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LOCALIZING FAMILIES FOR REAL FUNCTION ALGEBRAS

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Abstract: Let A be a real function algebra on (X,σ). A cover $\mathcal{R}$ of X by closed sets localizes A if from $f \in C(X,\sigma)$ and $f|_R \in A|_R^-$ for each $R \in \mathcal{R}$, it follows $f \in A$. Examples of such covers and some relations between them are given.

For a compact Hausdorff space X and a homeomorphism $\sigma : X \to X$, $\sigma \circ \sigma = \mathrm{id}$, $C(X,\sigma)$ is a real space of all complex continuous functions on X fulfilling $f(\sigma x) = \overline{f(x)}$ ([5]).

Let A be a real function algebra on (X,σ), i.e. a subalgebra of $C(X,\sigma)$ which is uniformly closed, separates points of X and contains real constants ([5]).

The well known Bishop theorem states that every uniform algebra A can be obtained by "gluing together" a family of antisymmetric algebras. In a sense, the class of antisymmetric algebras determines (forms a basis for) the class of uniform algebras. This idea of forming an algebra from more elementary "bricks" was precised by Arenson [1]. Following him we will define analogous notions for real function algebras.

Let $\mathcal{A}$ denote the class of all real function algebras (over all pairs

$(X,\sigma))$. For $A \in \mathcal{A}$, say $A \subseteq C(X,\sigma), R$ a closed subset of X, let $A|_R = \{ g|_R : g \in A\}$ and $A|_R^- :=$ the uniform closure of $A|_R$.

Definition 1. Let $\mathcal{R}$ be a cover of X by closed sets, $A \in \mathcal{A}, A \subseteq C(X,\sigma)$. We say that $\mathcal{R}$ *localizes* A if the conditions:

$$f \in C(X,\sigma) \text{ and } f|_R \in A|_R^- \text{ for each } R \in \mathcal{R}, \text{ imply } f \in A.$$

Definition 2. A subclass $\mathcal{B} \subseteq \mathcal{A}$ is called *basic* if for every $A \in \mathcal{A}$, say $A \subseteq C(X,\sigma)$, there exists a cover $\mathcal{R}$ of X such that
(i) $\mathcal{R}$ localizes A;
(ii) for $R \in \mathcal{R}, A|_R^- \in \mathcal{B}$.

More picturesquely, every $A \in \mathcal{A}$ can be obtained from algebras belonging to $\mathcal{B}$ by "gluing" them together in a specified way.

The Bishop theorem states then that the family of all maximal antisymmetric sets localizes A. Following [1], we will denote this family $\mathcal{R}_1$.

We will remind the definitions for real function algebras.

Definition 3. [6] Let A be a real function algebra on (X,σ). A nonempty subset R of X is called a *set of antisymmetry* if:
(i) $f \in A$ and $f|_R$ is real implies $f|_R$ is constant, and
(ii) $f \in A$ and $f|_R$ is purely imaginary implies $f|_R$ is constant.

Definition 4. [2] Let A be a real function algebra on (X,σ). A nonempty subset R of X is called a *set of r-antisymmetry* if:
(i) $f \in A$ and $f|_R$ is real implies $f|_R$ is constant, and
(ii) R is σ-invariant, i.e. $\sigma(R) = R$.

Note that if a set is σ-invariant then a function with nonzero imaginary part cannot be constant on it. It follows for example, that if $A = C(X,\sigma)$ then the only sets which are both antisymmetric and r-antisymmetric are the singleton fixpoints. So in general the notions of antisymmetric and r-antisymmetric sets are different.

In [2], Cor. 2.5. it was proved that if $A|_R$ is an algebra of real type (see [3]) then R is r-antisymmetric iff R is a set of antisymmetry for the complex algebra $A + iA$. From this fact and from [6], Lemma 2.12 and Th. 2.15 it follows that:

If $A|_R$ is an algebra of real type and $\sigma(R) = R$, then R is antisymmetric iff R is r-antisymmetric.

In general, if R is antisymmetric set for A, then $R \cup \sigma(R)$ is r-antisymmetric. We will soon use this fact.

From the analogue of Bishop theorem for real function algebras (see [6], Cor. 3.4. and Th. 3.6.), the cover $\mathcal{R}_1$ of X by maximal antisymmetric sets localizes A. The problem is, which other covers localize A, or, equivalently, which subclasses $\mathcal{B} \subseteq \mathcal{A}$ are basic.

