Mathematica Pannonica
2/2 (1991), 115 – 121

TWO FIXED POINT THEOREMS FOR ABSTRACT MARKOV OPERATORS

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Received June 1990

AMS Subject Classification: 47 D 07, 60 J 05

Keywords: Fixed point, Banach lattice, Markov operator, probability distribution

Abstract: Two fixed point theorems for Markov operators on (L)-spaces are proven: first under the assumption of existence of positive lower element h for the Markov operator P, i.e. $||(P^n d-h)^-|| \to 0$ for all probability distributions d, and second under the assumption of existence of prositive upper element h for P, i.e. $||h|| <2$ and $||(P^n d-h)^+|| \to 0$ for all probability distributions d. Both of them are abstract version of Lasota-Yorke theorems for Markov operators on L^1 but their proofs are something different.

Let L be a (L)-space, i.e. Banach lattice in which the norm has the following properties

$$|a| \leq |b| \Rightarrow ||a|| \leq ||b||,$$
$$a \geq 0,\ b \geq 0 \Rightarrow ||a+b|| = ||a|| + ||b||,$$

where

$$|a| = a^+ + a^-,\ a^+ = a \vee 0,\ a^- = (-a) \vee 0.$$

Lemma 1. *If* $a, b \in L$, $a \geq 0$, $b \geq 0$, *then*

$$||a-b|| = ||a|| - ||b|| + 2||(a-b)^-|| = ||b|| - ||a|| + 2||(a-b)^+||.$$

Proof. It is true because

$$||(a-b)^+|| + ||b|| = ||(a-b) + (a-b)^- + b|| = ||a|| + ||(a-b)^-||. \diamondsuit$$

Denote by

$$L_+ = \{a \in L : a > 0\}$$

the set of all *positive elements* of L and by

$$L_p = \{d \in L_+ : ||d|| = 1\}$$

the set of all *probability distributions* of L.

A linear mapping $P : L \to L$ is called a *Markov operator* on L iff

$$P(L_p) \subset L_p.$$

Every Markov operator P on a (L)-space L has the following properties

$$Pa > 0,\ ||Pa|| = ||a|| \text{ for } a \in L_+$$
$$Pa \le Pb \text{ for } a \le b$$
$$(Pa)^+ \le Pa^+,\ (Pa)^- \le Pa^-$$
$$|Pa| \le P|a|,\ ||Pa|| \le ||a||.$$

An element $h \in L_+$ will be called a *lower element* for the Markov operator P iff

$$\lim_{n\to\infty} ||(P^n d - h)^-|| = 0 \text{ for every } d \in L_p.$$

Denote by H_0 the set of all lower elements for a Markov operator P.

Theorem 1. *If the set H_0 is nonempty, then the Markov operator P has a unique fixed probability distribution d_0. Moreover*

$$P^n d \to d_0 \textit{ for all } d \in L_p.$$

We begin the proof of this theorem with a set of lemmas.

Lemma 2. *If $h \in H_0$, then $||h|| \le 1$.*

Proof. For $d \in L_p$ and $n \in \mathrm{N}$ we have

$$h \le P^n d + (P^n d - h)^-$$

and consequently

$$||h|| \le 1 + ||(P^n d - h)^-||. \diamondsuit$$

Lemma 3. *If $h \in H_0 \cap L_p$, then h is a unique fixed probability distribution for P and $P^n d \to h$ for all $d \in L_p$.*

Proof. By Lemma 1 we have for all $d \in L_p$

$$||P^n d - h|| = 2||(P^n d - h)^-|| \to 0.$$

So

$$P^n h \to h,\ P^{n+1} h \to h$$

and by continuity of P

$$P^{n+1} h \to Ph.$$

Hence $Ph = h$. Now if $\overline{h}$ is a fixed probability distribution too, then

$$\overline{h} = P^n \overline{h} \to h.$$

Finally $\overline{h} = h$. ◇

Lemma 4. *If $h_1, h_2 \in H_0$, then $h_1 \vee h_2 \in H_0$.*

Proof. It is obvious that $h_1 \vee h_2 \in L_+$. For the proof that

$$||(P^n d - h_1 \vee h_2)^-|| \to 0$$

it is enough to verify that

$$||(d - h_1 \vee h_2)^-|| \le ||(d - h_1)^-|| + ||(d - h_2)^-||$$

for $d \in L_p$. It is true because

$$(d - h_1 \vee h_2)^- = (h_1 \vee h_2 - d) \vee 0 \le (h_1 - d) \vee 0 + (h_2 - d) \vee 0 = \\ = (d - h_1)^- + (d - h_2)^-. \ \diamond$$

