 instead of Lemma 5.2.) ♢

**9.8 Corollary.** *A single Lodato merotopy* $\mathsf{M}_0$ *in a Lodato contiguity space has a Lodato extension iff* $\operatorname{int} \mathsf{c}_0^0$ *is a* $\Gamma$*-cover whenever* $\mathsf{c}_0 \in \mathsf{M}_0$. ♢

It is not enough to assume that the trace filters are Cauchy, not even when $X_0$ is open (take Example 5.18 b) with the Euclidean contiguity on $X$). In fact, the condition that $\operatorname{int} \mathsf{c}_0^0$ is a $\delta(\Gamma)$-cover (i.e. that there is a Lodato extension in $(X, \delta(\Gamma))$) is not sufficient either:
**Example.** With $X, X_1, \mathsf{M}_1$ from Example 5.19 a), and the Euclidean contiguity $\Gamma$ on $X$, $\operatorname{int} \mathsf{d}_1(\varepsilon)^0$ is a $\delta(\Gamma)$-cover ($\delta(\Gamma)$ is the same as $\delta$ in 5.19 a)), but it is not a $\Gamma$-cover (let $\mathsf{n}$ consist of three disjoint infinite subsets of $X_1^r$). ♢

**9.9 Theorem.** *A family of Lodato merotopies given on disjoint closed subsets in a Lodato contiguity space always has Lodato extensions;* $\mathsf{M}^0 = \mathsf{M}_L^0$ *is the coarsest one.*
**Proof.** $\mathsf{M}^0$ is an extension by Theorem 9.4. For any $c_i$-open $\mathsf{c}_i$, $\operatorname{int} \mathsf{c}_i^0 = \mathsf{c}_i^0$, thus $\mathsf{M}^0 = \mathsf{M}_L^0$; $\mathsf{M}_L^0$ is always Lodato. ♢

Example 9.5 shows that the statement of this theorem is false for intersecting sets, even for open-closed ones. $\mathsf{M}^0$ and $\mathsf{M}_L^0$ can be different in general; e.g. with $\{\mathsf{M}_1\}$ from Example 9.7 b), $\operatorname{int} \mathsf{c}_1(1)^0 \notin \mathsf{M}^0$.

## References

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