It is not difficult to prove that the cover $\mathcal{R}_1{}'$ by maximal r-antisymmetric sets localizes A. To this end let us show first:

Proposition 5. *Let A be a real function algebra on (X, σ) and let R be a maximal r-antisymmetric set for A. Then $A|_R$ is closed in $C(R, \sigma|_R)$.*

Proof. Consider two cases. First, if A is of complex type then R is maximal antisymmetric for a complex algebra A', where A' means A with the multiplication extended to complex scalars. Second, if A is of real type then R is maximal antisymmetric for a complexification $B = A + +iA$. In both cases the restriction algebras $A'|_R$ and $B|_R$ are closed in $C(R)$. Taking into account suitable inclusions it is easy to see that $A|_R$ is closed in $C(R, \sigma|_R)$. $\diamondsuit$

Now, Th. 3.3 from [6] (Machado theorem for real function algebras) states that for any $f \in C(X, \sigma)$ its distance from A is realized on some closed antisymmetric subset Y of X. Hence this distance is realized also on a r-antisymmetric set $Y \cup \sigma(Y)$ and repeating the proof of Cor. 3.4 in [6] we can show that the cover $\mathcal{R}_1{}'$ localizes A.

Let us consider other natural covers.

Definition 6. A closed set $F \subset X$ is a *peak set* for real function algebra $A \subset C(X, \sigma)$ if there exists $f \in A$ with $f = 1$ on F and $|f| < 1$ off of F. A closed set $E \subset X$ is a *weak peak set (p – set)* for A if E is an intersection of peak sets. If a function f equals 1 on a set (not necessarily closed) F and $|f| \leq 1$ off of F then we will say that f *peaks* on F.

Note that for any peak set, $F = \sigma(F)$ and that the countable intersection of peak sets is a peak set.

Definition 7. A real function algebra A on (X, σ) is called an *analytic (a weakly analytic) algebra* if from the fact that $f \in A$ and f is constant (f peaks) on an open subset of X it follows that f is constant on X.

It is clear that if A is analytic then it is weakly analytic.

We will call a closed set $R \subseteq X$ *(weakly) analytic* if the uniform closure $A|_R{}^{-}$ of the algebra $A|_R$ is (weakly) analytic. This means that any subset of R which is also a peak set (in weakly analytic case), or a set of constancy (in analytic case) for some $f \in A|_R$ is nowhere dense in R or coincides with R.

This definition is the same as for uniform algebras (see [1]). In the case of uniform algebras it is known that these types of algebras: antisymmetric, analytic and weakly analytic are all different. In [1] it is also proved that $\mathcal{R}_2 =$ the family of all weakly analytic sets, localizes A, while the family of all analytic sets does not.

Lemma 8. *If a set $F \subset X$ is weakly analytic, then it is antisymmetric.*

Proof. Let $f \in A$ be such a function that $f|_F$ is real. Suppose that $f|_F$ is not constant. Then the set $P(f)$ defined as the closure of the set of all polynomials of $f|_F$ contains a function g (defined on F) such that $\|g\| = 1, g \neq 1$ and $g^{-1}(1)$ contains a set which is open in F. This is impossible because F is weakly analytic. $\diamondsuit$

An easy consequence of this lemma is

Theorem 9. *If A is an analytic (weakly analytic) algebra, then it is also antisymmetric.*

From the lemma above, $\mathcal{R}_2 \subset \mathcal{R}_1$. We are going to show that $\mathcal{R}_2$ localizes A. First we define two smaller then $\mathcal{R}_2$ families of sets.

Given a probability measure ν on X we will consider A as a subspace in $L^p(\nu), 1 \leq p < \infty$ and denote its closure as $H^p(\nu)$. Also we define $H^\infty(\nu) = H^1(\nu) \cap L^\infty(\nu)$.

Definition 10. A probability measure ν is called an *antisymmetric measure* if every function in $H^\infty(\nu)$ that is real valued a.e. is constant a.e.

Let $\mathcal{R}_3$ denote the family of supports of antisymmetric measures.

Lemma 11. *The support of any antisymmetric measure is weakly analytic set.*

Proof. Let F be the support of any antisymmetric measure ν, let $f \in (A|_F)^{\bar{}}, \|f\| = 1$. Then $f \in H^\infty(\nu)$. If $G \subseteq f^{-1}(1)$ is open in F, then $\nu(G) > 0$ (because $F = \operatorname{supp} \nu$). It is easy to show that the sequence $((1+f)/2)^n$ converges a.e. to the characteristic function χ_H for some $H \supseteq G$. Since the measure is antisymmetric χ_H must be constant. $\diamondsuit$

It follows that $\mathcal{R}_3 \subset \mathcal{R}_2$.

Before defining the next cover we will remind some known facts.

Let $M(X, \sigma)$ be the set of all Radon self – conjugate measures on X, i.e.:

$$M(X, \sigma) = \{\mu \in M(X) : \mu = \overline{\mu} \circ \sigma\},$$

where $M(X)$ is the set of all Radon (= regular Borel) measures on X. We have:

Theorem 12. (Riesz type, [2]). *The mapping L defined by*

$$(L\mu)(f) = \int f d\mu \text{ for } \mu \in M(X,\sigma), f \in C(X,\sigma),$$

is a linear isometry from $M(X,\sigma)$ onto $C(X,\sigma)^*$.