Lemma 5. *If $h \in H_0$, then $Ph \in H_0$.*

Proof. It is obvious that $Ph \in L_+$. Now observe that

$$||(P^n d - Ph)^-|| \le ||P(P^{n-1} d - h)^-|| = ||(P^{n-1} d - h)^-|| \to 0. \ \diamond$$

Lemma 6. *If $h \in H_0$ and $Ph = h$, then $(2 - ||h||)h \in H_0$.*

Proof. Let $x = ||h||$ and assume that $x < 1$. For a given $d \in L_p$ consider the sequence

$$r_n = (1 - x)^{-1}(P^n d - h).$$

Since $h \in H_0$ we have $||r_n^-|| \to 0$ and (see Lemma 1)

$$||r_n|| = 1 + 2(1 - x)^{-1}||(P^n d - h)^-|| \to 1.$$

Therefore, for any given $\varepsilon > 0$ there exists $m \in \mathrm{N}$ such that

$$||r_m^-|| < \varepsilon/8,\ ||r_m|| < 1 + \varepsilon/4.$$

For

$$s = (|r_m|/||r_m||) - r_m$$

we have

$$||s|| = ||(|r_m|/||r_m||) - |r_m| + 2 \cdot r_m^-|| \leq (||r_m|| - 1) + 2||r_m^-|| < \varepsilon/2$$

and $r_m + s \in L_p$. Since $h \in H_0$ we have

$$||(P^n(r_m + s) - h)^-|| < \varepsilon/2 \text{ for } n \geq n_0(m).$$

Multiplication by $1 - x \in (0, 1)$ gives

$$||(P^{n+m}d - h + (1-x)P^n s - (1-x)h)^-|| < \varepsilon/2 \text{ for } n \geq n_0(m).$$

Consequently for $n \geq n_0(m)$

$$||(P^{n+m}d - (2-x)h)^-|| \leq \varepsilon/2 + (1-x)||P^n s|| \leq \varepsilon/2 + ||s|| < \varepsilon.$$

Finally $(2-x)h \in H_0$. ◇

Proof of Theorem 1. Let (see Lemma 2)

$$x_0 = \sup\{||h|| : h \in H_0\} > 0$$

and $\{\overline{h}_n\}$ be a sequence of lower elements such that $||\overline{h}_n|| \to x_0$. Replacing, if necessary, $\{\overline{h}_n\}$ by the sequence $\{h_n\}$ defined by

$$h_1 = \overline{h}_1,\ h_{n+1} = h_n \vee \overline{h}_{n+1},\ n \in \mathrm{N}$$

we get an increasing sequence of lower elements (see Lemma 4) such that $||h_n|| \to x_0$. Since

$$||h_m - h_n|| = ||h_m|| - ||h_n|| < \varepsilon$$

for $m \geq n \geq n_0(\varepsilon)$, there exists $h_0 \in L_+$ such that $h_n \to h_0$ and $||h_0|| = x_0$. Moreover $h_0 \in H_0$, because

$$||(P^n d - h_0)^-|| \leq ||(P^n d - h_k)^-|| + ||h_0 - h_k||$$

for all $d \in L_p$ and $n, k \in \mathrm{N}$. Finally h_0 is the largest element in H_0. Suppose it is not. Then there exists $h \in H_0$ such that the inequality $h \leq h_0$ is not true and for the lower element $\overline{h} = h \vee h_0$ we have $||\overline{h}|| > > x_0$ which is impossible. Now by Lemma 5, $Ph_0 \in H_0$ and consequently $Ph_0 \leq h_0$. Moreover $Ph_0 = h_0$, because the operator P preserves the norm on L_+. Therefore, according to Lemma 6, $(2 - x_0)h_0 \in \in H_0$. Hence $(2 - x_0)h_0 \leq h_0$ and consequently $(2 - x_0)h_0 = h_0$, because $x_0 \leq 1$ (see Lemma 2). Finally $||h_0|| = 1$ and applying Lemma 3 finishes the proof. ◇

An element $h \in L_+$ will be called an *upper element* for the Markov operator P iff $||h|| < 2$ and

$$\lim_{n\to\infty} ||(P^n d - h)^+|| = 0 \text{ for every } d \in L_p.$$

Denote by H^0 the set of all upper elements for the Markov operator P.

Theorem 2. *If the set H^0 is non-empty, then the Markov operator P has a unique fixed probability distribution d_0. Moreover $P^n d \to d_0$ for all $d \in L_p$.*

Lemma 7. *If $h \in H^0$, then $||h|| \geq 1$ and $Ph \in H^0$. If $h_1, h_2 \in H^0$, then $h_1 \wedge h_2 \in H^0$.*

The proofs of these facts are analogous to the proofs of Lemmas 2, 4 and 5. ◇

Lemma 8. *If the set H^0 is non-empty, then for all $d_1, d_2 \in L_p$*

$$\lim_{n\to\infty} ||P^n(d_1 - d_2)|| = 0.$$

Proof. Fix two arbitrary probability distributions d_1 and d_2. For $a = d_1 - d_2$ we have

$$||a^+|| = ||a^-|| = ||a||/2 = \alpha$$

because

$$||a^+|| + 1 = ||a^+ + d_2|| = ||a + a^+ + d_2|| = ||a^- + d_1|| = ||a^-|| + 1.$$

Assume for a moment that $\alpha > 0$ and $h \in H^0$. Then

$$\begin{aligned} ||P^n a|| &= \alpha||(P^n(a^+/\alpha) - h) - (P^n(a^-/\alpha) - h)|| \leq \\ &\leq \alpha(||P^n(a^+/\alpha) - h|| + ||P^n(a^-/\alpha) - h||). \end{aligned}$$

Since a^+/α, $a^-/\alpha \in L_p$ then there exists $n_1 \in \mathrm{N}$ such that

$$\begin{aligned} ||(P^{n_1}(a^+/\alpha) - h)^+|| \leq (2 - ||h||)/4, \\ ||(P^{n_1}(a^-/\alpha) - h)^+|| \leq (2 - ||h||)/4. \end{aligned}$$

Therefore, by Lemma 1

$$||P^n a|| \leq ||a||\,||h||/2.$$

For $\alpha = 0$ this inequality is obvious. Finally, for d_1, $d_2 \in L_p$ we have

$$||P^{n_1}(d_1 - d_2)|| \leq ||d_1 - d_2||\,||h||/2.$$

In the same way we can find a $n_2 \in \mathrm{N}$ such that

$$||P^{n_1+n_2}(d_1 - d_2)|| \leq ||P^{n_1} d_1 - P^{n_1} d_2||\,||h||/2 \leq ||d_1 - d_2||(||h||/2)^2$$

because P preserves the norm on L_+. After k steps we obtain

$$||P^{n_1+\dots+n_k}(d_1-d_2)|| \le ||d_1-d_2||(||h||/2)^k,$$

where $n_1,\dots,n_k$ are suitable choosen natural members. Hence

$$\lim_{n\to\infty} ||P^{n_1+\dots+n_k}(d_1-d_2)|| = 0$$

and since the sequence $\{||P^n a||\}$ is decreasing, for $a \in L$, we get

$$\lim_{n\to\infty} ||P^n(d_1-d_2)|| = 0. \ \diamondsuit$$

Proof of Theorem 2. For $h \in H^0$ we define the decreasing sequence $\{h_n\}$ of upper elements (see Lemma 7) by

$$h_1 = h,\ h_{n+1} = h_n \wedge Ph_n,\ n \in \mathbb{N}.$$

Since the sequence $\{||h_n||\}$ is decreasing and bounded, and

$$||h_m - h_n|| = ||h_m|| - ||h_n|| < \varepsilon$$

for $m \ge n \ge n_0(\varepsilon)$, there exists $h^0 \in L^+$ such that $h_n \to h^0$ and $||h^0|| < 2$. Moreover $h^0 \in H^0$, because

$$||(P^n d - h)^+|| \le ||(P^n d - h_k)^+|| + ||h_k - h^0||$$

for all $d \in L_p$ and $n,k \in \mathbb{N}$. An element h^0 is a fixed point of P, because from inequality $h_{n+1} \le Ph_n$ we have $h^0 \le Ph^0$ and because $||Ph^0|| = ||h^0||$ the inequality $h^0 < Ph^0$ is impossible. Finally $d^0 = h^0/||h^0||$ is a fixed probability distribution of P. Moreover by Lemma 8 we have

$$||P^n d - d^0|| \to 0 \text{ as } n \to \infty$$

for all $d \in L_p$, and consequently if d^1 is a fixed probability distribution of P too then

$$||d^1 - d^0|| = ||P^n(d^1 - d^0)|| \to 0,$$

which finishes the proof. $\diamondsuit$

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