Definition 13. ([2]) Let E be a subspace of $C(X,\sigma)$. A measure $\mu \in M(X,\sigma)$ is said to *annihilate (be orthogonal to) the subspace* E (in symbols $\mu \perp E$) if the functional F_μ represented by this measure fulfills the condition $F_\mu(f) = 0$ for every $f \in E$. The *annihilator* of $E, E^\perp$, is defined as the set of all measures orthogonal to E.

Definition 14. A Radon self-conjugate measure μ is an *extreme annihilating measure* for E if μ is an extreme point of the unit ball of $E^\perp, \mu \in \text{ext}B(E^\perp)$.

It is easy to prove ([2]) that if μ is an extreme annihilating measure then supp μ is an antisymmetric set.

Let $\mathcal{R}_4$ be the family of all supports of extreme annihilating measures along with all singleton subsets of X. The family $\mathcal{R}_4$ localizes A. (Let $f \in C(X,\sigma)$ be such that $f|_R$ belongs to $\overline{(A|_R)}$ for any $R \in \mathcal{R}_4$. From the Krein – Milman theorem, any $\mu \in B(A^\perp)$ annihilates f. Suppose that $f \notin A$. Then from the Hahn – Banach theorem there exists $\mu \in B(A^\perp), \mu(f) = 1$, a contradiction.)

We are going to show that $\mathcal{R}_4 \subseteq \mathcal{R}_3$. Let $\mu \in \text{ext}B(A^\perp)$. It suffices to show that $|\mu|$ is antisymmetric. Let $f \in H^\infty(|\mu|)$ be a real valued function. If $\varepsilon > 0$ is sufficiently small then $h = (1/2) + \varepsilon f$ fulfills $0 < h < 1$ and obviously $h\mu \in A^\perp$. We have

$$\mu = \|h\mu\| \frac{h\mu}{\|h\mu\|} + \|(1-h)\mu\| \frac{(1-h)\mu}{\|(1-h)\mu\|}.$$

But $\mu \in \text{ext}B(A^\perp)$, hence $h\mu = \|h\mu\|\mu$. It follows that $h = \|h\mu\|$ a.e., so f is constant.

Since $\mathcal{R}_4 \subseteq \mathcal{R}_3 \subseteq \mathcal{R}_2 \subseteq \mathcal{R}_1$ and $\mathcal{R}_4$ localizes A, hence each of $\mathcal{R}_i, i = 1,2,3,4$ does so.

We are now going to investigate the problem whether the natural cover of X consisting of supports of real part representing measures localizes A.

Lemma 15. *Let μ be a probability measure. Then μ is antisymmetric iff for every Borel set F such that $\chi_F \in H^\infty(\mu), \mu(F) = 0$ or $\mu(F) = 1$.*

Proof. The necessity is obvious. To prove sufficiency, let $f \in H^\infty(\mu)$ be a real function, $a = \text{ess inf} f(x), b = \text{ess sup} f(x)$. We are going to show that $a = b$. Take (P_n), a sequence of polynomials with real coefficients which is point convergent on $[a, b]$ to the characteristic function of $[a, (a+b)/2]$, and such that $\max_{t \in [a,b]} |P_n(t)| \leq 1$. Then $P_n(f)$ is a sequence of functions from $H^\infty(\mu), \|P_n(f)\| \leq 1, \lim P_n(f) = 1$ on a set $F := f^{-1}([a, (a+b)/2])$. This sequence has a subsequence which is w^*-convergent to some $g \in H^\infty(\mu)$. But $g \equiv 1$ on a set F and $\|g\| \leq 1$, so it is easy to see that $((1+g)/2)^n$ converges a.e. to χ_F. Hence $\chi_F \in H^\infty(\mu)$ and from the assumption $\mu(F) = 1$. It follows $(a+b)/2 = b$, so $a = b$. ◇

Recall (see [5]) that a probability measure μ is called a *real part representing measure* for $\phi \in \Phi_A$ (Φ_A denotes the *carrier space* of an algebra A) if:

- for all $f \in A, \int \text{Re } f \, d\mu = \text{Re}\, \phi(f)$, and
- for every Borel set $E, \mu(E) = \mu(\sigma E)$.

Remark 16. Note that the measure μ is multiplicative on Re $A \cap A$, since for $f \in \text{Re } a \cap A, \int f \, d\mu = \int \text{Re } f \, d\mu = \text{Re } \phi(f) = \phi(f)$. The last equality follows from the general fact that if an algebra B is of strictly real type, $\mathcal{R}_4$, then for every $\phi \in \Phi_B, \phi(f) \in \mathbb{R}$ for $f \in B$ - see [3] for details. It is obvious that Re $A \cap A$ is of R_4 type.

Lemma 17. *If μ is a real part representing measure for a homomorphism $\phi \in \Phi_A$ then it is antisymmetric.*

Proof. Take any real function $f \in H^\infty(\mu)$. We have to show that f is constant. By Remark 16 μ is multiplicative on the $L^1(\mu)$ – closure of Re $A \cap A$; we will denote this closure $H^1(\mu)^r$. Now take a Borel set F such that $\chi_F \in H^\infty(\mu)$. Then $\chi_F \in H^1(\mu)^r$, so $\mu(F)^2 = \mu(\chi_F)^2 = = \mu(\chi_F^2) = \mu(\chi_f) = \mu(F)$. Hence $\mu(F) = 0$ or $\mu(F) = 1$. From the preceding lemma, μ is antisymmetric. ◇

Let $\mathcal{S}'$ denote the cover of X by supports of real part representing measures. From the above lemma, $\mathcal{S}' \subseteq \mathcal{R}_3$. If $\mathcal{R}_4$ had been a subfamily of $\mathcal{S}'$,we would have known that $\mathcal{S}'$ localizes X. But $\mathcal{S}'$ cannot contain $\mathcal{R}_4$ because $\mathcal{S}'$ consists of σ- invariant sets only. In order to have a localizing family we will add to $\mathcal{S}'$ some other sets.

Definition 18. Let Y be any subset of X. If a set Y_σ fulfills $Y_\sigma \cup \cup \sigma(Y_\sigma) = Y$, we will call Y_σ a σ - *generating subset* for Y. If moreover, Y_σ does not contain any Z_σ with $Z_\sigma \cup \sigma(Z_\sigma) = Y$, we will say that Y_σ is *a minimal σ -generating subset* for Y.

Of course Y is σ-generating for itself.

Let now $\mathcal{S} = \{Y_\sigma : Y \in \mathcal{S}'\}$. We will prove that the family $\mathcal{S}$ localizes A if A is large enough.

Recall that there are various methods of defining a Shilov boundary of a real function algebra A. We will use the following. If A is a real function algebra on (X, σ) then $S \subseteq X$ is called a *boundary if* $S = \sigma(S)$ and if Re f assumes its maximum on S for all $f \in A$. The *Shilov boundary* $S(A)$ of A is defined as the smallest closed boundary of A.

It can be shown ([4]), Cor. 3.8) that the Shilov boundary of A coincides with the Shilov boundary of its complexification, $S(A) = = S(A + iA)$.

A complex function algebra B is said to be *relatively maximal* ([7]) if for any subalgebra B' of $C(\Phi_B)$ containing B and such that $S(B) = S(B')$ it follows $B = B'$. Following this definition we will call a real function algebra A *relatively maximal* if its complexification $B = A + iA$ is relatively maximal.

Remark 19. Let us call a real function algebra A *weakly relatively maximal* if for any subalgebra A' of $C(\Phi_A)$ containing A and such that $S(A) = S(A')$ it follows $A = A'$. It is easy to see that if A is relatively maximal then it is weakly relatively maximal. (For the proof take any $A' \supseteq A, A'$ a subalgebra of $C(\Phi_A), S(A) = S(A')$. Then $B' = A' + iA'$ is a complex function algebra, $B' \supseteq B = A + iA$ and from [4] Cor.3.8 $S(B') = S(A') = S(A) = S(B)$, so it follows $B = B'$ hence $A = A'$.) It is not clear whether the converse holds true.

Corollary 2 in [7] states that if a complex function algebra B is relatively maximal and $X = S(B)$ then the cover of X by supports of representing measures localizes A.

Theorem 20. *If a real function algebra A is relatively maximal and $X = S(A)$ then $\mathcal{S}$ localizes A.*

Proof. Let $B = A+iA$. B is relatively maximal and $S(A) = S(B) = X$ (by assumption and [4] Cor. 3.8). Hence by [7] Cor. 2 the family

$$\mathcal{U} = \{\text{supp}\mu : \mu \text{ is representing for } B\}$$

localizes B. Let μ_σ be a measure on X defined by $\mu_\sigma(E) = \mu(\sigma E)$ for all Borel subset of X and $m = (\mu + \mu_\sigma)/2$. m is a real part representing measure for A ([4], Cor. 3.4) and suppμ is a σ - generating subset for $Y = \text{supp}m$. It follows $\mathcal{U} \subseteq \mathcal{S}$ so $\mathcal{S}$ localizes B. Let $f \in C(X, \sigma), f|_S \in$

$\in A|_S^-$ for $S \in \mathcal{S}$. Then $f + if \in C(X), (f + if)|_S \in (A + iA)|_S^-$, so $f + if \in B$. Hence $f \in A$. $\diamondsuit$

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